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LEMNISCATES AND EQUIPOTENTIAL CURVES OF GREEN'S FUNCTION*

By J. L. WALSH, Harvard University

1. *Introduction.* It is our purpose in the present paper to discuss some geometric properties—such as involve multiple points, tangents and normals, convexity, centers of curvature—of lemniscates and of equipotential curves for Green's function of a plane region. These equipotential curves are fundamental in the study of conformal mapping; both sets of curves arise in many other investigations, such as the expansion of analytic functions in series of polynomials, and are also of some interest for themselves alone.

By a *lemniscate* we mean the locus of a variable point the product of whose distances to a finite number of fixed points is a positive constant. If the variable point is z and the fixed points are $z_1, z_2, \cdots, z_\nu$, the condition may be written in the form

$$|(z - z_1)(z - z_2) \cdots (z - z_\nu)| = \mu > 0; \tag{1}$$

the points z_k are called the *poles* of the lemniscate and need not be all distinct.

A region is an open connected point set. By *Green's function* with pole at infinity for an infinite region R whose boundary B is finite we mean the function $G(x, y)$ which is harmonic† at every finite point interior to R, continuous and equal to zero at each point of B, and such that $G(x, y) - \frac{1}{2}\log(x^2+y^2)$ approaches a finite limit when the point $z = x+iy$ becomes infinite. Green's function for an arbitrary region R (finite or infinite) with pole in the finite point $O:(x_1, y_1)$ of R is the function which is harmonic at every point of R (in the extended plane) other than O, continuous and equal to zero at each point of the boundary of R, and such that $G(x, y)+\frac{1}{2}\log[(x-x_1)^2+(y-y_1)^2]$ approaches a finite limit when (x, y) approaches O.

Green's function need not exist for a given region, but if existent is unique. Even when Green's function does not exist, a generalized Green's function may exist (see section 15 below), and we can deduce some properties of the latter.

Green's function is invariant under inversion of the plane in a circle, in the sense that if a region R and interior point O are transformed by inversion into a region R' and point O' respectively, then Green's function for R with pole at O is transformed into a constant multiple (not zero) of Green's function for R' with pole at O'.

2. *Topological properties of lemniscates.* Let the points z_k be given. It is clear

* Address delivered by invitation before the Association at Pittsburgh, December 31, 1934.

† A function $u(x, y)$ is *harmonic* at a finite point $P:(x, y)$ if it has continuous first and second partial derivatives throughout some neighborhood of that point and if it satisfies Laplace's equation in that neighborhood. A function $u(x, y)$ is harmonic at infinity if it is harmonic and uniformly limited exterior to some circle; it follows that $u(x, y)$ approaches a limit when the point (x, y) becomes infinite.

from (1) that through an arbitrary point z_0 distinct from the z_k passes one and only one of the lemniscates (1); namely the lemniscate

$$|p(z)| = |p(z_0)|, \qquad p(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_\nu).$$

Hence there is a family of curves (1) depending on one real parameter and completely covering the plane except the points z_k. No part of the plane is covered more than once.

It follows from a more detailed study of relation (1), using the implicit function theorem for real variables and the principle of maximum modulus for an analytic function, that a lemniscate consists of a set of a finite number of Jordan curves which are mutually exterior except that one or more subsets of these curves may intersect in a finite number of points. Each point z_k lies interior to one and only one of these Jordan curves. Any set of λ such curves can separate the plane into no more than $\lambda+1$ regions. A *Jordan curve*, of course, is a one-to-one continuous transform of a circumference; a *Jordan arc* is one-to-one continuous transform of a finite line segment.

The multiple points of the locus (1) are (by the implicit function theorem) points where the first partial derivatives of the left-hand member with respect to both x and y vanish. If we write (1) in the form

$$\sum_{k=1}^{\nu} \log\left[(x - x_k)^2 + (y - y_k)^2\right] = 2 \log \mu, \; z_k = x_k + iy_k,$$

this condition can be written

$$\sum \frac{x - x_k}{(x - x_k)^2 + (y - y_k)^2} = 0, \qquad \sum \frac{y - y_k}{(x - x_k)^2 + (y - y_k)^2} = 0.$$

This new condition is then precisely the condition for the vanishing of

$$\frac{p'(z)}{p(z)} = \sum \frac{1}{z - z_k} = \sum \frac{(x - x_k) - i(y - y_k)}{(x - x_k)^2 + (y - y_k)^2}.$$

Reciprocally, it is readily proved by a study of the analytic functions involved, that at an m-fold root of $p'(z)$ not a root of $p(z)$ the curve (1) has precisely $m+1$ branches, and the tangents to these branches are equally spaced. Thus the roots of $p'(z)$ distinct from the roots of $p(z)$ are just the multiple points of the curves (1), and are in number one less than the number of distinct points z_k, if each root of $p'(z)$ is counted according to its multiplicity.

When μ is small, the locus (1) consists of exactly as many ovals as there are distinct points z_k, say l; one oval surrounds each point z_k. The number of values of μ for which (1) has a multiple point is not greater than $l-1$. The number of curves composing the locus (1) decreases or remains unchanged when μ increases, and when the locus has no multiple point this number is equal to one more than the number of roots of $p'(z)$ exterior to the locus, each root counted

according to its multiplicity. When μ is sufficiently large, the locus (1) consists of a single Jordan curve.

3. *Topological properties of equipotential curves.* The equipotential (i.e., niveau or level) curves that we shall study are the curves $G(x, y)$ equal to a positive constant, where $G(x, y)$ is Green's function for a region and (x, y) lies interior to that region. For definiteness let us choose the region R as infinite but with finite boundary B, and choose the pole of $G(x, y)$ at infinity. Then the equipotential curves have very much the same properties in R as do the lemniscates in the entire plane.

We leave the details of the study to the reader. The chief tools are the implicit function theorem, and the theorem that a function harmonic and not identically constant in a region (open) can have no maximum or minimum there.

Through a given point $P:(x_0, y_0)$ of R passes one and only one of the loci

$$G(x, y) = \mu > 0, \tag{2}$$

namely the locus $G(x, y) = G(x_0, y_0)$. The locus (2) consists of a finite number of Jordan curves which are mutually exterior except that one or more subsets may intersect in a finite number of points. Each point of B lies interior to one and only one of these curves. Any set of λ such curves can separate the plane into no more than $\lambda+1$ regions.

The multiple points of (2) are precisely the points where both first partial derivatives of $G(x, y)$ vanish, that is to say, the *critical points* of $G(x, y)$. Let $H(x, y)$ denote a function (not necessarily single-valued) conjugate to $G(x, y)$ in R. A root of the derivative of the analytic function $G(x, y)+iH(x, y)$ of order m is called an m-fold critical point of $G(x, y)$, and at such a point the curve (2) has precisely $m+1$ branches, whose tangents are equally spaced. The function $G(x, y)+iH(x, y)$ is analytic in R and the zeros of its derivative [i.e., critical points of $G(x, y)$] can have no limit point in R.

The number of mutually exterior Jordan curves of which (2) consists is not greater than the number of components of the boundary B. If B has a finite number of components, and if μ is sufficiently small, the locus (2) consists of that same number of Jordan curves, one curve surrounding each component. The number of curves composing (2) decreases or remains unchanged when μ increases, and when the locus has no multiple point this number is equal to one more than the number of critical points of $G(x, y)$ exterior to the locus, each critical point counted according to its multiplicity. When μ is sufficiently large, the locus consists of a single curve.

Green's function with pole at infinity for the infinite region bounded by the locus (2) is obviously $G(x, y)-\mu$.

4. *A formula for Green's function.* In order to study in more detail the relation between lemniscates and equipotential curves, it is desirable to derive a representation of Green's function $G(x, y)$ with pole at infinity for an infinite region R whose boundary B is finite. We start with the familiar formula which

holds under suitable circumstances even if Green's function $G(x, y)$ is replaced by an arbitrary harmonic function:

$$G(x_0, y_0) = \frac{1}{2\pi}\int_{\Gamma_1}\left(\log r \frac{\partial G}{\partial n} - G\frac{\partial \log r}{\partial n}\right) ds; \tag{3}$$

here r denotes distance from $P:(x_0, y_0)$, and n denotes interior normal. Under the present circumstances we choose P as an arbitrary point of R, and choose Γ_1 as a locus $\Gamma: G(x, y) = \mu < G(x_0, y_0)$ together with a circle Γ' whose center is P and which contains Γ in its interior. We choose, as we may do, the locus Γ so that it passes through no critical point of $G(x, y)$. Then the locus Γ consists of a finite number of mutually exterior Jordan curves. The function $G(x, y)$ is harmonic throughout the closed region bounded by Γ and Γ', so (3) is valid, even though the circle Γ' depends on P.

The function $G(x, y)$ is constant on Γ, and P lies exterior to Γ, so we have

$$\int_\Gamma G\frac{\partial \log r}{\partial n} ds = \mu \int_\Gamma \frac{\partial \log r}{\partial n} ds = 0.$$

On Γ' we set $G \equiv G_1 + \log r$. The function $\log r$ is constant on Γ', and the function G_1 is harmonic on and exterior to Γ' even at infinity. Then we have

$$\int_{\Gamma'} \log r \frac{\partial G_1}{\partial n} ds = \log r \int_{\Gamma'} \frac{\partial G_1}{\partial n} ds = 0;$$

the latter equation is familiar for functions harmonic in finite regions, and follows in the present case by an inversion with P as center. By this same inversion and Gauss's mean value theorem we can write

$$-\frac{1}{2\pi}\int_{\Gamma'} G_1 \frac{\partial \log r}{\partial n} ds = \frac{1}{2\pi r}\int_{\Gamma'} G_1 ds = G_1(\infty).$$

The value $G_1(\infty)$ is independent of the point P, so (3) can now be written in the form

$$G(x_0, y_0) = \frac{1}{2\pi}\int_\Gamma \log r \frac{\partial G}{\partial n} ds + g, \tag{4}$$

where g is constant.

We note that on Γ we have $\partial G/\partial s = 0$; but Γ passes through no critical point of $G(x, y)$, so we must have $\partial G/\partial n \neq 0$; the vanishing of both $\partial G/\partial s$ and $\partial G/\partial n$ at a point would imply the vanishing of both $\partial G/\partial x$ and $\partial G/\partial y$ at that point. Everywhere exterior to Γ we have $G(x, y) > \mu$, whence $\partial G/\partial n > 0$ on Γ. By way of abbreviation we shall set on Γ

$$\frac{1}{2\pi}\frac{\partial G}{\partial n} ds = d\sigma, \qquad 0 \leqq \sigma \leqq \sigma_1, \qquad \sigma_1 = \int_\Gamma d\sigma, \tag{5}$$

and $d\sigma$ is positive.

5. *Relations between lemniscates and equipotential curves.* A far deeper connection exists between lemniscates and equipotential curves than the mere analogy indicated by comparison of sections 2 and 3. First we remark that *a set of lemniscates*

$$(6) \qquad |p(z)| = \mu > \mu_1 > 0, \qquad p(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_\nu),$$

is a set of equipotential curves of a Green's function. Indeed, Green's function $G(x, y)$ with pole at infinity for the region exterior to the lemniscate $|p(z)| = \mu_1$ is

$$\frac{1}{\nu} \log \frac{|p(z)|}{\mu_1},$$

as may be verified directly. The locus $G(x, y) = \mu' > 0$ is the lemniscate $|p(z)| = \mu_1 e^{\mu'\nu}$, which is equivalent to (6) when we set $\mu = \mu_1 e^{\mu'\nu}$.

By way of a reciprocal, we indicate rapidly* that *If R is an infinite region whose boundary B is finite and for which Green's function with pole at infinity exists, then B can be approximated as closely as desired by a lemniscate L which lies in R.* By virtue of (5), equation (4) can be written (this new n is entirely distinct from the previous n)

$$(7) \quad G(x_0, y_0) = \int_\Gamma \log r \, d\sigma + g = g + \lim_{n\to\infty} \frac{\sigma_1}{n}[\log r_{1n} + \log r_{2n} + \cdots + \log r_{nn}]$$

where r_{kn} indicates distance to (x_0, y_0) measured from the point of Γ at which $\sigma = k\sigma_1/n$. The limit in (7) is uniform in any closed limited region exterior to Γ. In particular, the locus $\Gamma_0: G(x, y) = \mu_0 > \mu$ is approximated by the locus

$$(8) \qquad g + \frac{\sigma_1}{n} \log (r_{1n} r_{2n} \cdots r_{nn}) = \mu_0, \qquad r_{1n} r_{2n} \cdots r_{nn} = \exp \frac{n}{\sigma_1} (\mu_0 - g),$$

which is a lemniscate. The locus Γ_0 can be chosen as near to B as desired, so B can also be approximated by a lemniscate. From the uniformity of the limit in (7), it also follows that equipotential curves (lemniscates) corresponding to the exterior of the locus (8) approximate the corresponding equipotential curves for R.

It follows from the foregoing that any set of a finite number of mutually exterior Jordan curves can be uniformly approximated by a lemniscate.

6. *Location of critical points.* With these preliminaries complete we are now in a position to study in some detail the curves which we have in mind.

There exists a large body of mathematical literature connecting geometrically the location of the roots of the derivative of a polynomial with the location

* For the details, the reader may refer in the case that R is simply connected to Hilbert, Göttinger Nachrichten (1897), pp. 63–70, and in the case that R is multiply connected to Walsh and Russell, Trans. Amer. Math. Soc., vol. 36 (1934), pp. 13–28.

of the roots of the original polynomial. From our present standpoint, these results are concerned with the location of multiple points of lemniscates of a given family, and these results can be used by virtue of our discussion of section 5 to study the location of critical points of Green's function.

By way of illustration, we state the classical Gauss-Lucas theorem:

THEOREM 1. *The smallest convex point set which contains the roots of a polynomial also contains the roots of the derived polynomial.*

A point set is said to be *convex* if it contains all points of the line segment joining any pair of points of the set.

We shall use Theorem 1 to prove its analogue:

THEOREM 2. *If R is an infinite region whose boundary B is finite, then all critical points of Green's function $G(x, y)$ (if existent) for R with pole at infinity lie in the smallest convex point set K which contains B.*

Let $L_1, L_2, \cdots$ be a sequence of lemniscates which lie in R and approach B monotonically. The corresponding Green's functions with poles at infinity $G_1, G_2, \cdots$ then approach G uniformly in any closed subregion of R, and the critical points of G in R are limits of critical points in R of the G_n; this last remark follows from Hurwitz's theorem by a study of analytic functions whose real parts are the G_n, and which approach uniformly a suitably chosen analytic function whose real part is G. The smallest convex point set K_n which contains L_n contains all the critical points of G_n and also contains K. No point exterior to K lies in all the K_n. No point exterior to K can be a limit point of critical points of the G_n, so Theorem 2 is proved.

Two other theorems analogous to each other are the following. The former is due to Jensen and the latter may be proved from it.

THEOREM 3. *Let $p(z)$ be a real polynomial and let the circles whose diameters are the segments joining the pairs of conjugate imaginary roots of $p(z)$ be called Jensen circles. Then all non-real roots of the derivative $p'(z)$ lie on or within the Jensen circles.*

THEOREM 4. *Let R be an infinite region whose boundary B is finite, let B be symmetric in a line L, and let the circles whose diameters are the segments joining the points of B symmetric in L be called Jensen circles. Then all critical points of Green's function for R with pole at infinity which do not lie on L lie on or within these Jensen circles.*

7. *Location of critical points, continued.* Still another pair of corresponding theorems is the following:

THEOREM 5. *Let the circles $C_1: |z-\alpha_1| = r_1$ and $C_2: |z-\alpha_2| = r_2$ contain respectively m_1 and m_2 roots of a polynomial $p(z)$ of degree m_1+m_2. Then all the roots of $p'(z)$ lie in C_1, C_2, and a third circle*

$$(9) \qquad C_3 : \quad \left| z - \frac{m_2\alpha_1 + m_1\alpha_2}{m_1 + m_2} \right| = \frac{m_2 r_1 + m_1 r_2}{m_1 + m_2}.$$

If C_k has no point in common with either of the other two circles, it contains precisely m_1-1, m_2-1, or one root of $p'(z)$ according as k is 1, 2, *or* 3.

THEOREM 6. *Let R be an infinite region whose boundary B is finite and let $G(x, y)$ be Green's function for R with pole at infinity. Let B be divided into two parts, B_1 and B_2, and let us introduce the notation*

$$(10) \qquad m_1 = \int_{J_1} \frac{\partial G}{\partial n}\, ds, \qquad m_2 = \int_{J_2} \frac{\partial G}{\partial n}\, ds,$$

where J_1 and J_2 are analytic Jordan curves in R which contain in their respective interiors all points of B_1 but no points of B_2 and all points of B_2 but no points of B_1. Let circles C_1: $|z-\alpha_1|=r_1$ and C_2: $|z-\alpha_2|=r_2$ contain B_1 and B_2 respectively in their interiors. Then all critical points of $G(x, y)$ lie in C_1, C_2, and the third circle (9). *If C_k has no point in common with the other two circles, and if C_k contains n_k components of B, then C_k contains precisely n_1-1, n_2-1, or one critical point of $G(x, y)$ according as k is* 1, 2, *or* 3.*

A criticism of Theorem 6 is that it is not purely geometric, but depends on the evaluation of the integrals (10). It frequently occurs that R has symmetry in a point or line, and in that case it may be possible to read off by inspection the equality of m_1 and m_2.

It would be of interest to obtain other conditions, depending merely on simple geometric considerations, of equality or inequality of the integrals (10).

It is worth noticing that although Theorem 2 is a consequence of Theorem 1, the reverse is also true, due to the interpretation made in section 5 of lemniscates as equipotential curves. Similarly, Theorem 3 is a consequence of Theorem 4, and Theorem 5 of Theorem 6.

For the original proofs of Theorems 2, 4, and 6, and for further study of the location of critical points, the reader may refer to two notes by the writer.† Practically all known theorems concerning roots of the derivative of a polynomial have non-trivial analogues in the study of critical points. Frequently the original proofs are valid with only minor modifications. This field is still a promising one for investigation.

8. *Properties of equipotential curves.* We turn now from a study of critical points to a consideration of tangents, normals, and centers of curvature. For the present we restrict ourselves to equipotential curves; results on lemniscates follow as corollaries. Our main new result‡ is

* For the type of reasoning used in proving this last statement, see a note by the writer in the Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 551–554.

† Bull. Amer. Math. Soc., vol. 39 (1933), pp. 775–782; Mathematica, vol. 8 (1933), pp. 185–190.

‡ Part (a) was presented to the American Mathematical Society, June 20, 1934.

If P: (x_0, y_0) in (a) is a critical point of $G(x, y)$, the normal at P to *any branch* of the locus $G(x, y)=G(x_0, y_0)$ extended from P in either sense must intersect K. A similar remark applies to Theorem 8 below.

THEOREM 7. *Let R be an infinite region whose boundary B is finite, and let $G(x, y)$ be Green's function (supposed to exist) for R with pole at infinity. Let K denote the smallest (necessarily closed) convex point set which contains B.*

(a) *If $P:(x_0, y_0)$ is an arbitrary point of R, then the normal at P to the curve $G(x, y)=G(x_0, y_0)$ extended from P in the sense of decreasing $G(x, y)$ must intersect K.*

(b) *Let A be the point of K on this normal which is nearest to P. Then the radius of curvature at P of the curve $G(x, y)=G(x_0, y_0)$ is not less than PA.*

(c) *If B lies interior to a circle C, then all points of inflection of the curves $G(x, y)=$ constant lie interior to a concentric circle C' whose radius is 2.7 times as large.*

Theorem 2 is essentially included in (a), for at a critical point $P:(x_0, y_0)$ of $G(x, y)$ the tangents (at least two in number) to the curves $G(x, y)=G(x_0, y_0)$ are equally spaced. If ϵ is numerically small, the curve $G(x, y)=G(x_0, y_0)+\epsilon$ near P has at least two branches, and the normals studied in (a) at suitably chosen points near P are half-lines through P approximately equally spaced. All of these half-lines intersect the convex point set K, so P must lie in K.

Likewise, Theorem 2 is included in (b), for in the neighborhood of a critical point of $G(x, y)$, there are points at which the radius of curvature of the equipotential curves is arbitrarily small. Then by (b) there are points A in every neighborhood of such a critical point, so such a critical point must belong to K.

It follows from (a) that the normal when extended from P must intersect B itself, provided R is simply connected.

It is a consequence of (a) (and of (b) and of Theorem 2) that any locus $G(x, y)=$ *constant* which lies exterior to K consists of a single curve and has no multiple point. In other words, the locus $G(x, y)=$ *constant greater than maximum of* $G(x, y)$ *on* K consists of a single Jordan curve.

Remark (b) holds whether the concavity of the curve $G(x, y)=G(x_0, y_0)$ at P is turned towards A or away from A. In either case the radius of curvature is not less than PA, which in turn is not less than the shortest distance from P to K.

We shall say that an analytic Jordan arc or curve is *convex* if it has no point of inflection. An arc of a spiral may thus be convex. However, a convex analytic Jordan arc the slope-angle of whose tangent varies no more than π, or a convex analytic Jordan curve, can be cut by no line in more than two points; this property is sometimes used as the definition of convexity. Remark (c) implies that any locus $G(x, y)=$ *constant* which lies exterior to C' is a convex Jordan curve, for such a locus consists of a single curve and has no multiple point or point of inflection.

Remark (b) can be stated in the following form:

(b′) *If B lies in a half-plane, then the center of curvature at P of the curve $G(x, y)=G(x_0, y_0)$ lies either in that half-plane or in the reflection in P of that half-plane.*

In (b′) the half-plane may clearly be taken as open or closed as desired.

9. *Study of normals*. We shall prove Theorem 7(a) by proving

(a′) *If B lies interior to a half-plane which does not contain $P:(x_0, y_0)$, then the normal at P to the curve $G(x, y)=G(x_0, y_0)$ extended from P in the direction of decreasing $G(x, y)$ must intersect this half-plane.*

It follows from (a′) that the normal extended from P must lie in the sector subtended by K at P, which implies (a) whenever P lies exterior to K. If P lies interior to K, the conclusion of (a) is trivial.

Denote by (α, β) the running coordinates in (4) and (5). Suppose B lies interior to the half-plane $\beta>0$; we choose Γ also in this half-plane, and choose P as the point $(0, b)$, $b<0$. We have from (4)

$$r^2 = (\alpha - x)^2 + (\beta - y)^2, \qquad r\frac{\partial r}{\partial x} = x - \alpha, \qquad r\frac{\partial r}{\partial y} = y - \beta, \tag{11}$$
$$b - \beta < 0, \qquad \left.\frac{\partial G}{\partial y}\right|_P = \int_\Gamma \frac{b-\beta}{r^2}\,d\sigma < 0.$$

Then the function $G(x, y)$ decreases in the direction vertically upward from P, which cannot be a critical point, so the normal considered must lie above the line $y=b$. Thus (a′) follows and hence (a).

Theorem 7(a) is indeed obvious from the physical interpretation of Green's function in two dimensions. The curve Γ is composed of positive matter attracting a particle at P according to the law of inverse distances. The force at P due to this attracting matter is in magnitude and direction $\partial G/\partial n$, taken in the sense of decreasing G, and the line of action of the force clearly cuts any convex region which contains Γ but not P.

10. *Centers of curvature*. In the proof of Theorem 7(b) we still assume B and Γ to lie in the half-plane $\beta>0$ and choose P as the point $(0, b)$, $b<0$. We have the usual formula for the ordinate of the center of curvature of the curve $G(x, y)=G(x_0, y_0)$ at P:

$$Y = y_0 + \frac{1+y_0'^2}{y_0''} = y - \frac{G_y[G_x^2 + G_y^2]}{G_y^2 G_{xx} - 2G_xG_yG_{xy} + G_x^2G_{yy}}. \tag{12}$$

From (4) and (11) we write the formulas for the derivatives at P; all integrals are to be taken over Γ.

$$G_x = \int \frac{-\alpha}{r^2}\,d\sigma, \qquad G_y = \int \frac{b-\beta}{r^2}\,d\sigma, \qquad G_{xx} = \int \left[\frac{1}{r^2} - \frac{2\alpha^2}{r^4}\right] d\sigma, \tag{13}$$
$$G_{xy} = -2\int \frac{-\alpha(b-\beta)}{r^2}\,d\sigma, \qquad G_{yy} = \int \left[\frac{1}{r^2} - \frac{2(b-\beta)^2}{r^4}\right] d\sigma,$$

We reduce the right hand member of (12) to a common denominator. The combined numerator reduces to

$$
\begin{aligned}
&\left(\int \frac{b-\beta}{r^2}\, d\sigma\right)^2 \int \left[\frac{\beta}{r^2} - \frac{2b\alpha^2}{r^4}\right] d\sigma \\
(14)\qquad &- 2b\left(\int \frac{-\alpha}{r^2}\, d\sigma\right)\left(\int \frac{b-\beta}{r^2}\, d\sigma\right)\int \frac{2\alpha(b-\beta)}{r^4}\, d\sigma \\
&+ \left(\int \frac{\alpha}{r^2}\, d\sigma\right)^2 \int \left[\frac{\beta}{r^2} - \frac{2b(b-\beta)^2}{r^4}\right] d\sigma,
\end{aligned}
$$

which is the sum of two terms involving β under the integral sign (and which are clearly positive) plus the product of $-2b$ and

$$
\int \left[A\frac{\alpha}{r^2} - B\frac{b-\beta}{r^2}\right]^2 d\sigma, \qquad A = \int \frac{b-\beta}{r^2}\, d\sigma', \qquad B = \int \frac{\alpha}{r^2}\, d\sigma'
$$

which is also non-negative. Hence (14) is positive. If the denominator in (12) is positive, we now have $Y>0$, the center of curvature lies in any half-plane containing Γ but not P, hence in any half-plane containing B but not P, and (b′) is established. If the denominator in (12) is negative, we compute $2b-Y$. We use the same common denominator as before; the numerator for $2b-Y$ reduces, when we set $r^2=\alpha^2+(b-\beta)^2$, to

$$
\begin{aligned}
&\left(\int \frac{b-\beta}{r^2}\, d\sigma\right)^2 \int \frac{2b(b-\beta)^2 - \beta r^2}{r^4}\, d\sigma \\
&- 2b\left(\int \frac{b-\beta}{r^2}\, d\sigma\right)\left(\int \frac{-\alpha}{r^2}\, d\sigma\right)\int \frac{2\alpha(b-\beta)}{r^4}\, d\sigma \\
&+ \left(\int \frac{-\alpha}{r^2}\, d\sigma\right)^2 \int \frac{2b\alpha^2 - \beta r^2}{r^4}\, d\sigma.
\end{aligned}
$$

This last expression can be written in the form

$$
2b\int \left[A\frac{b-\beta}{r^2} - B\frac{\alpha}{r^2}\right]^2 d\sigma, \qquad A = \int \frac{b-\beta}{r^2}\, d\sigma' \qquad B = \int \frac{\alpha}{r^2} d\sigma',
$$

plus other terms which contain β as factors under the integral sign and are obviously negative. Thus we have $2b-Y>0$, $Y<2b$, so Theorem 7(b′) is completely proved.

11. *Points of inflection.* In proving Theorem 7(c) we shall prove somewhat more, by showing *that all points where the curvature is zero lie interior to the circle C' concentric with C and whose radius is* 2.7 $(=csc\ 22\frac{1}{2}°)$ *times as large.* It follows from the proof (see section 10) that if P lies outside C', the center of curvature is finite and lies on the normal from P in the direction of decreasing $G(x, y)$.

Let B lie interior to C; then Γ can also be chosen interior to C. It is immaterial whether C lies in the upper half-plane or not. The denominator in the right hand member of (12) can be written in the form

$$
\begin{aligned}
&\left(\int \frac{b-\beta}{r^2}\,d\sigma\right)^2 \int \frac{(b-\beta)^2-\alpha^2}{r^4}\,d\sigma \\
(15)\qquad &\quad - 2\left(\int \frac{-\alpha}{r^2}\,d\sigma\right)\left(\int \frac{b-\beta}{r^2}\,d\sigma\right)\int \frac{2\alpha(b-\beta)}{r^4}\,d\sigma \\
&\quad + \left(\int \frac{\alpha}{r^2}\,d\sigma\right)^2 \int \frac{\alpha^2-(b-\beta)^2}{r^4}\,d\sigma,
\end{aligned}
$$

where P is still the point $(0, b)$, $b<0$. We can orient the plane so that at P we have $G_x=0$. It then follows from (13) that some values of α on Γ must be positive and some negative, so the axis of y cuts the circle C, since C contains Γ in its interior. Under these circumstances, expression (15) reduces to its first term. Whenever the point Q: (α, β) lies on or interior to C, and P lies exterior to C', the acute angle which PQ makes with the axis of y is numerically less than $45°$. Consequently for such a point Q we have $(b-\beta)^2-\alpha^2>0$, so expression (15) is actually greater than zero. Thus the radius of curvature $G(x, y)=G(x_0, y_0)$ at P cannot be infinite, so Theorem 7 is completely proved.

We have proved slightly more than previously noted:

If K subtends an angle less than $45°$ at P, then P cannot be a point of inflection of an equipotential curve of Green's function.

12. *Star-shaped curves.* An analytic Jordan arc or curve is said to be *star-shaped* with respect to a point O if no ray from O is tangent to the arc or curve. An arc of a spiral may thus be star-shaped with respect to a point. However, if the tangent never passes through O, then either an analytic Jordan arc whose angle subtended at O is less than 2π or an analytic Jordan curve, can be cut in at most one point by a ray extended from O; this latter property can be used as the basis of the definition.

We have in Theorem 7 some material for the study of star-shaped equipotential curves. If points O and P: (x_0, y_0) are arbitrary, and if all angles at P from OP to points of K are acute, it follows from Theorem 7(a) that OP cannot be tangent to the curve $G(x, y)=G(x_0, y_0)$ at P. If this condition is satisfied for a fixed O and for every point P of an equipotential curve or arc, then that curve or arc must be star-shaped with respect to O. In particular, it follows that *if K or B lies interior to a circle C whose center is O, then every equipotential arc or curve exterior to C is star-shaped with respect to O.*

Still more can be deduced. If the entire set K subtends an acute angle at the point P, then the tangent at P to the equipotential curve through P cannot cut K. Let M denote the closed point set at every point of which the set K subtends an angle greater than or equal to a right angle. It follows that *every equipotential arc or curve exterior to M is star-shaped with respect to every point of K.* Then if K lies interior to a circle C, every equipotential arc or curve exterior to the concentric circle C'' whose radius is $2^{1/2}$ times as great is star-shaped

with respect to every point of K and even with respect to every point on or within C.

Let O and P be arbitrary points, P in R, and let B lie interior to the half-plane which contains O and is bounded by the perpendicular bisector of the segment OP. It follows from Theorem 7(b) that *the circle of curvature at P of the equipotential curve through P cannot pass through O*, for the center of this circle cannot lie on the perpendicular bisector of OP. That is to say, no circle through O can have contact of order higher than the first with an equipotential curve at P.

13. *Green's function with finite pole.* As we have already pointed out, Green's function when transformed by an inversion of the plane is transformed into a constant multiple of Green's function for the new region; the original equipotential curves are transformed into the new equipotential curves. Consequently our previous results enable us to read off at once certain new results concerning equipotential curves of Green's function for a finite or infinite region with finite pole. Theorem 7 yields

THEOREM 8. *Let R be an arbitrary region whose boundary (not necessarily finite) is denoted by B, and let $G(x, y)$ be Green's function (supposed to exist) for R with pole in the finite point O.*

(a) *If R is simply connected and P an arbitrary point of R, then the circular arc from $P:(x_0, y_0)$ to O normal at P to the curve $G(x, y)=G(x_0, y_0)$ extended from P in the direction of decreasing $G(x, y)$—this circular arc must intersect B. Whether R is or is not simply connected, this circular arc can lie wholly in no circular region (half-plane, interior of a circle, or exterior of a circle) which contains O but contains no point of B.*

(b″) *Let a circular region bounded by a circle Q passing through O contain $P:(x_0, y_0)$ in its interior but contain no point of B in its interior. Denote by Q' the circle through P tangent to Q at O, by Q_1 the inverse of Q' in Q, and by Q_2 the reflection of Q_1 in Q'. Then the circle of curvature of the curve $G(x, y)=G(x_0, y_0)$ at P must cut either Q_1 or Q_2; the inverse of O in that circle of curvature cannot lie between Q and the inverse of Q in Q'.*

(c) *If B lies exterior to a circle C whose center is O, then any Jordan arc or curve $G(x, y)=G(x_0, y_0)$ which lies interior to a concentric circle whose radius is* $\sin 22\frac{1}{2}° = .38$ *times as large has the property that it cannot have contact of order higher than the first with any circle through O.*

In Theorem 8 and below when we deal with the extended plane, the term *circle* is used to include straight line.

Consequences of the first and last remarks respectively of section 12, found now by inversion, are of interest:

THEOREM 9. *Let R be an arbitrary region whose boundary is denoted by B, and let $G(x, y)$ be Green's function (supposed to exist) for R with pole in the finite*

point O. If B lies exterior to a circle C whose center is O, then any equipotential arc or curve of $G(x, y)$ which lies interior to C is star-shaped with respect to O.

THEOREM 10. *Let R be an arbitrary region whose boundary is denoted by B, and let $G(x, y)$ be Green's function (supposed to exist) for R with pole in the finite point O. Let B lie exterior to the circle whose center is an arbitrary point P and which passes through O; then P cannot be a point of inflection or even a point of zero curvature of an equipotential arc or curve of $G(x, y)$. Consequently, if B lies exterior to a circle C whose center is O, then no point interior to a concentric circle C_1 whose radius is half as large can be a point of inflection of an equipotential arc or curve. If an equipotential arc or curve lies wholly interior to C_1, it is convex.*

Of course it is well known that in the situation of Theorem 8, if B itself is star-shaped with rexpect to O or is convex (with R finite), so also are all the equipotential curves in R.* The reader can easily show likewise, by methods either of function theory or of potential theory, that in the situation of Theorem 7, if B is star-shaped with respect to some point S or is convex, so also are all the equipotential curves in R.

14. *Limit of center of curvature.* One further result is of interest in connection with the situation of Theorem 7:

THEOREM 11. *Under the hypothesis of Theorem 7, let the point (x_0, y_0) approach the point at infinity in any way whatever. Then the center of curvature at (x_0, y_0) of the equipotential curve through (x_0, y_0) approaches the point whose coordinates are*

$$\frac{\int_\Gamma \alpha d\sigma}{\int_\Gamma d\sigma}, \qquad \frac{\int_\Gamma \beta d\sigma}{\int_\Gamma d\sigma}, \qquad d\sigma = \frac{1}{2\pi}\frac{\partial G}{\partial n}ds. \tag{16}$$

Let us study as before the center of curvature of the curve $G(x, y) = G(0, b)$ at the point $(0, b)$. It is now immaterial whether Γ lies in the upper half-plane or not. We use again equation (12); the right-hand member reduces to the quotient of (14) and (15). In order to compute the limit of this quotient as b becomes infinite, we find it convenient to multiply both (14) and (15) by b^4. We notice that the limits of α/r, β/r, b/r are respectively 0, 0, 1, and the former two limits are uniform with respect to all α and β on Γ. The limit of the quotient of (14) and (15) is then seen by inspection to be

$$\frac{(\int d\sigma)^2 \int \beta d\sigma}{(\int d\sigma)^3},$$

equal to the second of the numbers (16).

We leave to the reader the care of studying the abscissa X of the center of curvature and showing that its limit is the first of the numbers (16).

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* See Bieberbach, *Lehrbuch der Funktionentheorie II* (Leipzig, 1927), Chapter I.

It would be of interest to connect the present results with known theorems on univalent (*schlicht*) functions, such as are given by Bieberbach. Much of the theory of univalent functions applies also to functions which are not single-valued but whose moduli are single-valued,

In the proof thus given, we notice now that the limits of X and Y are as stated independently of the orientation of the plane. In other words, there exists an upper bound for the distance between the point (16)—this point is a center of gravity and does not depend on the choice of axes—and the center of curvature; this bound can be expressed in terms only of the integral of $d\sigma$, and of max $|\alpha|$, max $|\beta|$, and $|b|$; this bound approaches zero as $|b|$ becomes infinite. Hence the center of curvature at P apptoaches the point (16) whenever the distance from the origin to P becomes infinite, so the proof is complete.

It obviously follows from Theorem 11 that the normal at P to the equipotential curve through P passes near the point (16), when P is remote from B.

It is a consequence of Theorem 11 that *the point* (16) *is independent of the particular choice of* $\Gamma: G(x, y)=\mu$, a fact which is by no means obvious.*

Of course the point (16) is a center of gravity for every Γ with the positive weight function $\partial G/\partial n$, so that (16) lies in K.

15. *Application to lemniscates.* We can now turn from the study of general equipotential curves to the study of lemniscates. It is entirely obvious that Theorem 7 can be applied to the study of lemniscates, with analogous results. This new theorem (Theorem 12) can be proved directly without difficulty; the integrals that appear in the proof of Theorem 7 are replaced by finite sums—indeed this fact explains in large measure the close connection between lemniscates and equipotential curves. Reciprocally, if Theorem 12 is proved without the aid of Theorem 7, it is of course true that Theorem 12 can be used to prove Theorem 7.

* It is a general theorem that if two sets of non-intersecting analytic Jordan curves Γ_1 and Γ_2 bound a finite region, and if the function $g(x, y)$ is harmonic in that region, continuous in the corresponding closed region and takes the constant values g_1 and g_2 ($\neq g_1$) on Γ_1 and Γ_2 respectively, then the centers of gravity of the sets of curves Γ_1 and Γ_2 weighted with the spread $\partial g/\partial n$ are identical:

$$\frac{\int_{\Gamma_1} x \frac{\partial g}{\partial n} ds}{\int_{\Gamma_1} \frac{\partial g}{\partial n} ds} = \frac{\int_{\Gamma_2} x \frac{\partial g}{\partial n} ds}{\int_{\Gamma_2} \frac{\partial g}{\partial n} ds}$$

where all normals are *interior* for the region. Indeed, the formula

$$\int_{\Gamma_1} \frac{\partial g}{\partial n} ds = - \int_{\Gamma_2} \frac{\partial g}{\partial n} ds$$

is known. In the general formula for harmonic functions $g(x, y)$ and $h(x, y)$

$$\int_{\Gamma_1} \left(h \frac{\partial g}{\partial n} - g \frac{\partial h}{\partial n} \right) ds = - \int_{\Gamma_2} \left(h \frac{\partial g}{\partial n} - g \frac{\partial h}{\partial n} \right) ds$$

we now set $h(x, y) \equiv x$, and make use of the obvious equations

$$\int_{\Gamma_k} g \frac{\partial h}{\partial n} ds = g_k \int_{\Gamma_k} \frac{\partial x}{\partial n} ds = 0, \quad k = 1, 2.$$

It is consequently true that under the hypothesis of Theorem 8, the point O is the center of gravity of every (weighted) equipotential locus which separates O from the point at infinity. The point O is the limit of the center of curvature at P when P approaches O.

The latter suggested method is in some respects more satisfactory than the former, for the following reason. Even if Green's function does not exist for a given region R, say for definiteness an infinite region whose boundary B is finite, we can consider a sequence of lemniscates $L_1, L_2, \cdots$ which lie in R, and which approach B monotonically. Green's function $G_n(x, y)$ with pole at infinity for the exterior of L_n then either becomes infinite at every point of R or approaches uniformly in every closed limited subregion of R a finite limit $G_0(x, y)$. This new function must coincide with Green's function for R with pole at infinity if the latter exists, and in the contrary case $G_0(x, y)$ is independent of the particular choice of the L_n and is called the *generalized* Green's function. The proof of Theorem 7 based on Theorem 12 holds even for the generalized Green's function if it exists, for the equipotential curves for the L_n approach monotonically the corresponding equipotential curves of $G_0(x, y)$. A similar remark applies to the other properties of Green's function which we have derived.*

It will be noted that the proofs of Theorem 7 and later theorems do not actually make use of the fact that $G(x, y)$ vanishes on B. The proofs apply equally well (with only obvious changes) to the study of equipotential curves of any finite distribution B or Γ of positive mass in the plane, whether surface, line, or point distribution, or a combination of them.

THEOREM 12. *Let L be a lemniscate with poles $z_1, z_2, \cdots z_\nu$, and let K denote a closed convex point set which contains all these points z_k.*

(a) *If $P:(x_0, y_0)$ is an arbitrary point of L, then the normal to L at P when extended from P in the direction of decreasing $|(z-z_1)(z-z_2)\cdots(z-z_\nu)|$ must intersect K.*

(b) *Let A be the point nearest to P which is a point of intersection of this normal with K. Then the radius of curvature of L at P is not less than PA.*

(c) *If the points z_k all lie in a circle C, then all points of inflection of L lie in a concentric circle whose radius is 2.7 times as large.*

The discussion of sections 12 and 14 obviously can be applied to the study of lemniscates; the reader will have no difficulty in stating the explicit results. In

* Geometric considerations of the monotonic approach of equipotential curves of the $G_n(x, y)$ enable us to study *points of inflection* of equipotential curves of $G_0(x, y)$ as in Theorem 7(c), but do not of themselves enable us to study *points of zero curvature* of equipotential curves of $G_0(x, y)$. However, Study has noticed that the condition for zero curvature $G_y^2G_{xx}-2G_xG_yG_{xy}+G_x^2G_{yy}=0$ which appears for $G(x, y)$ from (12) can be written in the form

$$\text{Real part of}\left[\frac{F''}{(F')^2}\right] = 0,$$

where $F(z)$ is an analytic function whose real part is $G(x, y)$. This new condition is in such a form that it cannot be satisfied for the limit function $G_0(x, y)$ unless it is identically satisfied (obviously impossible) or unless it is satisfied for all functions of the sequence $G_n(x, y)$ of sufficiently large index in the neighborhood of the point (x, y) considered.

particular, when Γ is a lemniscate whose poles are the z_k, the point (16) is precisely the center of gravity of the poles z_k.*

16. *Open problems.* A number of universal constants appear in some of our theorems; let us discuss their values.

In Theorems 7(b) and 12(b) we assert that the radius of curvature is not less than PA. It cannot be asserted in either of these theorems that the radius of curvature is not less than $k\cdot\overline{PA}$, $k>1$. For instance, to treat Theorem 7(b) we need merely let B be the circle $|z|=\epsilon>0$. The radius of curvature of the equipotential curve $|z|=r>\epsilon$ is r, and the distance PA is $r-\epsilon$. There exists no $k>1$ such that $r/(r-\epsilon)>k$ for all r and $\epsilon>0$.

Even if we restrict ourselves to the case that the point P separates A and the center of curvature, it cannot be asserted that the radius of curvature is not less than $k\cdot\overline{PA}$, $k>1$. For instance, if we take the situation of Theorem 12 and set $\nu=2$, $z_1=1$, $z_2=-1$, we find that the radius of curvature at the point $(0, b)$ divided by $PA(=|b|)$ is $|(b^2+1)/(b^2-1)|$, which approaches unity as b approaches zero.

The number *one-half* which appears in Theorem 10 in defining the radius of C_1 can be replaced by no larger number. We leave the verification of this fact to the reader. In the proof, it is convenient to return to the situation of section 12 to take the special situation of Theorem 12 just used where b is large and a circle of radius b is used to contain the points z_1 and z_2.

The number 2.7 which appears in Theorem 7(c) is certainly too large, and can be replaced by a smaller number. The exact determination of the smallest number is still open.

Most of the results that we have proved in detail concern *infinitesimal* properties of equipotential loci. What results can be proved in the large?

By the use of Theorem 7 and under the hypothesis of that theorem it is not difficult to prove that *an equipotential curve which cuts C can cut no concentric circle whose radius is* 6.3 *times as large.* What is the smallest number that can be used here to replace 6.3? Conjecture: 3.

From the method of proof of Theorem 11 follows the existence of a number

* F. Lucas (Bull. de la Soc. math. de France, vol. 17 (1888), pp. 17–69) states without proof that when μ increases indefinitely the locus $|p(z)|=\mu$, $p(z)\equiv(z-z_1)\cdots(z-z_\nu)$, approaches a large circle whose center is the center of gravity (say O) of the z_k. This statement is true whether considered to apply to the position of the center of curvature, or to the behavior of the curve in the large, for instance in the sense that the curve lies between two circles whose common center is O and the difference of whose radii can be chosen arbitrarily small. Let us prove the statement in the latter sense.

Let O be the pole of polar coordinates (r, θ). Let a locus $|p(z)|=\mu$ be chosen so that it consists of a single Jordan curve, so that everywhere on this curve r is large, and so that the center of curvature lies near O. Denote by ψ the angle at a point P of the curve from OP (extended) to the tangent to the curve at P. We have $\cot\psi=\epsilon(r, \theta)/r=dr/r\,d\theta$, and (by the analogue of Theorem 11) $\epsilon(r, \theta)$ is numerically uniformly small for all points of the curve. It thus appears that on the curve $r=\int\epsilon(r, \theta)d\theta$ cannot vary more than $2\pi[\max|\epsilon(r, \theta)|]$, which can be made as small as desired by choosing μ sufficiently large.

ρ such that (under the hypothesis of Theorem 11) *if B lies in a circle C, if P denotes an arbitrary point of R, if Z denotes the center of curvature at P of the equipotential curve through P, and if P lies exterior to a circle concentric with C whose radius is ρ times as large, then Z lies in C.* What is the smallest number ρ that will suffice?

The theorems established in the present paper are of considerable generality, for the restrictions on the point sets involved are light. With further restrictions, such as those of symmetry in a point or line, further results can be deduced, and this is an open field of investigation which should yield interesting results. For instance, in Theorem 7 if B is symmetric in the x-axis, then the equipotential curves can never have vertical tangents outside of the Jensen circles for B except on the x-axis; if also C is symmetric in the x-axis, then no point on the x-axis can be a point of zero curvature of an equipotential curve if it lies outside of a concentric circle C'' whose radius is $2^{1/2}$ times as large. The reader can easily formulate corresponding results for lemniscates.

ADVANCED PREPARATORY MATHEMATICS IN ENGLAND, FRANCE AND ITALY*

By W. D. CAIRNS, Oberlin College

The last reports on the status of mathematics in Europe and the United States were made in connection with the International Congress of Mathematics at Cambridge, England, in 1912. Frequent visits of university students and teachers furnish a continuing acquaintance with the developments of courses in mathematics in European universities, but for a long time no similar information has been available concerning the mathematics that corresponds to our lower college courses. An unexpected piece of good fortune supplied the opportunity to make this investigation last spring and early summer; the present paper is a record of this study. Grateful acknowledgement must be made to a considerable number of administrators and teachers who extended many courtesies to the writer. Since the extensive reports of the year 1912 are not always readily available to our American teachers,† this report will be made independently. This is accompanied by the display of a considerable assemblage of programs, textbooks and examination lists which will be available at this meeting and at the meeting of the Mathematical Association and of the National Council of Teachers of Mathematics at Pittsburgh in December.

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* A report presented to the Mathematical Association of America at the meeting at Williamstown, Mass., Sept. 3, 1934.

† The reports of 1912 were published as *The Teaching of Mathematics in the United Kingdom*, Parts I and II, Vols. 26 and 27, Board of Education, His Majesty's Stationery Office, London, 1912. Copies are available for inspection at this meeting. See also R. C. Archibald, *The Training of Teachers of Mathematics (in eighteen countries)*, Bull. 27, Bureau of Educ., U.S. Dept of Interior, 1917.

I. England

School Certificate. English boys and girls normally complete the first stage of their education by obtaining the "School Certificate" at the age of sixteen or a little earlier in one or another of the two hundred "public schools" (Winchester, Rugby, etc.). The examination on which this is based is not materially different from those given by our College Entrance Examination Board, except that it is more strenuous in the advanced part, as Exhibit A will indicate.

Exhibit A consists of the Oxford and Cambridge Schools Examination Board papers given in July 1933 (Oxford University Press, or Deighton, Bell and Co., Cambridge.) This consists of three papers in elementary mathematics, seven questions to be chosen out of ten in each list; three papers in additional mathematics, viz., geometry, algebra and trigonometry; statics and dynamics; coordinate geometry and elementary calculus, eight questions out of ten or eleven; and two questions in practical measurements (laboratory mathematics), in all, seven two-hour examinations. The first three lists contain easy questions in arithmetic, algebra, geometry and trigonometry; on the other hand they contain questions of greater difficulty as indicated by the following typical instances:

Show that, if $2s=a+b+c$,

(i)
$$s(s-a)+(s-b)(s-c)=bc.$$

(ii)
$$\frac{s^2-(s-a)^2}{(s-c)^2-(s-b)^2}=\frac{b+c}{b-c}.$$

State, without proof, two angle properties of circles, unconnected with tangents or quadrilaterals.

O is the centre of a circle QRS and T a point within the circle. A second circle passes through O and T and intersects the first circle at R and S, R being the nearer to T. OT is produced to meet the first circle at Q and TS, QS and RS are joined. Prove that $Q\hat{S}R=Q\hat{S}T$.

OX, OY are two lines at right angles to each other. On OX are taken two points P and Q such that $OP=0.9$ in. and $OQ=2.5$ in. Find, by calculation, the point of contact with OY of a circle passing through P and Q and touching OY.

Having found this point of contact, make an accurate full-size drawing of the circle and measure its radius.

Among the most difficult questions in the three additional papers are the following:

(i) Multiply $x^{2/3}+2+3x^{-2/3}$ by $x^{2/3}-2+3x^{-2/3}$ and find the value of the product when $x=8$.
(ii) If $pv^{1.4}=85.6$, find, with the aid of tables, the value of p when $v=0.6$.
On the same axes draw the graphs of $4\sin(20x-30)^\circ$ and $x/2$, for values of x from 0 to 8.
From your graphs find the values of x between 0 and 8 which satisfy the equation

$$8\sin(20x-30)^\circ=x.$$

The base $ABCD$ of a cube is horizontal and E, F, G, H are the angular points vertically above A, B, C, D, respectively. Each edge is 4 inches in length. If L is the mid-point of HG and K the mid-point of CG, find:
(i) the length of AL and its inclination to the horizon;
(ii) the inclination of the plane AKL to the plane $DCGH$.

A uniform ladder stands on a smooth floor and leans against a smooth wall, being held at an angle A with the vertical by a horizontal rope from the foot of the ladder to the nearest point of the wall. Show that the ratio of the tension of the rope to the weight of the ladder is $(\tan A)/2$.

This question is to be done in *two* ways:

(a) By writing down the three conditions of equilibrium of the ladder;

(b) By using a graphical method based on the proposition that if a body is in equilibrium under the action of three forces, the lines of action of these forces must go through a point.

A cyclist starts at the top of a hill at a velocity of 10 ft./sec., does 1000 ft.-lb. of work by pedalling, and then free-wheels, and arrives at the top of the next hill on a level with his starting-point, with a velocity of 5 ft./sec. The mass of man and bicycle together is 150 lb. and the length of road traversed is 600 ft. Find the retarding force of friction and air-resistance, taking it to be a constant.

Find the equation of the normal at the point $(a\mu^2, 2a\mu)$ of the parabola $y^2=4ax$. Also find the coordinates of the point on the parabola at which the tangent is parallel to the normal at the point $(a\mu^2, 2a\mu)$.

It is required to build a hall for the exhibition of pictures. The hall is to be square, the area of the inside surface of each wall is to be 400 square feet, and the roof is to be flat; the walls are to be 1 foot thick and the roof is to be 6 inches thick. Find the dimensions of the hall so that the area of the outside surface of a side of the hall is as small as possible.

Find the area and x coordinate of the centre of gravity of the lamina whose edges are formed by the lines $x=0$, $y=0$ and the part of the curve

$$y = (1 - x)(5 + 4x + x^2)$$

which is cut off by these lines in the first quadrant.

This description will indicate both the minimum and the maximum requirements for those who include mathematics in the school certificate. In the academic year before last 11,461 boys and girls (the girls being only about 6% of the whole) were examined by this Board; of these 6,335 (55% approx.) were awarded school certificates. Of this whole group 10,186 offered Elementary Mathematics, and 5,648 of these (55.5%) passed with credit; 2,201 offered Additional Mathematics, and 1,074 of these (49%) passed with credit.

Exhibit B is the set of papers given by this Board in July 1932.

Exhibit C is the Cambridge School Certificate Examination (Deighton, Bell and Co.) given in July 1933, eleven questions in arithmetic, seven in geometry, nine in algebra; two two-hour papers in additional mathematics, viz., algebra, geometry, trigonometry, coordinate geometry, calculus, mechanics; and two papers in statics and dynamics. Good answers to 60% of these questions will ordinarily bring "credit," considerably less than this will give a "pass."

Exhibit D is the Cambridge School Certificate regulations for 1934. This gives a statement of the general requirements in all subjects and, in particular, gives a mathematics syllabus for the school certificate and for the earlier "junior examination." The scope of the Cambridge examination will be indicated by giving here only the portion of the syllabus covering the "additional mathematics," for one can infer from this the portion pertaining to the school certificate:

Geometry. If two triangles have their sides parallel the lines joining corresponding vertices meet in a point.

Simple properties of the orthocentre and median point of a triangle; and of the circumference, incircle, excircles, and nine-point circle.

The locus of a point whose distances from two given points are in a constant ratio is a fixed circle (the "circle of Apollonius").

The feet of the perpendiculars from any point on the circumcircle of a triangle to the sides of the triangle lie on a straight line ("Simson's line"), and the converse.

The theorems of Ceva and Menelaus and their converses.

Elementary ideas in geometry of three dimensions with applications to lines, planes, regular tetrahedron, and spheres.

Algebra. Simple applications of the theory of permutations and combinations. The binomial theorem for a positive integral index. The remainder theorem. Simple surds. The sign of a quadratic expression and its turning value.

Trigonometry. Circular measure of angles; solution of plane triangles; solution of simple equations, graphically or by the use of the addition formulae. Questions may be set involving angles of any magnitude.

Analytical Geometry. Elementary treatment of the straight line; simple geometrical properties of the loci given by the equations

$$ax^2 + 2hxy + by^2 = 0, \qquad y^2 - kx = 0, \qquad x^2 + y^2 - r^2 = 0,$$

including the use of parameters. The use of rectangular Cartesian coordinates only will be required.

Differential Calculus. Differentiation from first principles; differentiation with respect to x of x^n, $\cos x$, $\sin x$, of the product or quotient of two functions of x, and of a function of a function of x. Simple applications to turning values, the motion of a point in a straight line and of a straight line rotating in a plane about a fixed point of the line, and the tangent and normal to a curve.

Applied Mathematics. Conditions of equilibrium of a rigid body under coplanar forces; simple cases of friction; mechanical advantage, velocity ratio, and efficiency of a machine; work, energy, power; simple principles of direct impact of inelastic bodies, and of projectiles considered as an example of the resolution of velocities and accelerations.

Pressure in a homogeneous fluid at rest under gravity, excluding centre of pressure. Archimedes' principle. Boyle's law.

Exhibit E contains a syllabus in geometry by the same examiners for the Cambridge School Certificate and the junior examination for 1934. The full statement of theorems and of proofs required will be of interest to some.

Exhibit F is the Cambridge "Previous Examination," Part II, for June 1933.

Exhibit G is the Oxford University Examination Papers for 1934 called "Responsions" (Oxford Univ. Press, 116 High St., Oxford), two papers of eight questions each to be chosen from nine questions in $2\frac{1}{2}$-hour examinations. The questions are all of an elementary character. This forms the entrance examination to the University.

Exhibit H is the Matriculation Examination Papers of the University of London for January 1934 (Eyre and Spottiswoode, 6 Great New St., London, E.C. 4), two three-hour papers in elementary mathematics and two other three-hour papers in more advanced mathematics and mechanics. This examination is an equivalent for the school certificate and on the whole is of about the difficulty of the Oxford and Cambridge examinations.

I have thus given a cross-section of the mathematics covered up to the age of sixteen by those who wish to elect mathematics, so far as sets of examination papers will show this. Such an examination is administered by several different

boards in Great Britain, these being coordinated by a commission acting under the central authority of the national Board of Education, for the purpose of entrance to the universities, technical schools, etc. For example, if a boy passes the school certificate examination with "credit" in five subjects, he is exempted from taking the Previous Examination at Cambridge. Any who have failed to win the school certificate, or for any reason have not had opportunity to try for one, may obtain admission to the university by taking the Previous Examination (the "little go") at Cambridge or the "Responsions" at Oxford. But it should be noted that this occurs on the average more than a year before the corresponding stage is reached in the United States.

Further information can be gained from Exhibit I, which is the greater part of a list of English texts used in various of the good public schools up to the school certificate.

Durell, *New Algebra for Schools*, Pts. I and II, G. Bell and Sons, 1933.

Godfrey and Siddons, *Elementary Algebra*, Pts. I, (1928) and II (1927), Cambridge Univ. Press.

Durell, *Elementary Geometry*, G. Bell and Sons, 1934.

Durell, *Concise Geometry*, G. Bell and Sons, 1933.

Durell, *Shorter Geometry*, G. Bell and Sons.

Godfrey and Siddons, *Modern Geometry*, Cambridge Univ. Press, 1928.

Godfrey and Siddons, *Practical and Theoretical Geometry*, Cambridge Univ. Press, 1926.

Durell and Wright, *Elementary Trigonometry*, G. Bell and Sons, 1933.

Durell, *School Mechanics*, Pts. I and II, G. Bell and Sons, 1933.

Wright, *Graded Problem Papers*, Cambridge University Press.

Higher Certificate. Those who purpose to specialize in mathematics in universities or the technical schools usually remain two years longer in the public schools, often but not always planning to obtain the Higher Certificate "with distinction." Such pupils devote the major part of their time to mathematics, one-fourth to one-third of their time being put upon other studies such as English, foreign languages, history, etc. Here the schools show their individuality, Eton for example emphasizing breadth of training, others allowing a boy to devote practically all of his time to his specialty in the two years from sixteen to eighteen. In the course of this two-year period the enrolment shrinks as pupils change their plans or become discouraged from studying mathematics, hence the more advanced classes are smaller in numbers and enjoy more individualized instruction. A student follows the "spiral" system of studies in that, as he advances into analytical trigonometry, analytic geometry, calculus and mechanics, he will go over familiar ground a second and even a third time, with greater emphasis on the abstract theory and on increased technical skill. As he approaches the advanced examination, he practices more and more diligently on examination lists of previous years which are available both in separate form and in the textbooks to an amount always amazing to an American user of English textbooks.

These Oxford and Cambridge examinations for the Higher Certificate (Exhibit J) consist of three $2\frac{1}{2}$-hour papers in arithmetic, algebra and trigonometry;

pure and analytic geometry; differential and integral calculus. About eight parts are to be chosen from a paper of nine or ten parts. There is also a three-hour paper in statics and dynamics, and three "mathematical distinction papers" covering advanced algebra, trigonometry, coordinate and modern geometry, calculus, differential equations, and more advanced mechanics. The following questions or problems will serve as examples to show the standard demanded at this stage in these fields. Quotations from the first four papers are separated by a line from those chosen from the last three. Keep in mind that this examination is taken at the age of eighteen and frequently earlier.

If α, β are the roots of the equation $ax^2+bx+c=0$, express the roots of the equation

$$ac(x^2 + 1) - (b^2 - 2ac)x = 0$$

in terms of α and β.

Prove that, if x_1, x_2 are roots of the equation

$$(x^2 + 1)(a^2 + 1) - max(ax + 1) = 0,$$

then

$$(x_1^2 + 1)(x_2^2 + 1) = max_1x_2(x_1 + x_2).$$

Prove that, in any triangle,

(i) $r=\Delta/s$;

(ii) if f, g, h are the lengths of the bisectors of the angles terminated by the opposite sides,

$$f = \frac{2bc}{b+c}\cos\tfrac{1}{2}A;$$

(iii)
$$\frac{1}{f\sin\frac{1}{2}A} + \frac{1}{g\sin\frac{1}{2}B} + \frac{1}{h\sin\frac{1}{2}C} = \frac{2}{r}.$$

Find, in its simplest form, the equation of the tangent to the rectangular hyperbola $xy=c^2$ at the point $x=ct$, $y=c/t$.

A tangent meets the asymptotes Ox, Oy at Q, R respectively. The lines through Q parallel to Oy, and through R parallel to Ox, meet the hyperbola at S and T. Prove that ST touches the rectangular hyperbola $16xy=25c^2$.

Prove that, if

$$y = \{x + \sqrt{(1 + x^2)}\}^a,$$

then

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2y = 0.$$

Deduce that, if y_n denotes the value of d^ny/dx^n when $x=0$, then $y_{n+2}=(a^2-n^2)y_n$, and obtain the expansion of y in ascending powers of x as far as the term x^4.

Integrate

$$\int\frac{xdx}{(1 + x^2)(1 - x)};\quad \int\frac{xdx}{\sqrt{(x^2 + 4x + 5)}};\quad \int\frac{x^5dx}{(a^2 + x^2)^2}.$$

Prove that, when a and b are positive,

$$\int_0^\pi \frac{\cos^2 xdx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{a(a + b)}.$$

Four uniform rods AB, BC, CD, DA each of length $2a$ and weight W are freely hinged at their ends, and rest with the upper rods AB, AD in contact with two smooth pegs in the same horizontal

line at a distance $2c$ apart. Prove that α, the inclination of the rods to the vertical in the equilibrium position, is given by the equation $c=2a\sin^3\alpha$, and determine the horizontal and vertical components of the reaction at B in terms of W and α.

A particle is projected with velocity u from the foot of a plane of inclination β, the direction of projection lying in the vertical plane through the line of greatest slope and making an angle α with the horizontal. Show that the range on the inclined plane is

$$\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

If the particle strikes the plane at right angles, prove that $1+2\tan^2\beta=\tan\alpha\tan\beta$.

Show that, if the coefficients of the equation

$$a_0x^{2n}+a_1x^{2n-1}+\cdots+a_{2n-1}x+a_{2n}=0$$

satisfy relations

$$a_{2n}=a_0,\ a_{2n-1}=a_1,\ a_{2n-2}=a_2,\ \cdots,$$

the solution of the equation may be made to depend on the solution of an equation whose degree is not greater than n.

Solve the equation

$$3x^6+x^5-27x^4+27x^2-x-3=0.$$

Establish the formula for $\tan n\theta$ in powers of $\tan\theta$, distinguishing the cases according as n is odd or even.

Prove that

$$\tan^2\frac{\pi}{17}+\tan^2\frac{2\pi}{17}+\tan^2\frac{3\pi}{17}+\cdots+\tan^2\frac{8\pi}{17}=136.$$

PQ is a chord of a parabola such that the circle on PQ as diameter touches the parabola at another point. Prove that PQ envelops an equal parabola, which is also the locus of the centre of the circle.

Prove that the equation of a conic circumscribed to the triangle of reference is of the form

$$fyz+gzx+hxy=0.$$

The tangents to a conic at A, B, C form a triangle PQR. Show that AP, BQ, CR are concurrent.

If the conic is a parabola, show that the point of concurrence lies on the ellipse touching the sides of the triangle ABC at their middle points.

Prove Euler's theorm (its statement and the definition of a homogeneous function of degree n were given).

Show that, for the case $n=1$, the determinant

$$\begin{vmatrix} \dfrac{\partial^2 u}{\partial x^2} & \dfrac{\partial^2 u}{\partial x\partial y} & \dfrac{\partial^2 u}{\partial x\partial z} \\ \dfrac{\partial^2 u}{\partial y\partial x} & \dfrac{\partial^2 u}{\partial y^2} & \dfrac{\partial^2 u}{\partial y\partial z} \\ \dfrac{\partial^2 u}{\partial z\partial x} & \dfrac{\partial^2 u}{\partial z\partial y} & \dfrac{\partial^2 u}{\partial z^2} \end{vmatrix}$$

vanishes identically.

Transform the equation

$$\frac{d^2u}{dz^2}-\frac{2z}{1-z^2}\frac{du}{dz}+\frac{u}{(1-z^2)^2}=0$$

to one in which y is the dependent and x the independent variable, where

$$u(1-z^2)^{1/2} = y \text{ and } (1+z^2)/(1-z^2) = x,$$

obtaining the result in the form

$$(x^2-1)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} + \frac{1}{2}y = 0.$$

Find the reduction formula for

$$\int \frac{dx}{(ax^2+2bx+c)^{n+1}}.$$

Prove that, if a and $ac-b^2$ are positive and n is a positive integer, then

$$\int_{-\infty}^{\infty} \frac{dx}{(ax^2+2bx+c)^{n+1}} = \frac{2n-1}{2n}\cdot\frac{a}{ac-b^2}\int_{-\infty}^{\infty}\frac{dx}{(ax^2+2bx+c)^n}$$
$$= \frac{1.3\cdots(2n-1)}{2.4\cdots 2n}\cdot\frac{\pi a^n}{(ac-b^2)^{n+1/2}}.$$

A uniform rod can turn in a vertical plane about one end which is fixed. The rod is in equilibrium in the horizontal position, supported by a fine string attached to its other end passing over a smooth peg vertically above the middle point of the rod and then carrying a suitably chosen weight. Prove that the horizontal position is one of stable equilibrium if the inclination of the string to the horizontal is less than 45°.

For the intelligent interpretation of this examination it should be said that although the scale of grading papers varies somewhat from one examining body to another, it is safe to say that in order to obtain "distinction" a pupil must answer well five questions out of each list, while he may obtain a "pass" by answering well two or three in each list.

Exhibit K is the Cambridge Higher School Certificate Examination given in July 1933.

Exhibit L is the regulations for the 1934 higher certificate examination as prescribed by the Cambridge examination board (Deighton, Bell and Co.). It is of interest as giving the conditions in advanced mathematics and the mathematics syllabus for this group.

Exhibit M is the "Higher School Examination" papers for the University of London given in the midsummer of 1933 (Eyre and Spottiswoode), six three-hour papers of nine questions each chosen from a list of ten. A second set is given with equal emphasis on pure and applied mathematics. The order of difficulty is about the same as that of the Oxford and Cambridge examinations.

When a Winchester College boy aims at a scholarship examination he pays no attention to the higher certificate, but takes a local college examination, the Duncan Examination. (Exhibits N and O). A group of boys at Marlborough College, one year beyond the school certificate, took this summer the Marlborough College examination as an equivalent of the higher ("pass") certificate (Exhibit P). These boys ranged from sixteen to seventeen and two-thirds in age. The group of boys at Christ's Hospital, the old "Blue-coats School," who were in the last year, "Grecians," and were specializing in mathematics, were examined in June 1934 by an external examiner (Exhibit Q) who set the papers, graded them and interviewed each boy with respect to various parts of his papers.

It will be seen at once in a very concrete way that this portion of the public schools covers the mathematics of at least the first two college years in the United States. And we can scarcely adopt a more scientific criterion as to the standards current in our own individual colleges and universities than to consider whether at the close of the sophomore year our students majoring in mathematics attain this level. Again we should note that while the English student is not required, as with us, to spend the majority of this time in gaining breadth in his education, he has on the other hand attained this advanced stage in his specialty at an earlier age than with us.

Criticize the strenuous examination plan though we may on various grounds, it has the great advantage of making this material immediately available.

While the school certificate with several "credits" is sufficient for entrance to a university, each college within the university prescribes and administers its own admission conditions, on the basis of the school certificate and other evidences of preparation, such as the higher certificates. If the boy desires to better his record by obtaining a "distinction," he must take three extra papers and, roughly speaking, must answer 40% of the questions on the regular papers and, 60% of those on the extra papers.

I am showing the further character of the preparation for this stage of mathematical education by Exhibit R, part of a collection of textbooks currently used in the nine high grade public schools which I visited. Some will be recognized as old books, yet these furnish the material for training in the technique of problem-solving even in the present-day examinations.

Hall and Knight, *Higher Algebra*, Macmillan.
C. Smith, *Treatise on Algebra*, Macmillan.
Durell and Robson, *Advanced Algebra*, Vol. I, G. Bell and Sons.
Oakes, *School Coordinate Geometry*, Pitman, 1932.
Loney, *Coordinate Geometry*, Pt. I. *Cartesian Coordinates*, Macmillan, 1931.
C. Smith, *Conic Sections, Coordinate Geometry*, Macmillan, 1924.
Carslaw, *Plane Trigonometry*, Pt. I, Macmillan, 1930.
Durell and Robson, *Advanced Trigonometry*, G. Bell and Sons, 1930.
W. M. Baker, *Calculus for Beginners*.
Edwards, *Calculus for Beginners*, Macmillan.
Durell and Robson, *Elementary Calculus*, Vol. I with appendix, G. Bell and Sons, 1933.
Caunt, *Introduction to Infinitesimal Calculus*, Oxford Press.
J. M. Child, *Coordinate Geometry*, Macmillan, 1933.
Durell, *Modern Geometry*, Macmillan.
A. C. Jones, *Algebraic Geometry*, Oxford Univ. Press, 1912.
Humphrey, *Intermediate Mechanics*, Statics, Dynamics, Longmans.
Durell, *School Mathematics*, Pt. III, G. Bell and Sons.

Scholarship Examinations. A still higher stage of excellence is attained by those who expect to compete for scholarships such as will ordinarily cover their college expenses. These students often waive the higher certificate and spend more than two years beyond the school certificate at the public schools. There is naturally a small number of these students in a school and they have more attention from the mathematical master. And yet to a surprising extent

these boys are put "on their own," they spend most of their time working at their own pace and repair to the master only when they meet with difficulties in these endless lists of hard problems. Exhibit S gives the Cambridge examination papers for December 1933–March 1934 for "scholarships and exhibitions," the latter being partial or consolation scholarships but being regarded nevertheless as very honorable. For this purpose the colleges combine into five different groups. It will be a revelation to examine the questions set for these groups. That set by King's College, Trinity College and others comprises three-hour papers in algebra, trigonometry and calculus; geometry; statics and dynamics; pure and applied mathematics; general questions (foreign languages, English composition, a paraphrase); and an English essay. Some of the typical questions from these papers are given here:

Pure Mathematics

Sum the series, n being a positive integer:

(i) $$\frac{1}{(2n!)^2}+\frac{1}{(2n-2)!(2n+2)!}+\frac{1}{(2n-4)!(2n+4)!}+\cdots+\frac{1}{2!(4n-2)!}+\frac{1}{(4n)!},$$

(ii) $$1+x\cos\theta+x^2\cos 2\theta+\cdots+x^n\cos n\theta.$$

Prove that, if $0<\alpha<\pi$, then

$$\int_0^{\pi/2}\frac{d\theta}{1+\cos\alpha\cos\theta}=\frac{\alpha}{\sin\alpha}\cdot$$

What is the value of the integral when $\pi<\alpha<2\pi$?

If $I_p=\int(x^2+a)^p dx$, show that $(2p+1)I_p-2paI_{p-1}=x(x^2+a)^p$, and, hence or otherwise, evaluate

$$\int_0^\infty\frac{dx}{(x^2+1)^{3/2}}\cdot$$

Prove that the geometric mean of n positive numbers does not exceed their arithmetic mean.

Show that if a, b are positive, and p, q are positive rational numbers satisfying $1/p+1/q=1$, then

$$ab\leqq\frac{a^p}{p}+\frac{b^q}{q}\cdot$$

Geometry

Find the equation of the line joining the two points P and Q in which the circles $(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$ intersect. Show that the circle described on PQ as diameter is $(a^2+b^2)(x^2+y^2)=2ab(bx+ay)$. Find the length of the tangent from the point $(\lambda b, \lambda a)$ to any one of these circles.

Show that the equation of the normal at a point (α, β) of the curve $f(x, y)=0$ is

$$(x-\alpha)\frac{\partial}{\partial\beta}f(\alpha,\beta)=(y-\beta)\frac{\partial}{\partial\alpha}f(\alpha,\beta).$$

Hence show that from any point (ξ, η) three normals, of which either one or all three must be real, can be drawn to the parabola $y^2=4ax$.

Prove that the area of the triangle formed by joining the feet of these three normals is

$$\left\{4a(\xi-2a)^3-27a^2\eta^2\right\}^{1/2}$$

and deduce the equation to the locus of centres of curvature of the parabola.

Show that the inverse of a circle C with respect to a circle Γ is a circle C', and that if C cuts Γ at right angles, then $C'=C$.

C_1 and C_2 are two circles which cut at right angles; C_3 is a circle touching C_1 and passing through its centre; C_4 is the inverse of C_3 in C_2, and C_5 the inverse of C_4 in C_1. Show that C_5 is a circle touching C_1 and passing through the centre of C_2.

Statics and Dynamics

A uniform heavy chain rests on a smooth cycloidal curve in a vertical plane, the base of the cycloid being horizontal and its vertex uppermost and the chain extending from one cusp to the next. Prove that the pressure on the curve at any point is proportional to the radius of curvature.

Two unequal masses m_1 and m_2 are fixed to the ends of a light helical spring of natural length l and elastic modulus λ. The system is placed on a smooth horizontal table and the spring is compressed through a distance d and the system is then released at both ends simultaneously. Investigate the subsequent motion of the system. It may be assumed that the axis of the spring remains straight during the motion and that the masses remain on the axis.

A simple pendulum of length l is initially at rest. Its point of suspension is suddenly set moving with constant velocity u in a horizontal straight line. Find the conditions (i) that the bob may describe complete circles relatively to the point of suspension, (ii) that the string may oscillate about the vertical remaining taut throughout.

Exhibit T is a similar Oxford group, Balliol and other colleges, for December 1933, with a "detailed approximate syllabus" for this examination. This consisted of five papers in mathematics and a general paper. I was informed that a scholarship might be obtained by answering ably four to six and one-half questions on each of the five papers, but that this varies from year to year.

Exhibit U is a similar examination and syllabus from a second group of Oxford colleges, Merton and others.

Exhibit V is part of a list of books commonly used in preparing for scholarship examinations:

Durell and Robson, *Elementary Calculus*, Vol. II, with appendix, G. Bell and Sons, 1934.
Joseph Edwards, *Introduction to Differential Calculus*, Macmillan.
Lamb, *Infinitesimal Calculus* (most of this), Cambridge Univ. Press, 1913.
Durell, *Projective Geometry*, Macmillan.
Hardy, *Pure Mathematics*, Macmillan.
Dockeray, *Elementary Treatise on Pure Mathematics*, G. Bell and Sons, 1934.
Ramsey, *Statics*, Cambridge Unkv. Press, 1934.
Ramsey, *Dynamics*, Cambridge Univ. Press, 1934.

It is very common to find young men who have won a scholarship in December two and one-half years after the school certificate and who remain at the school for the remainder of the school year working on still more advanced parts of mathematics; they are either perfecting their technique or are anticipating the first year material of the university.

The First Year University Examination. Although the university courses in Europe are better known than the earlier courses, it will be well briefly to present here examination lists taken at the end of the first and third years of the university. Since this is not a part of the present topic, no quotations will be made here.

Exhibit W is Part I of the Cambridge Mathematical Tripos given in May-June 1933.

Exhibit X is the Oxford First Public Examination Mathematics given in May-June 1933.

The Third Year University Examination.

Exhibit Y is Part II of the Tripos, May–June 1933, and Exhibit Z is the Oxford Second Public Examination Mathematics, May-June 1933. Details as to grading the papers can be furnished if desired.

Conclusions. It will be obvious that this study will have only a remote bearing on general mathematical education in America. In Great Britain as in the United States the great mass of children are educated in the free schools, and the methods, conditions and difficulties are not greatly different from our own. But the training of selected students in a special subject cannot but have important lessons for us in both high schools and colleges in our objective of developing mathematical scholars in America in an effective manner. In the first place we do not with sufficient zeal single out at an early age the particular boy or girl who is interested in mathematics and develop this natural interest for its own sake. I asked one of the strongest mathematical masters in England, "Why do you give the boys so very extended work in advanced trigonometry, an amount as extensive now as it was in the days of Todhunter?" He replied, "Because we all like it." In the second place we allow such pupils to be held back by our system of class promotion instead of examining them frequently and assigning them to courses where they really belong. A new alignment is made every term at the English "public schools" in each subject. In the third place we have swung away too far from the examination system in our o'erweening desire to conserve interest as though, forsooth, this can be done by easing up on the pupil's responsibility and activity. I saw in these boys nothing of resentment or ennui, but rather a steady progress into great matematical power through constant acquaintance with examination papers and a frequent requirement to face these examinations. The system involves much delving, for example, in portions of advanced trigonometry never used in later courses, but it produces men who in their university years are masters of all the preliminary material which they have had, and this we cannot claim for so large a percentage of our men and women sent to the American universities.

II. France*

Through the kindness of the director and the professors of mathematics, I was able to visit classes in one of the strongest lycées of Paris. One class in kinematics was composed of thirty boys about nineteen years old enrolled for the naval course. The professor in charge lectured for fifty minutes on translation and forty-five minutes on rotation. He then questioned several at the board,

* The reports of 1912 appeared as *Commission internationale de l'enseignement mathématique; sous-commission française.* The one which concerns the present study is *Rapports: Vol. II. Enseignement secondaire.* Libr. Hatchett et Cie, Paris, 1911.

one at a time, for the remainder of the two-hour session; one was asked to interpret the path:

$$x = a \cos^2 \phi, \quad y = a \sin \phi \cos \phi, \quad z = a \sin \phi,$$

another to give the xy and the yz projections. In another class of fifty-five boys in "mathématiques spéciales" A, the last year of the lycée, the professor in charge lectured on involution for the entire two-hour period with a style and a precision as perfect as though it were a printed book, watching as he lectured to see that the boys were all attentive and only interrupting his lecture now and then to ask a boy some question of detail. Most of them took notes busily but a a few consulted lecture notes already prepared; the subject matter was available in printed form but not in just the form used in this course. The professor in charge showed me elaborate exercises, or "problèmes" (graphs, geometrical constructions, etc.), prepared in a very neat fashion. The rector of a large provincial university and the president of the lycée for boys permitted me to visit the classes of the senior professor of mathematics in mathématiques spéciales; the lecturer occupied about an hour and a third, and two boys used the remainder of the period in presenting two topics from their recent study. Both here and in Paris it was evidently required that the recitations be in a finished form and that constant review should keep them ready for the impending examinations.

Exhibit a is the "Programme des classes de Mathématiques spéciales et de Mathématiques spéciales préparatoires," (Libr. Vuibert, Paris, 1932, 2 fr. 50.) The extent of the mathématiques spéciales will be shown by quotations indicating some of the more advanced topics in this syllabus.*

Algèbre et analyse

Nombres complexes. Opérations. Formule de Moivre. Application à la multiplication et à la division des angles. Résolution trigonométrique des équations binômes.

Séries. Calcul approché de la somme d'une série convergente; limite supérieure de l'erreur commise.

Définition du nombre e; ce nombre est incommensurable.

Séries à termes complexes.

Fonctions d'une variable réelle. Infiniment petits et infiniment grands. Comparaison des infiniment petits. Application à l'étude des séries. Comparaison des infiniment grands.

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* Reference should also be made to the following:

R. C. Archibald, "*Mathematical Instruction in France,*" Proc. and Transac., Royal Soc. of Canada, 3rd Ser., Vol. IV, 1911, pp. 89–152.

Publications de la Faculté des Sciences de Marseille. Bulletin Mathématique des Facultés des Sciences et des Grandes Écoles, Gap, France, vol. 1, No. 1, Jan. 1934, 32 pp.

This new monthly periodical, edited by Professor C. E. Traynard of the faculty of sciences of the University of Marseilles, and J. Pérès, professor at the Sorbonne and examiner at the École Polytechnique, Paris, is designed in part to take the place of Nouvelles Annales de Mathématiques. It is intended to cover the fields of the licence and agrégation examinations in mathematics, and to contain memoirs on scientific and pedagogical questions in these fields.

Théorème de Rolle. Formule des accroissements finis; interprétation géométrique. Formule de Taylor.

Séries entières, à coefficients réels, d'une variable réelle. Intervalle de convergence. A l'intérieur de l'intervalle de convergence, on obtient la dérivée ou une primitive de la fonction définie par la série, en dérivant ou intégrant terme à terme. (On pourra admettre cette règle, mais on établira la convergence des séries auxquelles elle conduit.)

Fonctions de plusieurs variables réelles indépendantes. Dérivées partielles. Dérivée d'une fonction composée. Formule de Taylor. Dérivée d'une fonction implicite. Identité d'Euler pour les fonctions homogènes.

Différentielles. Différentielle première d'une fonction d'une variable. Différentielle totale, définie par la formule $df=f_x'dx+f_y'dy+\cdots$ Transformation de cette expression lorsqu'on remplace $x, y\cdots$, par des fonctions d'autres variables. Applications. Calculs sur les différentielles.

Propriétés générales des équations algébriques. Relations entre les coefficients et les racines. Calcul d'une fonction entière et symétrique des racines, en fonction des coefficients de l'équation. Élimination d'une inconnue entre deux équations au moyen des fonctions symétriques.

Intégrales. Dérivée d'une intégrale définie considérée comme fonction de sa limite supérieure. Intégrale définie.

Changement de variables. Intégration par parties.

Intégration des différentielles rationnelles et de celles qui deviennent rationnelles par un changement de variable simple.

Équations différentielles. Intégration des équations différentielles du premier ordre: 1° dans le cas où les variables se séparent immédiatement; 2° dans le cas où l'équation est homogène ou linéaire.

Intégration de l'équation différentielle linéaire du second ordre à coefficients constants sans second membre; cas où le second membre est un polynome ou une somme d'exponentielles de la forme Ae^{ax}.

Calcul numériques. Calcul approché d'une racine d'une équation par la méthode de Newton, par celle des parties proportionnelles, par celle des approximations successives.

Calcul approché d'une intégrale définie par la méthode des trapèzes.

Géométrie analytique

Vecteurs. Somme géométrique. Produit scalaire et produit vectoriel de deux vecteurs. Moments.

Coordonnées homogènes. Notions sur le points à l'infini et sur les éléments imaginaires. Rapport anharmonique de quatre points alignés et de quatre droites concourantes. Homographie. Involution.

Courbes définies par l'expression des coordonnées du point courant en fonction d'un paramètre. Exemples de construction. Les courbes du second ordre et celles du troisième ordre à point double sont unicursales.

Courbes définies par une équation non résolue. Recherche des asymptotes sur des exemples numériques simples, tels que des courbes du second ou du troisième ordre. Asymptote considérée comme tangente en un point à l'infini.

Courbes du second ordre. Division en trois genres d'après la nature des points à l'infini; asymptotes.

Homographie et involution sur une conique.

Notions succinctes sur les coniques appartenant à un faisceau linéaire; ces coniques déterminent sur une droite quelconque une involution.

Coniques homothétiques.

Enveloppes. Définition d'une courbe par l'équation générale de sa tangente.

Courbes gauches. Tangente, plan osculateur. Courbure. Application à l'hélice circulaire.

Surfaces du second ordre. Conditions pour que la surface possède un ou plusieurs points doubles, à distance finie ou à l'infini. Démontrer que pour toute surface du second ordre il existe au moins trois directions conjuguées rectangulaires.

Géométrie descriptive

Cônes et cylindres. Plan tangent; contours apparents; ombres. Intersection avec une droite. Section plane; développement.

Surfaces réglées du second ordre. Hyperboloïde de révolution et paraboloïde hyperbolique. Modes de génération. Intersection avec une droite. Plan tangent; contours apparents; ombres. Section plane.

Intersections de surfaces: deux cônes ou cylindres; cône ou cylindre et surfaces de révolution; deux surfaces de révolution dont les axes sont dans un même plan.

Mécanique

Cinématique d'un système invariable. Translation. Rotation autour d'un axe fixe. Mouvement hélicoïdal.

Statique des systèmes. Démontrer qu'il existe six conditions nécessaires d'équilibre indépendantes des forces intérieures. Ces six conditions sont suffisantes pour les systèmes invariables.

Équilibre d'un solide invariable qui n'est pas libre. Cas d'un point fixe, d'un axe fixe avec ou sans glissement le long de cet axe, d'un, deux ou trois points de contact avec un plan fixe. Réactions.

Dynamique du point. Théorème de la force vive. Énergie cinétique et énergie potentielle d'un point placé dans un champ de force.

Homogénéité. Dimensions d'une vitesse, d'une accélération, d'une force, d'un travail, d'une force vive.

According to the information received, pupils who are admitted to study for the "Baccalauréat" should spend two or three years in the study of mathematics in order to prepare for the competition of the "Grandes Écoles de l'État," that is, École Normale Supérieure, École Polytechnique, etc. This demands a thorough knowledge of "mathématiques spéciales," physics and chemistry. Those who choose may enter other schools where the competition is less difficult. It will be kept in mind that only a minority of the lycées are under such national control as will assure mathematical education of the high order here indicated.

Exhibit *b* is the textbooks used in this particular lycèe in Paris:

Solutions de problèmes de mathématiques, Baccalauréat première partie, Sér. A, A', B, Programme du 30 Avril 1931, 5e éd., par A. Tétrel, Libr. Croville-Morant, Paris.

Solutions de problèmes de mathématiques élémentaires, Baccalauréat seconde partie, 2e éd., par A. Tétrel, Libr. Croville-Morant, Paris.

Cours complet de mathématiques spéciales, Tome I, algèbre et analyse, 2e éd., par J. Haag, Gauthier-Villars, Paris, 1926.

Exercices de cours de mathématiques spéciales, Tome I, algèbre et analyse, 2e éd., par J. Haag, Gauthier-Villars, Paris, 1927.

These can be ordered through Libr. Flammarion, 4 Rue Rotrou, Paris.

III. Italy*

After five years in the primary grades and four years in the "gymnase" a boy or girl may enter the "liceo scientifico" rather than the "liceo classico," at the age of thirteen or fourteen for four years. (After this the scientific student

* The reports for 1912 were published as the *Atti della sottocommissione italiana* and may be obtained, with other reports of the Comm. Internat. de l'Enseignm. Math. from Libr. Georg et Cie, Geneva, Switzerland.

spends four years in the university or five years in the polytechnic course or six years in the medical school.) The present status of the curriculum in mathematics as in other studies is the result of the revision of the national system of education by Senator Giovanni Gentile, who was appointed minister of education by Signor Mussolini shortly after the march on Rome.

I was fortunate in being able to visit classes in the best liceo scientifico in Italy through the kind intervention of Professor Bompiani and the courteous permission of the president of the liceo. The senior professor, quiet in manner but effective and thorough, taught a third-year class (III A) of seventeen boys and two girls. They had been previously assigned and had studied the problem: Given a semicircle of radius r on a diameter AD, also a chord BC parallel to AD, to find the condition that the sum of the segments BC and CD is a maximum. The pupils found both necessary and sufficient conditions through a discussion involving inequalities, the roots of an appropriate quadratic equation, functional notation and simple calculus. A second professor, very lively, sharp in manner but good natured, conducted a quiz in a third-year class consisting of seventeen boys and one girl. Two recited at the board at a time, alternately, and each was responsible when the other needed correction. Apparently recalling recent problems, two were asked for a graphical solution of the system $x^2+y^2+2x=0$, $x+2y=0$. Why is the first a circle? Where is its center? Why? Etc. The instructor gave to two others (without reference to a text) $2 \log x - \log (x-4) = \log a$; they developed as consequences $x>4$, $a>16$. To two others he gave for discussion $f(x)=ax^2+bx+c$, $\alpha \leqq x \leqq \beta$; they ultimately came to $a^2 f(\alpha) f(\beta) <0$. A third professor had a fourth-year class of sixteen boys and two girls, most of whom were evidently eighteen, a few of whom however seemed young as fourteen. He lectured for half of the hour in advanced theory and used the other half for quizzing several at the board. They explained the summation method of finding the volume of a solid of revolution and applied it to the cone and the sphere.

I had the distinct impression that all of these students were on tip-toe and were reciting just as strenuously and circumspectly as though they were actually in the presence of the examiner. Each professor carries a group of pupils through the four years and instructs them in both mathematics and physics. Each is left to his own detailed plans, only the final examination requirements being prescribed. The second teacher above referred to said to me that he develops almost all the program during the first three years, so as to have the fourth year for applications. As to the mortality of the enrolment, about 50% of those entering the liceo drop out during the first two years, and only about 5% of those of the fourth class fail.

The character of the four-year course will be shown by Exhibit c, the current "Programmi didattici" with its syllabus of mathematics for the various types of schools. Here again quotations covering the more advanced part of the third year and all of the fourth year illuminate the high character of this liceo.

Disuguaglianze. Disequazioni di 1° e 2° grado. Sistemi misti di equazioni e disequazioni di 1° e 2° grado.

Parti di spazio e intersezioni di due piani. Retta e piano paralleli. Piani paralleli. Retta e piano perpendicolari. Distanze. Diedri e loro sezioni normali. Diedri uguali. Piani perpendicolari. Distanza di due rette sghembe. Sezioni ugualmente inclinate di diedri uguali. Angolo di due rette sghembe. Triedri. Triedri polari. Triedri uguali. Angoloidi. Piramidi. Prismi. Parallelepipedi. Cilindro indefinito; cilindro finito. Cono indefinito; cono finito. Tronco di cono. Sfera. Intersezione di rette e piani con sfere e di sfere con sfere.

Identità ed equazioni gonimetriche. Funzioni circolari di un angolo. Applicazioni ai triangoli piani; teoremi dei seni, di Nepero, di Delambre, delle proiezioni, di Carnot; varie espresson dell'-area del triangolo. Applicazione al quadrangolo inscritto. Applicazione ai poligoni regolari. Tavole trigonometriche; loro costruzione ed uso. Risoluzione di triangoli nei casi fondamentali. Applicazione ai triedri. Applicazione ai triedri rettangoli. Cenno della risoluzione dei triedri.

Studio analitico del triedro: Il teorema dei seni; le formule di Nepero, di Delambre, di Eulero, di proiezione, delle cotangenti, dei semidiedri. Le formule dei triedri rettangoli dedotte da quelle del triedro qualunque. Cenni di risoluzione dei triedri (esclusa qualsiasi discussione). Esercitazioni su detta risoluzione nei quattro casi fondamentali.

Studio analitico del triangolo sferico. Cenni di risoluzione dei triangoli sferici.

Il numero figurato ${}_nC_m$; sue proprietà elementari. Binomio di Newton. Proprietà elementari dei coefficienti binomiali. Formola ricorrente pel calcolo della somma delle k.esime potenze dei primi numeri naturali. Sua applicazione al calcolo di detta somma per $k=2$; $k=3$.

La nozione di limite di una funzione. Teoremi fondamentali che vi si riferiscono.

La nozione di derivata di una funzione di una variable e suoi significati geometrico e cinematico. La derivata di una somma, di un prodotto, di una funzione di funzione. Le derivate di x, sen x, cos x, tang x.

Massimi e minimi col metodo delle derivate.

Nozione di integrale. Area della parte di piano limitata dal grafico della funzione $f(x)$, dall'asse della x e delle perpendicolari a quest'asse condotte per i punti di ascisse a e b. Volume del solido generato da detta parte di piano in un giro completo intorno all'asse x.

Calcolo di aree e volumi relativi a cilindro, cono, sfera e sue parti.

Aree paraboliche. Volumi paraboloidici.

Discussione del sistema misto $ax^2+bx+x=0$; $m<x<n$.

Discussione del problemi di 2° grado.

Elementi di teoria dei numeri. Divisibilità. Numeri primi. M. C. D. e M. M. C. L'indicatore $\phi(n)$. Congruenze.

Analisi indeterminata di 1° grado (a due o più incognite).

Calcolo combinatorio.

Exhibit d is the texts used in this liceo:

Bisconcini, *Algebra Elementare*, Vol. 2, 11th Ed., Signorelli, Rome, 1933.
Bisconcini and Freda, *Trigonometria Piana*, 3rd Ed., Signorelli, Rome, 1932.
Severi, *Elementi de Geometria*, Vol. II,. Vallecchi, Florence, 1931.
Bini, *Analisi Matematica*, Vols. I and II, Vallecchi, Florence, 1931.

The first two are used in the first two years, the other two in the third and fourth years. There is a great uniformity in the standards over Italy under the present administration, although the strongest schools are rather naturally in Rome. The preparation demanded for professorships in the licei is severe, candidates studying general mathematics for two years and "complementary" mathematics and physics for two years. These positions are obtained only through a keen mathematical competition; as a feature of this, each candidate, given a

topic the preceding day, gives a formal presentation of a lesson to the examiners, who listen but say nothing whatever by way of commendation or criticism, the candidate learning of his work only when the final decision is announced.

The writer has chosen to utilize the available space for a presentation of the facts, leaving to the reader the comparison of these standards with those prevailing in America and the profound question of the ways in which the mathematical training of our better students can be more effectively provided.

DISCUSSION OF THE CAIRNS REPORT*

By C. W. WATKEYS, University of Rochester

The material which Professor Cairns has gathered and discussed in his paper on "Advanced Preparatory Mathematics in England, France and Italy" is amazing. To any one interested in mathematical instruction in the United States the questions at once arise, "What is the course of training which enables students to meet such tests?" and "Can any of these methods be adapted to our system?"

Last year I visited schools at different levels in these countries and in Germany in the effort to understand the educational process which differentiates the mathematical preparation of their students so much from ours.

At Oxford, the lectures in Advanced Analytic Geometry for first year students dealt with higher plane curves, and besides plane analytics assumed a knowledge of projective geometry and transformations such as inversion, polar reciprocation. The members of the class were of the same age as our freshmen, about eighteen or nineteen, but their mental alertness indicated that they were a much more highly selected group and possessed considerably greater mathematical maturity.

The difference between the extent of the mathematical knowledge of first year students in an English university and that of our freshmen interested in the subject is due to differences in the training of teachers of mathematics, in the method of selection of students, and in the philosophy of education which markedly affects programs, methods of training students, and attitudes developed in them.

There are approximately 170,000 teachers employed in the elementary schools of England and Wales, of which 75% have been certified by the Board of Education. The certified teacher has had four years in a training college or three years in a university followed by one year of special training or has completed the course in a secondary school and has had some practical experience. In all cases the candidate must pass an examination in his specialty set by the Board. As rapidly as retirement through length of service permits, uncertified teachers are being replaced by certified.

The report of the Board for 1932 states that of 22,000 teachers in secondary

* This discussion was presented at the Williamstown meeting of the Mathematical Association of America following the presentation of the report by Professor Cairns.

schools approximately 68% are university graduates. In boys schools 82% of the men are graduates. This means that the teachers of mathematics in the secondary schools are for the most part specialists in their subject. Further they regard teaching as a profession for life rather than a bridge to something else, with the consequence that a large proportion of the teachers are experienced and know their subject.

At an elementary private school in Oxford which prepares boys for the "public schools," classes in arithmetic, algebra and geometry were taught in a way to arouse interest in pure mathematics. The class was alert and responsive. There were some applied problems, but the emphasis was laid on the consideration of the mathematical relations involved.

At the Oxford high school the six teachers in the department of mathematics were all graduates, four having taken honor degrees and two pass degrees. The head of the department, who had taken his degree with first class honors in two subjects, conducted a class in algebra for boys about thirteen years of age. The Remainder Theorem was under discussion and after the class had responded to this assigned topic the instructor raised the question of the determination of the remainder obtained by dividing a cubic by a quadratic function. Under the leadership of the teacher the class worked out a method of obtaining the coefficients of the linear function which forms the remainder. The question was an "original" and it was a pleasure to witness the surprising ability of boys of their age in handling this abstract problem. The teacher told me he was much more interested in pure mathematics than in applied and he was trying to develop a similar liking in his pupils.

While the English have been making their elementary mathematics less academic and relating it more to the social environment there is still a clear emphasis on mathematics as a system of logical thinking.

The attitude toward the study of mathematics not only in England but on the Continent is well expressed by the statement in the *Suggestions to Teachers*, a publication of the British Board: "Every course should aim at developing in the pupil an appreciation of the meaning of a coherent system of mathematical ideas and the realization of the subject as an instrument of scientific and social progress."

The method of teaching mathematics in the secondary school consists in covering a considerable amount of ground in the first year, expanding topics and introducing others in the next and following years. The development of skill in mathematical manipulation takes place slowly. In the first year there is no separation of arithmetic, algebra and geometry and whenever possible their relations are emphasized. At the end of the first year backward pupils are placed in one form and the bright ones may follow a "quick" line. In the second year, the accelerated group covers the greater part of arithmetic, algebra and geometry of the syllabus without much detail. In succeeding years details are filled in and by the end of the fourth year all the topics have been introduced. The fifth year is spent in review and discussion of the more difficult portions of the work. Each topic has been considered in several different years. Numerical

calculus is studied in the fourth year. The rate of change of a function is found by means of the slope of the graph and the function by means of the area under the graph of the derivative function. In the fifth form the formulas for the derivative of x^n, $n=1, 2, 3, -1, -2, -3$ are obtained and applications made to problems of maxima and minima, approximations, areas and volumes, center of gravity, pressure and work.

The selective process of the examinations results in there being in school only 15.3% of the population in the age group 14–18 in contrast with 57% for the same age range in New York State. In the age range 16–17 about 8.6% of the population complete the secondary school as far as the leaving certificate while in New York State 18.8% of those in the age range 17–18 were still in school. About 10% of the students who obtain the leaving certificate continue for the higher certificate. Twenty out of thirty periods are devoted to pure and applied mathematics. The whole ground is covered the first year with a follow up course the second, which fills in details. Candidates spend considerable time working over problems and old examination papers. The classes are small, the student is independent and shows initiative, the supervision is individual, and students are encouraged to use the mathematical library and become acquainted with topics beyond the mathematics of the syllabus. All the elements of the tutorial system which the teachers themselves have experienced are present in the instruction. The candidate for a scholarship may spend an additional year of intensive training for the competitive examinations of the University Board, under the criticism of teachers who are interested not only for the sake of the student but also for the reputation of the school. The British are aware of the wastage of their highly selective system and intend to extend some form of secondary education to all, but in developing new types of schools for the average they do not propose to abandon the types of secondary schools which have produced such scholarship as they have. They are not satisfied with the preparation of some of their teachers, which is considered inferior to that of teachers of France and Germany. The work for the higher certificate is regarded as too specialized. The examinations do not permit of as much experimentation and adaptation as is desirable and various types of examinations are advocated. Some believe that the examination at the end of the primary stage should be delayed two years. In spite of criticism the fact remains that the student interested in mathematics studies the subject continuously from his entrance to the elementary school through his work for the higher certificate. He gradually becomes accustomed to an increasing amount of home work which requires two and a half hours each day in the fifth form. He acquires the power to work independently with concentrated effort and withal enjoys his pure mathematics.

An English inspector who visited schools in New York State and in Indiana said that "American education is like a broad highway with lines for many kinds of traffic but no arrangement for rapid transit." What we need is a "third lane" which will permit the abler students to proceed at their own pace and not at the pace of the average. We have been the first to provide secondary education for

everybody and have made admirable progress in constructing programs which will interest the average and the inferior. Our successes in these fields have been studied by educators abroad and some parts have been adapted to their systems, but in a way that does not interfere with the selection and training of the elite.

More than anything else we need to offer our abler students as good an opportunity relatively for them as we now offer our average students, that is, we need quality in the educational opportunity, not identical opportunity. Only in this way can we be truly democratic.

The project method has been carried to such extremes that the program in arithmetic reads like a course in elementary sociology from an arithmeticnl point of view. The experience abroad makes it clear that abler students are iaterested in pure mathematics and science if given the chance. The junior high school mathematics has real value and if it were followed up by continuous study throughout the senior high school, a course containing algebra, geometry, trigonometry and graphical calculus, without increasing the time usually given to the subject, we could recover much of the ground we have lost.

The requirement for the certificate to teach mathematics in New York State has recently been made 15 semester hours of college mathematics. This will tend to improve the teaching, but such an amount of training will not approximate the extent of the training of a secondary school teacher abroad.

The story of the secondary schools on the Continent is similar to that in England. There is a movement toward extending secondary education to everybody with the organization of a variety of schools for the various needs but there is no intention of abandoning a selective system which is rigorously enforced or an educational philosophy that believes that able minds can be trained by exercising in exacting fields of knowledge.

BIBLIOGRAPHY

The following publications are issued by the British Board of Education:

1. *An Outline of the Structure of the Educational System in England and Wales* (1933).
2. *The Education of the Adolescent* (1926).
3. *Handbook of Suggestions for Teachers* (1929).
4. *Secondary Education in the States of New York and Indiana* (1928). This pamphlet contains some interesting comparisons of the systems in these states with the system in Great Britain.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON FOURIER SERIES OF CONTINUOUS FUNCTIONS

By OTTO SZÁSZ, Massachusetts Institute of Technology and Brown University

It is known by the theorem of Riemann and Lebesgue, that the Fourier

coefficients of an integrable function converge to zero as n increases. Several writers have treated the question, as to whether anything can be said about the rate of decrease if one knows that the function is continuous. The answer is negative. More precisely, given any function $u(n) > 0$, such that

$$\lim_{n\to\infty} u(n) = 0,$$

there exists a continuous function

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{-inx},$$

such that

$$\lim_{n\to\infty} \sup \frac{|c_n|}{u(n)} > 0.$$

A recent paper of Randels* contains a rather complicated example of such a function. Hence, it might be of interest to give a simple example that I have used sometimes in my lectures.

By assumption we can determine a sequence of integers $n_1 < n_2 < \cdots$ such that $\sum_{\nu=1}^{\infty} u(n_\nu) < \infty$ (for instance $u(n_\nu) < 1/\nu^2$). Now the series

$$\sum_{1}^{\infty} a_n \cos nx \equiv \sum_{1}^{\infty} u(n_\nu) \cos n_\nu x$$

is absolutely and uniformly convergent, hence represents a continuous function, and

$$\lim_{n\to\infty} \sup \frac{a_n}{u(n)} = 1.$$

Moreover the conjugate series is also absolutely and uniformly convergent. If the n_ν are determined in such a way that $u(n_\nu) < 1/\nu^2$, $\nu = 1, 2, 3, \cdots$, then evidently

$$\sum_{1}^{\infty} \frac{\cos n_\nu x}{\nu^2}$$

is an example of the same kind.

A similar remark holds for series of orthogonal functions if the orthonormal set is uniformly bounded.

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* William Randels, *A remark on Fourier series of continuous functions.* This Monthly, vol. 40 (1933), pp. 97–99.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N.Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

A History of Mathematics in America before 1900. By David Eugene Smith and Jekuthiel Ginsburg. Carus Mathematical Monographs, No. 5. Chicago, The Open Court Publishing Company, 1934. viii+210 pages. $2.00; to members of the Association, $1.25 (directly through the Secretary's office).

Differential Equations. By H. B. Phillips. Third Edition. New York, John Wiley and Sons, 1934. vi+126 pages. $1.75.

Analytical Geometry of Three Dimensions. By D. Y. Sommerville. Cambridge University Press, 1934. xvi+416 pages. $4.75.

Mathematics of Finance. By L. L. Smail. New York, McGraw-Hill Book Company, 1934. xiv+274 pages. $2.75.

Mathematics Essential for Elementary Statistics. A Self-Teaching Manual. By Helen M. Walker. New York, Henry Holt and Co., 1934. xiv+246 pages. $1.50.

An Introduction to the Teaching of Science. By E. R. Downing. University of Chicago Press, 1934. viii+258 pages. $2.00.

The Mathematics of Finance. By C. N. Hulvey. New York, The Macmillan Company, 1934. xii+306 pages. $3.00.

Differential and Integral Calculus. By C. E. Love. Third Edition. New York, The Macmillan Company, 1934. xvi+384 pages. $2.75.

Analytic Geometry. By F. S. Nowlan, New York, McGraw-Hill Book Co., 1933. xii+352 pages. $2.25.

Plane and Spherical Trigonometry. By C. I. Palmer and C. W. Leigh. Fourth Edition. New York, McGraw-Hill Book Co., 1934. xiv+230 pages. $1.50.

Solid Mensuration. By W. F. Kern and J. R. Bland. New York, John Wiley and Sons, 1934. viii+74 pages. $1.25.

Essentials of Plane Trigonometry. By A. H. Sprague. New York, Prentice-Hall, 1934. Paper, viii+124 pages. 80 cents.

Plane Trigonometry and Analytic Geometry. By A. H. Sprague. New York, Prentice-Hall, 1934. x+228 pages. $1.80.

Analytical Geometry. By V. C. Poor, New York, John Wiley and Sons. 1934. vi+234 pages. $2.25.

Das Spiel der 30 Bunten Würfel. Ein Mathematischer Zeitvertreib für Jedermann. By F. Winter. Leipzig, B. G. Teubner, 1934. 128 pages. Rm. 3.60.

Statics. By A. S. Ramsey. Cambridge University Press, 1934. xii+296 pages. $3.00.

Introduction to Theoretical Physics. By J. C. Slater and N. H. Frank. New York, McGraw-Hill Book Co., 1934. xx+576 pages.

Synthetische Geometrie. By H. Liebmann. Leipzig, B. G. Teubner, 1934. viii +120 pages. Rm. 5.60.

Easily Interpolated Trigonometric Tables with Non-Interpolating Logs, Cologs and Antilogs. By F. W. Johnson. San Francisco, The Simplified Series Publishing Company, 1933. Semilooseleaf, $1.70; bound, $3.50.

Advanced Calculus. By F. S. Woods, Boston, Ginn and Co., 1934. x+398 pages. $4.60. (New edition, with slight modifications and additional exercises.)

Carrês magiques au dêgrê n, Séries numérales de G. Tarry. By Général E.Cazalas. Paris, Herman et Cie, 1934. 192 pages. 40 francs.

Reihenentwickelungen in der mathematischen Physik. By Josef Lense. Berlin, Walter de Gruyter, 1933. 178 pages. Rm. 9.50.

Lectures on Matrices. By J. H. M. Wedderburn. Colloquium Publications of the American Mathematical Society, volume XVII. New York, 1934. viii+200 pages. $3.00

Differential and Integral Calculus. Volume 1. By R. Courant. London, Blackie & Son Ltd., 1934. xiv+568 pages. 20 shillings.

An Elementary Treatise on Pure Mathematics. By N. R. Culmore Dockeray. London, G. Bell & Sons Ltd., 1934. xiv+566 pages. $5.00.

Elementary Statistics, an Introduction to the Principles of Scientific Method. By J. G. Smith. New York, Henry Holt and Co., 1934. x+518 pages. $3.50.

Gli Elementi d'Euclide e la critica antica e moderna. Edited by Federigo Enriques with diverse collaborators. Vol. 1, Rome, Alberto Stock, 1925. Books I–IV, 324 pages. Vols. 2 and 3, Bologna, Nicola Zanichelli, 1930 and 1932. Books V–IX, 356 pages. Book X, 336 pages.

The Poetry of Mathematics and Other Essays. By David Eugene Smith. The Scripta Mathematica Library, No. 1. New York, Scripta Mathematica, 1934. vi+91 pages.

Actualités scientifiques et industrielles. Paris, Herman et Cie, 1928–1934.

We have received no general descriptive announcement of this series of monographs in mathematics and the physical and biological sciences. The *Exposés de géométrie* are published under the direction of E. Cartan, the *Exposés sur l'analyse mathématique et ses applications* under J. Hadamard, the *Exposés mathématiques* to the memory of Jacques Herbrand, the *Exposés d'analyse générale* under Maurice Fréchet. The mathematical titles which have been received are listed below, the first bearing the date 1933. These monographs will not be placed for review, except that at the request of readers who are interested in writing brief reviews of particular ones, we shall gladly send them.

92. *Les éspaces métriques fondés sur la notion d'aire.* E. Cartan. 48 pages, 12 fr.
77. *Questions non résolues de géométrie algébrique.* Lucien Godeaux. 24 pages, 8 fr.
79. *Les éspaces de Finsler.* E. Cartan. 42 pages. 12 fr.
80. *La métrique angulaire des éspaces de Finsler, et la géométrie différentielle projective.* P. Delens. 36 pages, 12 fr.

109. *Ueber gewisse Ideale in einer einfachen Algebra*. Helmut Hasse. 16 pages, 4 fr.
114. *Sur quelques propriétés des polynomes*. J. Dieudonné. 24 pages, 6 fr.
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148. *Zerfallende verschränkte Produkte und ihre Maximalordnungen*. Emmy Noether, 16 pages, 5 fr.
149. *Sur les suites stationnaires*. N. Lusin. 20 pages, 5 fr.
157. *Sur les ds_2 d'Einstein a symétrie axiale*. M. Delsarte. 28 pages, 7 fr.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Hunter College of the City of New York

The mathematics club of Hunter College exists for the purpose of promoting interest in the field of mathematics, in itself, and in its relation to other fields and the outside world.

The club carries on a variety of activities. Each year, welcoming teas for the freshmen and newcomers to the Main Building are held, as well as socials given by each of the branch clubs to the other branches. Twice a year, the semi-annual affairs are held for which all the branches unite. Each party is featured by purely mathematical entertainment and games.

This year, an added feature completed the social program: The mathematics club hike, which was held on April 29th, and was greatly enjoyed by the participants.

Professor Lao Genevra Simons, Head of the Mathematics Department at Hunter College, is a member, ex officio, of the club, and Assistant Professor Marguerite Darkow was faculty adviser for the school year 1933–1934.

The academic program has consisted of talks by students, members of the faculty, and outside speakers, given at bi-weekly meetings of the club. The program for the school year 1933–1934 was as follows:

September 29, 1933: Election of officers.
October 3, 1933: "Proof of the partial fraction expansion theorem" by Henrietta Chafets.
October 17, 1933: "Handling of the election vote" by William J. Henderson, General Manager of the City News Association.
October 31, 1933: "Mathematics in business calculations" by Professor William S. Schlauch of New York University.
November 14, 1933: "Differential equations of planetary orbits" by Anna Scanlon.
November 28, 1933: "Calculus of finite differences as applied to problems in probability" by Emma Spaney and Roxee Ward.
December 12, 1933: "Use of mathematics in Astronomy" by Professor Jean Conklin.
January 2, 1934: "Problems in Statistics" by Miriam K. Stern.
February 27, 1934: "Alternative interpretations of the Hilbert postulates for Euclidean geometry" by Bella Manel.
March 13, 1934: "Magic numbers" by Professor Ernst Riess, Head of the Classics Department of Hunter College.
March 27, 1934: "Concurrency proofs of linear dependence" by Rose Bloom.
April 10, 1934: "The game of nim" by Lillian Hunvald.
April 24, 1934: "Crystallography" by Assistant Professor Robert Balk of the Geology Department of Hunter College.
May 8, 1934: "The nature of logic" by Dr. Helen Schlauch Adams.
May 15, 1934: "A new method of constructing tangents to curves" by Frances Rosenfeld.

The officers for the academic year 1933–1934 were: Gertrude B. Stern, President; Margaret LeVien, Vice President; Anatalie Goldstein, Secretary; Dolores Ferentz, Treasurer; Roxee Ward, Publicity Manager.

ANATALIE GOLDSTEIN, *Secretary*

The Mathematics Club of the College of Liberal Arts of Boston University

The purpose of the club is (1) to promote good fellowship among the students interested in mathematics; (2) to encourage the study of mathematics in the university; (3) to discuss the more interesting aspects of mathematics.

Any person interested in higher mathematics is eligible for membership in the club.

The officers for the academic year 1933–1934 were: George Livermore, President; Lily Cravitz, Vice President; Marion Peridier, Secretary; George Gibson, Treasurer; Mr. Lucien Taylor, Faculty Adviser.

The activities of the club for the academic year 1933–1934 were:

October 26, 1933: "The school of Athens" by Professor Mode.
November 9, 1933: "The normal frequency curve and DeMoivre" by Maryanna Watras.
November 23, 1933: "The compound interest law" by Thomas Mariner.
November 27, 1933: The mathematics club held a joint social meeting with the Urania (astronomy) club.
December 7, 1933: "Jewel cutting" by Marion Peridier.
February 8, 1934: "Curve fitting" by Professor Dow.
March 1, 1934: "Vector analysis" by Elmore Lundgren.
April 12, 1934: "Some curves in analytic geometry" by George Livermore.
April 26, 1934: "The modern mind of the ancient Greeks" by Mr. George W. Evans.
May 10, 1934: "Mathematics vs. chemistry" by Margorie Parker; Election of officers.

MARION PERIDIER, *Secretary*

The Mathematics Club of the University of Buffalo

The mathematics club of the University of Buffalo is open to all students of mathematics. Its monthly programs consist in general of the presentation by students or guests of topics in mathematics followed by a social period. The club was organized in 1929 and has at present about thirty members.

The faculty adviser for the academic year 1933–1934 was Harriet F. Montague and the student officers were: Charles Strobel, President; Lois Plummer, Vice President; Genevieve Grotjan, Secretary-Treasurer.

The programs for the year were as follows:

October, 1933: "Mathematicians of the past" (Student reference work followed by short talks).

December 1933: "Mongean method in geometry" by Assistant Professor Carlos E. Harrington.

February 1934: A sleigh ride followed by the selection of special problems to be presented at the next meeting.

March 1934: Student presentation of the solutions of special problems.

May 1934: Final meeting of the year. A picnic was held at Lincoln Park.

GENEVIEVE M. GROTJAN, *Secretary*

The Mathematics Club of The George Washington University

The Mathematics Club of The George Washington University was organized on October 17, 1933 for the purpose of developing a creative spirit in mathematics and fostering social contact between mathematics students of the university. Its membership is limited to those students of the university who have completed a course in the differential calculus and who have an active interest in mathematics or in its applications.

The officers for 1933–1934 were: J. Harvey Edmondston, President; Marion Fowler, Secretary-Treasurer; Professor Frank M. Weida, Faculty Adviser.

Fourteen meetings were held during the school year at which talks were given by the following members and guests on the indicated topics:

October 17, 1933: Preliminary organization.

October 24, 1933: Adoption of By-laws and the election of officers.

November 7, 1933: Installation of the faculty adviser, Dr. Frank M. Weida.

November 21, 1933: "Some properties of digital systems of notation" by J. Harvey Edmonston.

December 8, 1933: The club gave a banquet in honor of Professor Arnold Dresden of Swarthmore College who spoke on the topic: "The aims and purposes of the Mathematical Association of America."

December 19, 1933: "An appreciation of mathematics" by Robert Bray.

January 9, 1934: "Some properties of infinite series" by Charles M. Lennahan.

February 20, 1934: "Some properties of the trapezoid" by Mr. Lee Gilbert from the department of mathematics of Central High School, Washington, D. C.

March 6, 1934: "Analysis of Variance" by Walter Hendricks.

March 20, 1934: "Mathematics of life insurance" by John Lathrop.

April 10, 1934: "On conformal mapping" by Thomas Berry.

April 17, 1934: "Errors and laws of error" by Professor Frank M. Weida.

May 1, 1934: Banquet at the Cosmos Club in Washington, D. C.

May 15, 1934: Election of officers for 1934–1935.

MARION FOWLER, *Secretary-Treasurer*

The Mathematics Club of Bryn Mawr College

A mathematics club was organized by the undergraduates in January of this year. The membership included the faculty members of the department of mathematics and all students with an interest in mathematics. The meetings were held regularly, and one topic was presented at each

meeting. The meetings were followed by informal discussions at which refreshments were served.

The programs were as follows: "Mengenlehre" by Elizabeth Monroe; "Paradoxes in mengenlehre" by Dr. W. W. Flexner; "Simple operations with prime numbers" by Emmaleine A. Snyder; "Finite geometries" by Dr. Marguerite Lehr; "Analysis Situs" by Alma I. Waldenmeyer; "Regular polygons" by Dr. Anna Pell Wheeler.

It is hoped that sufficient interest was stimulated in order that the club may be reorganized next year. The club is to exist only because of the interest of the students.

ALMA I. WALDENMEYER, *President*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 131. *Proposed by W. B. Clarke, San Jose, California.*

In a general plane triangle a line is drawn from each vertex to the point which is half way around the perimeter from that vertex. These lines are concurrent in the point V. Then a line is drawn from the midpoint of each side to the point which is half way around the perimeter from that midpoint. These three lines are concurrent in the point M. If G is the centroid of the triangle and I the incenter, prove that V, M, G and I are collinear, and that the segments VM, MG and GI are in the ratio $3:1:2$.

E 132. *Proposed by H. T. R. Aude, Colgate University.*

Show that if $2a$ is the harmonic mean of the two rational numbers b and c, then the sum of the squares of the three numbers, a, b and c, is the square of a rational number.

E 133. *Proposed by L. S. Johnston, University of Detroit.*

The center and one vertex of a conic are given, as well as the focus nearer to the given vertex. The only available instrument is a draftsman's ordinary right isosceles triangle. It is required to construct the center of curvature of the conic at the given vertex. (Is it possible, under these same conditions, to construct the ends of the minor axis?)

E 134. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the circumcenter, the point of contact of side BC with the escribed circle corresponding to side AC, and the intersection of BC produced with the bisector of the exterior angle at A.

E 135. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

While dealing regularly in an ordinary bridge game, South dropped some of the undealt cards onto the floor, but retained the rest in his hand. He then observed that the number of cards on the floor was two-thirds of the number he had already dealt to West, and that the number already dealt to East was two-thirds of the number of undealt cards still in his hand. How many cards had been dealt?

SOLUTIONS

E 100 [1934, 390]. *Proposed by G. R. Livingston, State Teachers College, San Diego, California.*

In two concentric circles, locate parallel chords in the outer circle which are tangent to the inner circle, by the use of compasses only, finding the ends of the chords and their points of tangency.

Solution by Claude Shannon, University of Michigan.

It is assumed that the common center of the two circles is given. If it is not given, it may be located by the methods of Mascheroni.

The point diametrically opposite a known point on a circle of known radius may be located by "stepping off" the radius three times as a chord from the known point around the circle.

Construction. Select a point P on the inner circle for one point of tangency, and locate its diametrically opposite point as the other point of tangency. With P as center, draw a circle through C, the center of the given circles, and locate Q diametrically opposite C on this new circle. With Q as center and the radius of the larger given circle as radius, draw a circle cutting the larger given circle at S and T. ST is tangent to the smaller given circle at P. With radius CQ and centers S and T draw arcs cutting the larger arc of the larger given circle at U and V, which will be the ends of the second chord which is tangent to the smaller given circle.

Proof. By construction, CQS and CQT are congruent isosceles triangles, so that ST is the perpendicular bisector of CQ, and hence tangent to the smaller given circle at P. Furthermore, if the circle centered at Q be "translated" to center at C, it will coincide with the larger given circle, with S and T each traveling a distance equal to CQ, and hence falling at U and V respectively. Consequently, $STVU$ is a rectangle of width CQ, and UV is tangent to the smaller given circle.

Also solved by A. K. Fuller, J. Rosenbaum, Aldo Scandurra, E. P. Starke, Simon Vatriquant and G. A. Williams.

E 101 [1934, 390] *Proposed by R. S. Underwood, Texas Technological College.*

Prove that the sum of the squares of the integers in the nth row of the Pascal triangle is equal to the nth number in the $(2n-1)$st row.

Solution by Harry Siller, College of the City of New York.

In the Pascal triangle, the numbers in the nth row are the successive coefficients of the powers of x in the expansion of $(1+x)^{n-1}$. Let

$$(1) \qquad (1+x)^{n-1} = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}.$$

Then $\sum_{i=1}^{n-1}a_i^2$ is the sum of the squares of the numbers in the n-th row of the Pascal triangle. To evaluate this sum, we note that $a_i = a_{n-1-i}$, so that (1) may be rewritten as

$$(2) \qquad (1+x)^{n-1} = a_{n-1} + a_{n-2}x + a_{n-3}x^2 + \cdots + a_1x^{n-2} + a_0x^{n-1}.$$

Multiplying (1) by (2), we find that $\sum_{i=1}^{n-1}a_i^2$ is equal to the coefficient of x^{n-1} in $(1+x)^{2n-2}$. But this coefficient is the n-th number in the $(2n-1)$st row of the Pascal triangle.

Also solved by Frank Ayres, Jr., Hansraj Gupta, D. W. Hall, M. A. Heaslet, Roy MacKay, Claude Shannon, R. S. Underwood, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 102 [1934, 390]. *Proposed by Roy MacKay, Albuquerque, New Mexico.*

If P is a point on the Euler line of a triangle whose sides are a, b and c, one-kth of the distance from the circumcenter O, to the orthocenter H, then $OP^2 = (9R^2 - a^2 - b^2 - c^2)/k^2$, where R is the circumradius of the triangle.

Solution by W. E. Buker, Leetsdale, Pa.

If M is the centroid of the triangle, we find from page 175 of Johnson's Modern Geometry that $9\overline{OM}^2 = 9R^2 - (a^2+b^2+c^2)$. Therefore, since $\overline{OH}^2 = 9\overline{OM}^2$, and $\overline{OP}^2 = \overline{OH}^2/k^2$, the relationship is obviously true.

Also solved by W. B. Clarke, L. M. Kelly, C. W. Trigg, Simon Vatriquant and the proposer.

E 103 [1934, 390]. *Proposed by Harry Langman, Cooper Union, N. Y.*

Suppose the earth is a sphere of radius four thousand miles. A flexible belt is constructed around the equator, but large enough to encircle a sphere with radius one inch greater. If a vertical pole is placed under the belt at one point, drawing it taut, how high must it be, and how far from this pole does the belt first touch the earth?

Solution by E. C. Kennedy, University of Texas

Let BC be the vertical pole, with OB the radius of the earth, and CA a tangent. If the angle $COA = \theta$, is measured in radians, then the length of the arc AB is $r\theta$, where r, the radius of the earth, is 4000 miles. The length of CA is $r \tan \theta$, and $r \tan \theta - r\theta = \pi$ inches.

Using the first two terms of the expansion of tangent θ, we get

$$\theta^3 = (3\pi)/(4000\cdot 5280\cdot 12), \qquad \text{whence } \theta = .003338 \text{ radians}.$$

This gives $AB = AC = 13.352$ miles, and $BC = \frac{1}{2}(13.352)(.003338)(5280) = 117.7$ ft. as the height of the pole.

Also solved by Claude Shannon, W. J. Thome, Simon Vatriquant and the proposer.

E 104 [1934, 390]. *Proposed by L. S. Johnston, University of Detroit.*

Show that the coordinates of the center (X, Y) of the circle through the points $P_i(x_i, y_i)$ $(i=1, 2, 3)$ are given by

$$X = \frac{D[r^2, y, 1]}{2D[x, y, 1]} \qquad Y = \frac{D[x, r^2, 1]}{2D[x, y, 1]},$$

and that the length t of the tangent to the circle from the origin is given by $t^2 = -D[x, y, r^2]/D[x, y, 1]$, where $r_i^2 = x_i^2 + y_i^2$, and

$$D[x, y, z] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Solution by Maud Willey, Long Beach, Mississippi.

The equation of the circle with center (X, Y) and radius R is

$$x^2 + y^2 - 2Xx - 2Yy + X^2 + Y^2 - R^2 = 0. \tag{1}$$

The equation of the circle through the points $P_i(x_i, y_i)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{2}$$

A comparison of equations (1) and (2) shows that, if the determinant be expanded by the elements of its first row and the resulting equation divided through by $D[x, y, z]$, the coefficients of x and y give

$$X = D[r^2, y, 1]/2D[x, y, 1], \quad \text{and} \quad Y = D[x, r^2, 1]/2D[x, y, 1].$$

Similarly, the square of the tangent from the origin, by the Pythagorean theorem, is

$$t^2 = X^2 + Y^2 - R^2 = -D[x, y, r^2]/D[x, y, 1].$$

Also solved by L. J. Adams, Frank Ayres, Jr., Roy MacKay, E. P. Starke, C. W. Trigg, Simon Vatriquant, J. A. Ward and the proposer.

E 105 [1934, 391]. *Proposed by W. R. Ransom, Tufts College.*

Laugh this off: $AHAHA + TEHE = TEHAW$. It resulted from substituting a code letter for each digit of a simple example in addition, and it is required to identify the letters and prove the solution unique.

Solution by Hansraj Gupta, Gov't College, Hoshiarpur, India.

A comparison of the units and hundreds columns of this addition shows that there must be a carry from the tens column. The tens column then tells us that $A < H$, so that there can be no carry from the units or hundreds columns. The five columns then give us the following five equations:

$$A + E = W \tag{1}$$

$$2H = A + 10 \tag{2}$$

$$H = W + 1 \tag{3}$$

$$H + T = E + 10 \tag{4}$$

$$A + 1 = T. \tag{5}$$

The five linear equations in five unknowns, if solved simultaneously, produce the unique solution, $A=4$, $T=5$, $H=7$, $W=6$ and $E=2$, so that the original example in addition was $47474+5272=52746$.

Also solved by W. E. Buker, M. L. Constable, Daniel Finkel, L. M. Kelly, Theodore Lindquist, F. L. Manning, Aldo Scandurra, Claude Shannon, E. P. Starke, W. J. Thome, C. W. Trigg, M. J. Turner, Simon Vatriquant, J. A. Ward, W. W. Weber, B. C. Zimmerman and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3717. *Proposed by H. D. Ruderman, James Madison High School, Brooklyn, New York.*

How many different necklaces can be made with 6 red beads, 12 yellow, and 18 blue, if all the beads are to be used in each?

3718. *Proposed by Frank Morley, Johns Hopkins University.*

Show that the ellipse through the points given by the complex numbers a, b, c and with center $(a+b+c)/3$ has semi-axes whose lengths are

$$|a + \omega^2 b + \omega c|/3 \pm |a + \omega b + \omega^2 c|/3$$

where $\omega = (-1+i\sqrt{3})/2$.

3719. *Proposed by Morgan Ward, Institute for Advanced Study.*

Prove that

$$\sum_{r=0}^{n} {}_nC_r/(x+r)(x+r+1)\cdots(x+r+n) = 2^n/x(x+2)(x+4)\cdots(x+2n).$$

3720. *Proposed by C. J. Coe, University of Michigan.*

In transforming coordinates from the rectangular system $OX_1Y_1Z_1$ to the congruent rectangular system $OX_2Y_2Z_2$, we have the following table of cosines:

	X_1	Y_1	Z_1
X_2	λ_1	μ_1	ν_1
Y_2	λ_2	μ_2	ν_2
Z_2	λ_3	μ_3	ν_3

the determinant of the array having the value unity. Prove that

$$\lambda_1 + \mu_2 + \nu_3 \geqq -1.$$

SOLUTIONS

3641 [1933, 561]. *Proposed by J. A. Bullard, University of Vermont.*

Prove, for positive integers p and q, the following summations for binomial coefficients:

$$\text{(a)} \qquad \sum_{h=0}^{p} \frac{(-1)^h}{2h+1}\, {}_pC_h = 2^{2p}(p!)^2/(2p+1)!$$

$$\text{(b)} \qquad \sum_{h=0}^{p+q} (-1)^h\, {}_{2p}C_h\, {}_{2q}C_{p+q-h} = (-1)^p(2p)!(2q)!/(p+q)!p!q!.$$

If $p \neq q$ in (b) the terms which are undefined are to be omitted in the summation.

Solution by Frank Ayres, Jr., Dickinson College.

(a) This sum is a special case ($s=t=1$) of

$$\sum_{h=0}^{p} \frac{(-1)^h}{{}_t\Pi_s}\, {}_pC_h \qquad \text{where} \qquad {}_t\Pi_s = \prod_{j=t}^{s} (2h+2j-1).$$

For the general case, we have

$$\begin{aligned}
\sum_{h=0}^{p} \frac{(-1)^h}{{}_t\Pi_s}\, {}_pC_h &= \sum_{h=0}^{p-1} (-1)^h \left(\frac{1}{{}_t\Pi_s} - \frac{1}{{}_{t+1}\Pi_{s+1}} \right) {}_{p-1}C_h \\
&= 2(s-t+1) \sum_{h=0}^{p-1} \frac{(-1)^h}{{}_t\Pi_{s+1}}\, {}_{p-1}C_h \\
&= 2^2(s-t+1)(s-t+2) \sum_{h=0}^{p-2} \frac{(-1)^h}{{}_t\Pi_{s+2}}\, {}_{p-2}C_h \\
&= \cdots\cdots\cdots\cdots\cdots \\
&= \frac{2^p(s-t+1)(s-t+2)\cdots(s-t+p)}{(2t-1)(2t+1)\cdots(2s+2p-1)} \\
&= \frac{2^{s-t+2p}(s-t+p)!(s+p-1)!(2t-2)!}{(s-t)!(t-1)!(2s+2p-1)!}.
\end{aligned}$$

(b) We can write

$$\sum_{h=0}^{p+q} (-1)^h\, {}_{2p}C_h\, {}_{2q}C_{p+q-h} = (-1)^{p-q} \frac{(2p)!(2q)!}{(p+q)!^2} \sum_{h=0}^{2q} (-1)^h\, {}_{p+q}C_h\, {}_{p+q}C_{2q-h}.$$

Then,

$$\sum_{h=0}^{2q} (-1)^h\, {}_{p+q}C_h\, {}_{p+q}C_{2q-h} = (-1)^q\, {}_{p+q}C_q$$

as can be seen by equating the coefficients of x^{2q} in $(1-x)^{p+q}(1+x)^{p+q} = (1-x^2)^{p+q}$ and the desired result follows.

Solved also by M. Dresher, Jewell C. Hughes, Roy MacKay, E. P. Starke, and the proposer.

Editorial Note. The remark at the top of page 456 of volume 41 of this MONTHLY is applicable to (a) and to the above generalization. Using the notation of the solution, we have at once

$$\sum_{h=0}^{p} \frac{(-1)^h}{{}_t\Pi_s}\, {}_pC_h = (-1)^p \left[\Delta^p\left(\frac{1}{{}_t\Pi_s}\right)\right]_{h=0},$$

where Δ applies to h in ${}_t\Pi_s$. The evaluation of the pth difference is easy; and, after setting $h=0$, in this difference, we find for the right side the result given in the solution.

The other solutions of (a) were obtained by setting the sum equal to

$$\int_0^1 (1-x^2)^p dx = \int_0^{\pi/2} \cos^{2p+1}\theta d\theta,$$

or by processes which employed similar integrals.

The proposer's solution of (b) used the formula

$$\int_0^{\pi/2} \sin^{2p} x \cos^{2q} x dx = \frac{\Gamma\left(\frac{2p+1}{2}\right)\Gamma\left(\frac{2q+1}{2}\right)}{2\Gamma(p+q+1)} = \frac{(2p)!(2q)!\pi}{2^{2(p+q)+1}(p+q)!p!q!},$$

together with another evaluation of the left side given in his article *On the evaluation of certain trigonometric integrals* in this MONTHLY, vol. 40 (1933), p. 162.

$$\frac{(-1)^p}{2^{2(p+q)}} \Delta_{p+q}^{2p} \frac{\pi}{2} = \frac{(-1)^p \pi}{2^{2(p+q)+1}} \sum_{h=0}^{p+q} (-1)^h\, {}_{2p}C_h\, {}_{2q}C_{p+q-h}.$$

The combination of these two formulae gives the desired result.

Morgan Ward states that (b) gives a solution of a problem, attributed to Catalan, on page 138, vol. 2 of Polya and Szegö's *Aufgaben und Lehrsätze aus der Analysis*: To prove that $(2a)!(2b)!$ is exactly divisible by $a!b!(a+b)!$. However, a simpler proof is given by another method which he sketched.

The remaining solutions of (b) made use of the comparison of powers of x in the identity $(1-x)^m(1+x)^m=(1-x^2)^m$. Miss Hughes referred to a similar problem in Crystal's *Algebra*, Part 2, second ed., p. 210, ex. 11. The method indicated in the first lines of this note may also be applied to (b) after a slight transformation. Here we reach a result which requires the evaluation of

$$\Delta^{p+q}{}_{p+q}C_{2p-0} = [\Delta^{p+q}{}_{p+q}C_{2p-h}]_{h=0}.$$

Since this evaluation appears to be the essence of (b) and since the various solutions have in substance accomplished this evaluation, the result will be set down for future reference. If m and n are positive integers, and ${}_mC_k=0$ when k is negative or $k>m$,

$$\begin{aligned}\Delta^m{}_mC_{n-0} &= (-1)^{m-(n/2)}{}_mC_{n/2}, \quad \text{if } n \text{ is even},\\ &= 0 \qquad\qquad\qquad\quad, \quad \text{if } n \text{ is odd}.\end{aligned}$$

3644 [1933, 561]. *Proposed by J. Rosenbaum, The Milford School, Milford, Connecticut.*

Prove that in a tetrahedron, the three conditions:

1. The altitudes are concurrent,
2. The sums of the squares of the pairs of opposite edges are equal, and
3. The opposite edges are perpendicular to each other,

are equivalent (i.e., anyone of the conditions implies the other two).

I. *Solution by J. W. Clawson, Ursinus College.*

Let $ABCD$ be the tetrahedron. Take the X-axis along BD, B at the origin, the XY-plane coinciding with BCD. Thus B is (0, 0, 0), D is $(a, 0, 0)$, C is $(b, c, 0)$, A is (p, q, r).

1. Then the equations of the altitude from A to the opposite face are $x=p$, $y=q$.

The equations of the altitude from B to the opposite face are

$$\frac{x}{cr} = \frac{y}{(a-b)r} = \frac{z}{ac+bq-aq-cp}.$$

The equations of the altitude from C to the opposite face are

$$x = b, \qquad \frac{y-c}{r} = -\frac{z}{q}.$$

The equations of the altitude from D to the opposite face are

$$\frac{x-a}{-cr} = \frac{y}{br} = \frac{z}{cp-bq}.$$

These eight equations are all true if and only if $p=b$ and $cq+b^2=ab$.

2. The second condition of the problem gives

$$p^2 + q^2 + r^2 + (a - b)^2 + c^2 = (b - p)^2 + (c - q)^2 + r^2 + a^2$$
$$= (a - p)^2 + q^2 + r^2 + b^2 + c^2.$$

These equations also reduce to $p=b$, $bp+cq=ab$.

3. Applying the condition that the sums of the products of corresponding direction numbers must be zero, if the lines are to be perpendicular,

$$p(a - b) - cq = 0, \qquad (b - p)a = 0, \qquad (p - a)b + qc = 0.$$

These equations likewise reduce to $b=p$, $b^2-ab+cq=0$. Hence the three conditions reduce to the same analytical statement.

II. *Solution by A. S. Householder, Washburn College.*

Let the vertices be A_i $(i=1, 2, 3, 4)$ determined by the vectors $\mathbf{a}_i$ from any given origin. The plane through A_i perpendicular to A_kA_l contains the altitude h_i, and its equation is

(1) $$\pi_{ij} \equiv (\mathbf{a}_k - \mathbf{a}_l)\cdot(\mathbf{x} - \mathbf{a}_i) = 0.$$

Similarly the plane $\pi_{ji}=0$ passes through A_j, perpendicular to A_kA_l, and it contains h_j. A necessary and sufficient condition that h_i and h_j intersect is $\pi_{ij}-\pi_{ji}=0$, or

(2) $$(\mathbf{a}_k - \mathbf{a}_l)\cdot(\mathbf{a}_j - \mathbf{a}_i) = \mathbf{a}_j\mathbf{a}_k + \mathbf{a}_i\mathbf{a}_l - \mathbf{a}_i\mathbf{a}_k - \mathbf{a}_j\mathbf{a}_l = 0.$$

But this means that A_kA_l is perpendicular to A_iA_j. It also means that

(3) $$(\mathbf{a}_i - \mathbf{a}_l)^2 + (\mathbf{a}_j - \mathbf{a}_k)^2 = (\mathbf{a}_i - \mathbf{a}_k)^2 + (\mathbf{a}_j - \mathbf{a}_l)^2, \text{ or}$$
$$(A_iA_l)^2 + (A_jA_k)^2 = (A_iA_k)^2 + (A_jA_l)^2.$$

A necessary and sufficient condition that all the altitudes meet in a point is that each pair of h_i, h_j, h_k meet in a point; or that the conditions of the problem (2) and (3) be satisfied. Hence each of these three conditions implies the other two.

Solved also by Roy MacKay, A. Pelletier, Maud Willey, W. P. Udinski, and S. Vatriquant.

Editorial Note. The solution by Udinski used vector equations in a manner different from the solution II above. The remaining solutions were geometric. The first part of solution II showing the equivalence of (1) and (3) can be stated in a simple geometric manner without the use of vectors. Also the rest of the proof can be replaced by a simple geometric proof, which may not be so elegant as the vector proof. It was observed by some of the solvers that (3) of the problem is equivalent to saying that opposite edges are perpendicular for two pairs of such edges. A similar remark applies to (2); but these facts are obvious from the proofs.

The method used by Vatriquant in part of his proof is of interest. It is as follows:

Let x, x', y, y', z, z' be the lengths of pairs of opposite edges of any tetrahedron. Each pair of opposite edges, say x, x', determines uniquely a pair of

parallel planes, one containing x and the other, x'. The tetrahedron can thus be inscribed in a parallelopiped with edges x'', y'', z''. We have

$$x^2 + x'^2 = 2(y''^2 + z''^2); \quad y^2 + y'^2 = 2(z''^2 + x''^2); \quad z^2 + z'^2 = 2(x''^2 + y''^2).$$

If then condition (2) of the problem is satisfied, $x''=y''=z''$, and the parallelopiped has faces which are rhombi, and therefore the diagonals of these faces are perpendicular. Thus (3) is true. If (3) is true, the diagonals of the faces of the parallelopiped are perpendicular, and these faces are rhombi. Hence $x''=y''=z''$ and the above relations give (2).

The following properties of orthogonal tetrahedrons were given by Vatriquant without proof:

(a) The common perpendiculars to opposite edges meet in the same point as the altitudes.

(b) The midpoints of the edges and the feet of the common perpendiculars for pairs of opposite edges are equidistant from the centroid of the tetrahedron.

(c) If d_1, d_2, d_3 are the lengths of the shortest distances between opposite edges, then $xx'd_1=yy'd_2=zz'd_3$, where x, x' are the lengths of a pair of opposite edges corresponding to d_1, etc.

(d) The sum of the squares of the products of the lengths of opposite edges is four times the sum of the squares of the areas of the faces.

(e) The sum of the six dihedral angles and of the twelve angles between edges and faces is equal to twelve right angles.

3645 [1933, 562]. *Proposed by Paul S. Dwyer, Antioch College.*

Show that the value of the determinant formed by deleting the kth column from the array

$$\left\|\begin{matrix} 1 & 1 & 1 & 1 & 1 \cdots 1 & 1 \\ 0 & {}_1C_1 & {}_2C_1 & {}_3C_1 & {}_4C_1 \cdots {}_nC_1 & {}_{n+1}C_1 \\ 0 & 0 & {}_2C_2 & {}_3C_2 & {}_4C_2 \cdots {}_nC_2 & {}_{n+1}C_2 \\ 0 & 0 & 0 & {}_3C_3 & {}_4C_3 \cdots {}_nC_3 & {}_{n+1}C_3 \\ 0 & 0 & 0 & 0 & {}_4C_4 \cdots {}_nC_4 & {}_{n+1}C_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 \cdots {}_nC_n & {}_{n+1}C_n \end{matrix}\right\|$$

is ${}_{n+1}C_{k-1}$ (*not* ${}_nC_{k-1}$).

I. *Solution by E. P. Starke, Rutgers University.*

From the binomial expansion of $(1-1)^{s-r}$ we have $\sum_{t=0}^{s-r}(-1)^t\cdot{}_{s-r}C_t=0$, $r<s$. Upon multiplying each term by ${}_{s-1}C_{r-1}$, and noting that $(s-1)!\cdot{}_{s-r}C_t/(r-1)!(s-r)!={}_{r+t-1}C_t\cdot{}_{s-1}C_{r+t-1}$, we have

$$(1) \qquad \sum_{t=0}^{s-r}(-1)^t{}_{r+t-1}C_t\cdot{}_{s-1}C_{r+t-1} = 0 \text{ for } r<s, \text{ but } = 1 \text{ for } r=s.$$

Let us transform the given matrix by taking the rows in order ($r=1, 2, \cdots, n+1$) and adding to the elements of each row the corresponding elements of the succeeding rows multiplied respectively by $-{}_rC_1, +{}_{r+1}C_2, -{}_{r+2}C_3, \cdots$. By (1), we shall have zero for every element for which $r<s\leqq n+1$, where s represents the column and r, the row in which the element is situated. If $r=s\leqq n+1$, the element is 1. In the last column ($s=n+2$) for each element the sum (1) lacks its last term. Hence the element in the rth row of the last column is $(-1)^{n-r+1}{}_{n+1}C_{n-r+2}=(-1)^{n+r+1}{}_{n+1}C_{r-1}$. The matrix is now such that, if we delete the last column, there is left a square matrix with unity in the principal diagonal and all other elements of this square matrix are zero.

These transformations will not affect the value of any determinant formed by the deletion of a column.

If now the kth column ($k\neq n+2$) is deleted, all elements of the kth row of the resulting determinant will be zero, except the last, $(-1)^{n+k+1}{}_{n+1}C_{k-1}$. The cofactor of this last element is easily seen to be $(-1)^{n+k+1}$. Hence the expansion of the determinant according to elements of the kth row is ${}_{n+1}C_{k-1}$. If however the $(n+2)$th column is deleted, the resulting determinant has clearly the value 1, which equals ${}_{n+1}C_{n+1}$.

II. *Solution by J. Williamson, The Johns Hopkins University.*

From the given array form a square matrix A of $n+2$ rows and columns by adding a last row whose elements are all zeros except ${}_{n+1}C_{n+1}$ in the last column. This matrix is non-singular of determinant unity, and its inverse B has the same principal diagonal as A with zeros below this diagonal. The remaining elements of B are those of A but with signs alternately $-$ and $+$above the principal diagonal, i.e., $b_{ij}=(-1)^{i+j}{}_{j-1}C_{i-1}$.

Proof. If d_{ij} denote the element in the ith row and jth column of AB, we see immediately that $d_{ij}=0$ if $i>j$ and that $d_{ii}=1$ ($i, j=1, 2, \cdots n+2$). If $j=i+r$ where r is positive,

$$\begin{aligned} d_{i+1,j+1} &= (-1)^{i+j}[{}_iC_i\,{}_jC_i - {}_{i+1}C_i\,{}_jC_{i+1} + \cdots + (-1)^r{}_jC_i\,{}_jC_j] \\ &= (-1)^{i+j}{}_jC_i[1 - {}_rC_1 + {}_rC_2 - \cdots + (-1)^r{}_rC_r] \\ &= (-1)^{i+j}{}_jC_i(1-1)^r \\ &= 0. \end{aligned}$$

If a_{ij}, b_{ij} denote the elements in the ith row and jth column of the matrices A and B respectively, the cofactor of $a_{n+2,k}$ is $b_{k,n+2}=(-1)^{n+2+k}D_k$, where D_k is the determinant of the matrix obtained from A by removing the last row and the kth column. Hence $D_k={}_{n+1}C_{k-1}$.

The proposer of this problem might be interested in a note by A. C. Aitken, *Note on dual symmetric functions*, Proc. Edin. Math. Soc., 2, 1930–31, page 166. The determinants there considered are similar to the one in the problem.

III. *Solution by J. H. M. Wedderburn, Princeton University.*

Set ${}_nC_r=0$ for $n<r$. If we add a row to the array, it becomes the matrix

$E=\|e_{ij}\|$, $e_{ij}={}_jC_i$ $(i,j=0,1,\cdots,n+1)$ whose determinant is 1. If this matrix is regarded as operating on the polynomial basis $(1,x,x^2,x^3,\cdots)$ its effect is to change x into $x+1$; the inverse is therefore $E^{-1}=\|f_{ij}\|$, $f_{ij}=(-1)^{i+j}e_{ij}$, which changes x into $x-1$, and the minors required in the problem are therefore, apart from sign, the elements in the last column of this matrix.

Solved also by Frank Ayres, Jr., J. A. Bullard, Harry Langman, W. Weisberg and W. P. Udinski.

Editorial Note. In the solutions of Bullard and Weisberg the required determinant was reduced to its minor of order $n-k+2$ in the lower right-hand corner. This determinant was next evaluated by successive combinations of the rows. Ayres also used the reduction to the determinant of order $n-k+2$, but the final evaluation was obtained by combining all the other columns with the last, using for that purpose an identity similar to that in the solution above. Langman's solution employed a set of linear equations with binomial coefficients, using a similar identity.

Udinski considered a generalization of the problem by deriving from the formula for Δ^{n+1}, where the unit difference in the operator Δ is h, an identity

$$\sum_{i=0}^{n-k+1}(-1)^{k+i}\binom{n+1}{k+i}\binom{k+i}{k}P_{n-k}[(k+i)h]=0, \qquad m\leqq k\leqq n.$$

This gives a system of linear equations in

$$\binom{n+1}{m},\ \binom{n+1}{m+1},\ \cdots,\ \binom{n+1}{n+1}.$$

The matrix

$$M=\left\|\binom{m+i}{m+s}P_{n-m-s}[(m+i)h]\right\| \qquad \begin{matrix}0\leqq s\leqq n-m\\ 0\leqq i\leqq n-m+1,\end{matrix}$$

was then considered, where s denotes the row and i the column. If M_j denotes the determinant obtained by deleting the $(j+1)$st column from M, then

$$M_j=\binom{n+1}{m+j}P_{n-m}(mh)P_{n-m-1}[(m+1)h]\cdots P_0(nh), \qquad 0\leqq j\leqq n-m+1.$$

By setting all of the P's equal to unity and $m=0$, we have the result for the given problem.

Solution III is interesting. Since it may appear rather condensed to readers unfamiliar with substitutions, some details will be given which may be of aid in reading it, and also in seeing the relation between the various solutions. Consider the substitution

$$(1) \qquad x_m\sim\sum_{t=0}^{m}{}_mC_tx_t, \qquad m=0,1,\cdots,n+1,$$

which means that x_m is replaced by the linear expression on the right, where the x's are independent variables. This substitution is defined completely by the matrix

$$A = (a_{ij}) = ({}_{i-1}C_{j-1})$$

which is the given matrix when made square and its rows are replaced by its columns. The cofactor of $a_{k,n+2}$ is $(-1)^{n+k}\Delta_k$, where Δ_k is the determinant to be evaluated. The determinant of A is unity, and, since it is not zero, a second matrix B, or substitution, can be found so that A followed by B gives the identical substitution $x_m \sim x_m$; or $AB = I$, where I is the matrix all of whose elements are zero except those in the prcinipal diagonal which are unity.

Since the matrix B is independent of the special values of x, we may set $x_m = x^m$: and then (1) becomes

$$x^m \sim (x+1)^m, \quad \text{or} \quad x \sim (x+1).$$

Hence B must be of such a form that in this case

$$x \sim (x-1), \quad \text{or} \quad x^m \sim (x-1)^m.$$

On returning to the general case of the x's we must have for B

$$x_m \sim \sum_{t=0}^{m} (-1)^{m+t}{}_mC_t x_t, \qquad m = 0, 1, \cdots, n+1 \tag{2}$$

$$B = (b_{ij}) = (-1)^{i+j}{}_{i-1}C_{j-1}.$$

The product AB is the matrix

$$AB = (a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{i,n+2}b_{n+2,k}), \tag{3}$$

where the element is zero if $i \neq k$, and unity if $i = k$. Then $n+2$ equations of (3) for $i = 1, 2, \cdots, n+2$ give $(-1)^{n+k}{}_{n+1}C_{k-1} = b_{n+2,k} =$ the cofactor of $a_{k,n+2}$ in the determinant of A, i.e., $(-1)^{n+k}\Delta_k$, or $\Delta_k = {}_{n+1}C_{k-1}$.

This method is also a proof of the identity which has been used in a number of the solutions. For, if in (3) we replace the a's and b's by their respective values, we have

$$\sum_{t=0}^{n+1} (-1)^{t+k}{}_iC_t\, {}_tC_k = 0, \quad \text{if } i \neq k$$

$$= 1, \quad \text{if } i = k.$$

3646 [1933, 610]. *Proposed by N. A. Court, University of Oklahoma.*

Determine the surface generated by the common perpendicular of two skew lines a and b, when a describes a flat pencil while b remains fixed.

Solution by S. Vatriquant, Brussels, Belgium.

The trivial case where b is parallel to the plane α of the pencil a will be considered first. Let c be the projection of b on α, and let A be the vertex of the

pencil a. If A lies on c, then c is an element of the pencil and there are infinitely many common perpendiculars lying in the plane of b and c. Of these perpendiculars the one at A is also the common perpendicular for any other line a of the pencil. If A does not lie on c, there is one element c' of the pencil a which is parallel to both c and b; and there are infinitely many common perpendiculars lying in the plane of b and c'. The remaining elements of a give infinitely many distinct common perpendiculars lying in the plane of b and c.

In the contrary case let b cut α in O; let β be the plane through O perpendicular to b; let x be the common perpendicular to b and the element a, cutting the latter in P; and let the projections of A and P on β be A' and P'. The projection of x on β is then OP', and OP' is perpendicular to $A'P'$. The locus of P' is a circle with the diameter OA', and this circle is the base of a right circular cylinder with the elements b, $A'A$, and the variable element $P'P$. Since P lies on the cylinder and also in the plane α, the locus of P is an ellipse Γ through O and A. The generator x of the required surface meets the two skew straight lines b and i, the line at infinity in β, and also the ellipse Γ. From Salmon's theorem concerning the order of such ruled surfaces, the locus of x is a cubic surface with the nodal line b.

The same result may be obtained by solid analytic geometry. Let O be the origin of rectangular coordinates; b, the z-axis; and the intersection of α and β, the y-axis. The equation of α is of the form $z=kx$, and the coordinates of A are then (a, b, ka). The equations of an element a are then

$$z = kx, \tag{1}$$

$$y - b = m(x - a), \text{ the vertical plane through } AP. \tag{2}$$

The vertical plane through b perpendicular to (2) has the equation

$$my + x = 0. \tag{3}$$

The coordinates of P are obtained by solving (1), (2), and (3) simultaneously, and its $z=P'P$ is given by

$$(1 + m^2)z = km(am - b). \tag{4}$$

Eliminating m from (3) and (4), we obtain the equation of the surface

$$(x^2 + y^2)z = kx(ax + by),$$

generated by the common perpendicular x.

Solved also by W. C. Janes, W. P. Udinski, and F. Underwood.

Editorial Note. The solutions by Janes and Underwood used rectangular coordinates in a different manner, while Udinski's solution made use of vector equations. The line b is the locus of the feet of common perpendiculars to the generating elements x of the required ruled surface. Such a line is called in general the line of striction for the skew ruled surface. In the general case of ruled skew surfaces any given generator and a neighboring generator determine a

common perpendicular. The limiting position of the foot on the given generator is called the *central point* of the given generator; and the locus of the *central points* is the *line of striction*. The ruled surface of the problem gives a simple illustration of a theorem by Bonnet. If a curve on a ruled surface cuts all the generators, then, if it has two of the following properties, it has the third also.

(1) It is a geodesic.
(2) It cuts the generators at a constant angle.
(3) It is a line of striction.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio

Eleven phases of social insurance, arising out of the chief objective of the national administration for the "security of the men, women and children of the nation," are to be studied and reported on by special committees under auspices of the President's Committee on Economic Security, which consists of Secretaries Perkins, Morgenthau and Wallace, Attorney General Cummings and the Federal Relief Administrator, H. L. Hopkins.

The committee, in its first public report last week, announced the personnel of the eleven special groups together with a sub-committee of actuaries to counsel with the administration on means of financing the program, which is considered to be the most difficult problem of the tentative proposals.

This actuarial committee is headed by Professor J. W. Glover of the University of Michigan, who has resigned from the chairmanship of the university's department of mathematics but is continuing as a professor. Others on the committee are: M. A. Linton, president of the Provident Mutual Life; Professor H. L. Rietz, University of Iowa; and Professor A. L. Mowbray, University of California.

Beginning with the October 1934 issue, the Mathematics News Letter, published in Baton Rouge, Louisiana, changed its title to National Mathematics Magazine. The new magazine was also enlarged by the addition of two new departments; the Teacher's Department under the editorship of Joseph Seidlin and W. P. Webber, and the Notes and News Department edited by I. Maizlish. The aims of the new magazine as stated by the editorial board are:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to the groups most interested high-class papers of research quality representing all mathematical fields.

The Southern Intercollegiate Mathematics Association, which was organized at Centenary College in October, 1933, will hold its second annual meeting in May, 1935, at Centenary College, Shreveport, La. It is hoped that additional institutions will participate in this year's contests (see this MONTHLY, vol. 41, p. 58). Any institution in the states of Louisiana, Arkansas, Texas, Mississippi, and Oklahoma may join the S.I.M.A. by writing to the secretary, Miss Frances White, Louisiana Polytechnic Institute, Ruston, La.

The Engineer's Council for Professional Development is sponsoring a series of comprehensive examinations in mathematics covering arithmetic, high-school algebra, plane geometry, solid geometry and trigonometry, to be given to incoming freshmen in colleges of engineering. About a dozen representative institutions have been asked to cooperate in this experiment which, it is hoped, will eventually furnish sufficient data to determine criteria which may be used by student advisers in secondary schools.

Professor D. J. Struik, of the Massachusetts Institute of Technology, delivered a series of lectures on differential geometry and on probability at the National Academy of Sciences, "Antonio Alzate," in Mexico City, in July, 1934, on the occasion of the celebration of the fiftieth anniversary of that Academy. Professor Struik was elected to honorary membership in the Academy.

Dr. B. E. Mitchell of Millsaps College is chairman of a committee appointed by the mathematics section of the Mississippi Education Association. This committee selected a standard test of mathematical training to be given to college freshmen throughout the state.

Dr. L. M. Blumenthal is spending his second year as a National Research Fellow at the University of Vienna, working with Professor Karl Menger.

Dr. A. T. Craig has been promoted to an assistant professorship at the University of Iowa.

Associate Professor A. H. S. Gillson has been promoted to a professorship at McGill University.

Dr. G. D. Gore has been promoted to a professorship at the Central Y.M.C.A. College, Chicago.

Dr. W. I. Miller has been promoted to an assistant professorship at the University of Pittsburgh.

Dr. Gordon Pall, lecturer at McGill University, has been promoted to an assistant professorship.

Dr. W. V. Parker, formerly professor and head of the department of mathematics at the Mississippi State Teachers College, has been appointed to an assistant professorship at the Georgia School of Technology.

Dr. H. H. Pixley has been promoted to an assistant professorship at Wayne University. During the year 1933–34 Dr. Pixley served as mathematical economist in the N.R.A.

Assistant Professor I. R. Pounder has been promoted to a professorship at the University of Toronto.

Professor Gabriel Szegö, of Königsberg, has been appointed visiting professor of mathematics at Washington University, St. Louis.

Dr. H. L. Turrittin, of the University of Wisconsin, has been appointed adjunct professor at the College of Mines and Metallurgy, El Paso, Texas.

Professor A. E. Whitford, who was co-head of the department of mathematics at Alfred University, has been promoted to the deanship of the College of Liberal Arts; Professor Joseph Seidlin, who was also co-head of the department, is now the departmental executive.

Associate Professor W. L. G. Williams has been promoted to a professorship at McGill University.

The following appointments to instructorships in mathematics have been announced;

Columbia University: George Komentz;
Lehigh University: Dr. D. H. Lehmer;
College of the City of New York: Dr. Selby Robinson;
Rennselaer Polytechnic Institute: Dr. D. B. Ames;
Rice Institute: Dr. E. F. Beckenbach;
Rockhurst College: B. R. Wicker;
College of St. Francis (Joliet, Illinois): Dr. F. C. Smith.

Dr. C. R. Wylie of Cornell University has been appointed to an assistantship at the Ohio State University.

C. L. Arnold, professor emeritus of the Ohio State University, died November 8, 1934. Professor Arnold had been a member of the department of mathematics at that institution for a period of forty years. He was a charter member of the Association.

Professor O. J. Bond, of the Citadel, The Military College of South Carolina, died October 1, 1933, at the end of thirty-seven years service there as professor of mathematics. He was a member of the Mathematical Association.

Professor R. A. Wells, head of the department of mathematics and astronomy at Park College, died October 8, 1934. He was a charter member of the Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. B. Carver, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. Johnson, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

Allegheny Mountain.
Illinois.
Indiana, Hanover, May 3-4.
Iowa.
Kansas.
Kentucky.
Louisiana-Mississippi, Pineville, La., March.
Maryland-District of Columbia-Virginia, May 11.
Michigan, Ann Arbor, Mar. 9.
Minnesota.
Missouri.
Nebraska.
Ohio, Columbus, Apr. 4.
Oklahoma, Tulsa, Feb. 1.
Philadelphia, Easton, Pa., Nov. 30.
Rocky Mountain.
Southeastern, Decatur, Ga., March.
Southern California, Los Angeles, Mar. 2.
Texas.
Wisconsin.

Affiliated Organizations: The New England Association of Teachers of Mathematics, The National Council of Teachers of Mathematics.

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THE FIFTEENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The fifteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois College, Jacksonville, Illinois, on Friday and Saturday May 4–5, 1934. Professor W. C. Krathwohl, chairman of the Section, presided.

The attendance was about eighty, including the following forty-six members of the Association: Beulah Armstrong, Edith I. Atkin, H. W. Bailey, G. A. Baker, O. K. Bower, D. F. Campbell, C. C. Carter, J. W. Cell, Laura E. Christman, A. B. Coble, C. E. Comstock, A. R. Crathorne, H. B. Curtis, D. R. Curtiss, J. E. Davis, W. M. Davis, Elinor B. Flagg, A. E. Gault, R. M. Ginnings, G. D. Gore, H. W. Haggard, W. W. Hart, M. C. Hartley, Martha Hildebrandt, Mildred Hunt, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, M. I. Logsdon, H. J. Miles, E. B. Miller, C. N. Mills, G. E. Moore, E. J. Moulton, Mary W. Newson, Rufus Oldenburger, O. E. Olson, R. G. Sanger, E. W. Schreiber, E. E. Scott, H. E. Slaught, C. A. Stone, V. B. Teach, M. E. Wescott, F. E. Wood.

The following officers were elected for the coming year: Chairman, Professor E. B. Miller, Illinois College; Vice-Chairman, Professor E. C. Kiefer, James Millikin University; Secretary-Treasurer, Professor C. N. Mills, Illinois State Normal University. The next meeting of the Section will be held at James Millikin University on Friday and Saturday, May 3–4, 1935.

The following papers were presented at the Friday sessions:

1. "Some observations concerning points in the projective plane with applications to curve tracing" by Dr. G. E. Moore, University of Illinois.

2. "The theory of probability and its paradoxes" by Professor A. R. Crathorne, University of Illinois.

3. "On some recent developments regarding the foundations of probability" by Dr. Luise Lange, City Junior College, Chicago.

4. "Mental grading of line segments" by Professor R. M. Ginnings, Western Illinois State Teachers College.

5. "A review of the fifth Carus Monograph *History of Mathematics in America before 1900* by Smith and Ginsburg" by Dr. R. G. Sanger, University of Chicago.

6. "Fundamental principles involved in a well regulated pension system" by Dr. D. F. Campbell, Chicago.

Abstracts of some of these papers follow, numbered in accordance with their listing above:

1. The projective coordinates of a point P in the plane were defined as usual in terms of cross-ratios, with the four base points $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$, and the unit point. The pencil of lines on A_1 was chosen as $x_2-\mu x_3=0$; on A_2 as $x_1-\lambda x_3=0$, and on A_3 as $x_2-\nu x_1=0$. Then λ, μ, ν were the projective coordinates of the point P, and x_1, x_2, x_3 the homogeneous projective

coordinates of the same point, subject to the relationship $\lambda\nu=\mu$. A large drawing was exhibited showing one hundred lines of the pencils through each of the vertices of the triangle of reference. This formed a coordinate system, whereby curves might be graphed in homogeneous or projective coordinates with as much ease as curves are graphed in rectangular coordinates. Methods of interpolation for the position of points was discussed.

2. Professor Crathorne spoke briefly on the increase of interest in the theory of probability due to its close connection with the newer developments in physical science. He then took up some of the criticisms that have been made against the theory, commenting on the various definitions of probability, the Bertrand paradoxes, Buffon's needle problem, the St. Petersburg problem and Bayes theorem, and sketched briefly some of the attempts to put the theory of probability on a surer foundation. In particular he commented on the rise of a sort of law of relativity in probability.

3. This paper discusses chiefly the work on the basic concepts of probability of two men at the University of Berlin, R. von Mises (Math. Zeitschrift, 1919) and Reichenbach (Math. Zeitschrift, 1932).

According to von Mises the probability of a class P in a given class O is defined as the limit of the relative frequency of the elements of P in O as the number of elements of O increases indefinitely. Moreover elements of P must be distributed randomly in O. Such classes O are called Kollektivs. The task of the calculus of probability is: given the frequency distribution in the original Kollektiv, to calculate frequency distributions in the derived Kollektivs; but *not* to find the original frequency distribution. This latter is a separate task, usually of empirical character. Contrasted with traditional views this theory is found to differ: (1) in its negation of the subjective interpretation of probability; (2) in denying any meaning in the probability of non-recurring events; (3) in its rejection of the traditional definition of probability as the quotient of the number of favorable to that of all equally possible cases. This definition is rejected as either not referring to frequency of occurrence and hence incapable of predicating anything about them; or, if referring to frequency of occurrence, as a mere special case of the general frequency definition; as such too narrow, as shown by failure to deal with so-called statistical cases.

Some of the paradoxes are discussed in the light of these views.

The use of the limit concept is criticized. On this point Reichenbach has reached the conclusion that ordinary true-false logic is not capable of answering the question concerning this limit; he finds that the question is answered by another probability judgment. In the opinion of the speaker the type of problems in which the long run relative frequency is clearly known a priori rather than found empirically should be set apart with greater emphasis and the principle used therein explicitly coordinated with the rest of the theory.

4. Twenty-six line segments representing the ability or achievement of twenty-six students were graded by one hundred nine college students into five ranks. Very superior was ranked A, superior B, medium C, inferior D and very

inferior E. The longest line segment represented the greatest ability or achievement.

The frequency distribution of the line segments was 14–6–4–1–1 but was graded by the students 4.3, 9.3, 7.8, 3.3, and 1.4. This grading showed a clear attempt to arrange the grades in a bell shaped distribution but was skewed in the direction of the actual distribution.

One hundred six college students likewise graded twenty-six line segments with the frequency distribution just reversed; viz., 1–1–4–6–14 with a frequency distribution 1.8, 5.2, 7.7, 6.7, and 2.8. This is more like the usual bell shaped distribution than the first set. However, the distribution was skewed somewhat in the direction of the actual distribution. In both cases a badly skewed (near J-shaped) distribution was made into a bell shaped distribution somewhat like the probability distribution.

Conclusion:—Unless mental ability or achievement can be graded mentally more reliably than these line segments, it is possible the usual frequency distributions in high schools and colleges as a whole can be accounted for mainly if not wholly by the way the human judgment acts mentally in grading students work.

The Saturday morning session was devoted to a symposium conducted by Dean E. J. Moulton of Northwestern University on "The preparation of teachers of mathematics for colleges and secondary schools." Dean Moulton is chairman of a commission appointed by the Mathematical Association of America to study the training and utilization of advanced students in mathematics, and the subject of this symposium is one of the problems being studied. The present session was planned to obtain a free discussion of this problem by those engaged in teaching mathematics. The discussion was led by Professor Mayme I. Logsdon after the following papers were read:

1. Introductory remarks by Dean E. J. Moulton.
2. "Mathematics taught in state teachers colleges," by Professor R. M. Ginnings, Western Illinois State Teachers College.
3. "A high school teacher's view on the preparation of teachers," by Martha Hildebrandt, Proviso Township High School, Maywood.
4. "Relations between high school and college mathematics," by Professor C. A. Stone, Central Y.M.C.A. College, Chicago.
5. "Experiences in the preparation of teachers of mathematics," by Professor W. W. Hart, University of Wisconsin.

Abstracts of some of these papers follow:

3. The preparation of a good high school teacher should begin in the high school, continue to the bachelor's degree and as many years thereafter as necessary. Its foundation should be a well-rounded cultural background. It should include, in mathematics, not only some of the differential and integral calculus, but also as many as possible of the introductory courses in the various fields of mathematics, taught to meet the needs of the prospective high school teacher as well as the prospective research worker. It should not neglect work in the

allied fields of economics, physics, mechanics, astronomy, etc. In the field of professional training, beside a comprehension of the general purposes of education, a developed skill in teaching is best begun by placing the greatest stress on the subject matter of secondary school mathematics, particularly its content, aims, organization, methods of teaching and in the interrelationship of the algebra and geometry with each other and other subjects. It is most important that the courses in education be taught by people capable of good teaching and with experience in the problems of the secondary school. There would be better teaching in the secondary schools if in all the departments of the college, including mathematics and education, the courses were taught by teachers with enthusiasm for and power in the subjects which they teach. The thorough preparation necessary to be a good teacher does not always pay financially, but usually it pays in satisfaction in results achieved.

4. In this paper it was pointed out that mathematics has been gradually pushed out of the secondary school curriculum by educators who claim that the subject has no educational value. To refute their claims cases were cited to show that mathematics is essential in all walks of life and that it is an essential part of any high school curriculum. The author also stated that Perry, Klein, Moore, The Committee of Ten, and The National Committee on Mathematical Requirements made many recommendations which if followed would have resulted in the elimination of the criticisms of mathematics.

It was also brought out that the tremendous increase in enrollment in the secondary schools of this country resulted in an influx of a great number of pupils who could not assimilate the subject matter of the high school. This produced a great number of failures with the result that mathematics became a rehashing process from year to year. This extended to trigonometry, college algebra and analytic geometry with the result that too much time is now spent in reviewing high school mathematics rather than teaching the actual content of these courses. It was further shown that high school deficiencies were enhanced by the poor attitude and poor instruction on the part of teachers who were totally unqualified to teach mathematics.

In order to eliminate the above difficulties the following recommendations were made:

(a) Raise the requirements of mathematics in the secondary school. Every teacher should have at least a master's degree or its equivalent in mathematics, including a sound knowledge of the tools of his profession—algebra and geometry.

(b) Add a teacher well trained in mathematics in the pedagogy of the subject for the purpose of training mathematics teachers.

(c) Experts in mathematics should determine the qualifications of teachers rather than administrators who know nothing about the subject.

(d) Reorganize the mathematics of the secondary school, as far as placement is concerned, to meet the needs of the various groups of high school pupils. A general course should be organized for those who do not plan to go on to

college and a concentrated course of $1\frac{1}{2}$ units of algebra in the third year of the high school and $1\frac{1}{2}$ units of geometry in the fourth year of the high school for those who plan to go to college. This would eliminate the need for rehashing in the college.

(e) Start a campaign of publicity in order to sell mathematics to the students and the public.

(f) Secure a better articulation between high school and college mathematics.

5. Professor Hart said that students who are preparing to teach mathematics need some courses in the department of mathematics which are closely related to the mathematics which they will teach in high school. In the professional training there should be an integration of the general courses in education and the special methods course usually taken by the students. The latter course should be in charge of an instructor who is sympathetic with the aims of the department of mathematics and who has major responsibility in directing the practice teaching in the demonstration school.

In the professional training, as well as in the schools, a moratorium on curriculum-tinkering and much greater attention to skill in teaching is needed. The over-emphasis upon selection of that which appears "useful" should be replaced by attention to "certain psychological requirements in curriculum making . . . (which) are infinitely more important for the future of the race than are the practical adjustments of trade and industry"* or of social science. Curriculum-tinkering has already interfered with desirable instruction in elementary and secondary mathematics; and the present tendency to abandon mathematics as a requirement for entrance to colleges will result in lowering possible standards of university and professional training, and will still further weaken instruction in elementary and secondary mathematics.

C. N. MILLS, *Secretary*

THE CAUCHY PROBLEM FOR LAPLACE'S EQUATION IN THREE DIMENSIONS

By L. H. JOHNSON, JR., Rice Institute

1. *Introduction.* It is well known that the Cauchy data, if not given analytically, need not insure a solution of a partial differential equation; in fact if a harmonic function $u(xyz)$ takes on zero values on a portion σ of the x, y plane, it may be extended harmonically across σ so as to be harmonic and analytic in a three dimensional region which includes in its interior part of σ.† Hence $\partial u/\partial z$ will also be analytic on this part of σ and therefore cannot take

* Professor Charles Hubbard Judd, University of Chicago.

† This most important fact about harmonic extension is well known to advanced students, but is not given in the texts which are likely to be at hand. Hence a short discussion of it is given in the appendix to this paper.

on the values of an arbitrary continuous function. In this paper we seek conditions which we may impose on non-analytic data in this particular problem so as to insure a solution. We have the following theorem:—

THEOREM. *If $f(x, y)$ and $f_1(x, y)$ are two functions continuous in a bounded region σ of the x, y plane and vanishing continuously on the boundary and outside of σ, and if the first and second partial derivatives of $f(x, y)$ are continuous, then*

(1) *there cannot be more than one function $u(x, y, z)$, harmonic for $z>0$ in the neighborhood* of σ for which*

$$\lim u(x', y', z') = f(x, y) \qquad (1)$$
$$\lim \frac{\partial u(x', y', z')}{\partial z'} = f_1(x, y)$$

as (x', y', z') approaches $(x, y, 0)$ in σ, and

(2) *there will be one such solution if and only if $f_1(x, y)$ has the value*

$$f_1(x, y) = F(x, y)$$

$$-\int_\sigma \left\{\frac{\frac{\partial^2 f}{\partial x^2}(x-x_M)^2 + \frac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \frac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2+(y-y_M)^2}\right\} \left\{\tfrac{1}{2}\log\left[(x-x_M)^2+(y-y_M)^2\right]\right\} dxdy, \qquad (2)$$

where $F(x, y)$ is an analytic function of x, y for x, y in σ.

2. *Proof of part* (1). It is almost immediately evident that there cannot be two different solutions which satisfy the given conditions. If there were two such solutions $u_1(x, y, z)$ and $u_2(x, y, z)$, their difference $U(x, y, z)$ would take on zero values in σ. But a harmonic function which takes on continuously zero values on a plane portion of the boundary of the region in which it is harmonic is in fact harmonic on this plane portion and can be extended across it uniquely as a harmonic function. Hence $U(x, y, z)$ is analytic at interior points of σ and vanishes there. Hence $\partial U/\partial z$ is also analytic at interior points of σ and vanishes there since $\partial U/\partial z = \partial u_1/\partial z - \partial u_2/\partial z$. But we have the statement of the Cauchy problem with analytic data. There is one and only one $U(x, y, z)$ analytic in the neighborhood of σ such that

$$U(x, y, 0) = 0, \quad \frac{\partial U(x, y, 0)}{\partial z} = 0, \quad \text{for } (x, y) \text{ in } \sigma$$

and this function is evidently $U(x, y, z) \equiv 0$. Accordingly we cannot have $u_1(x, y, z)$ different from $u_2(x, y, z)$.

* A neighborhood of σ is a region T such that if (x_0, y_0) is any point in the interior of σ, then a sphere can be drawn with center (x_0, y_0) and radius sufficiently small so that the whole sphere is contained in T.

3. *Determination of a function* $\bar{u}(x, y, z)$. In order to demonstrate the second part of our theorem, it is first necessary to find a function related to $u(x, y, z)$ which can be extended across the plane $z=0$. Let $\bar{g}(x, y)$ be a function which is bounded and continuous over the x, y plane. We find first a function $\bar{u}(x, y, z) = \bar{u}(M)$ which takes on continuously the values $\bar{g}(x, y) = \bar{g}(Q)$ as M approaches Q in the x, y plane. For this purpose we employ the Green's function $g(M, P)$ for the domain $z>0$, and write

$$\bar{u}(M) = \frac{1}{4\pi}\int_W \frac{\partial g(M, P)}{\partial z_P}\,\bar{g}(P)\,d\sigma_P \tag{3}$$

where W indicates integration over the whole x, y plane and P is the variable point in the x, y plane in terms of which the integration is effected. The integral is convergent since $\partial g/\partial z_P$ vanishes at infinity like $1/MP^2$. In fact we have $g(M, P) = 1/r - 1/r'$, where $r = MP$, $r' = M'P$, and M' is the reflexion of M in the plane $z=0$. To calculate $\partial g(M, P)/\partial z$ we write r and r' in terms of the coordinates of M, M', and P.

$$r = \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2}$$

$$r' = \sqrt{(x_P - x_{M'})^2 + (y_P - y_{M'})^2 + (z_P - z_{M'})^2}$$

$$\therefore \quad \frac{\partial g(M, P)}{\partial z_P} = -\frac{z_P - z_M}{r^3} + \frac{z_P - z_{M'}}{r'^3}$$

and since

$$z_{M'} = -z_M,$$

$$\therefore \quad \frac{\partial g(M, P)}{\partial z_P} = -\frac{z_P - z_M}{r^3} + \frac{z_P + z_M}{r'^3}.$$

As z_P approaches zero, MP approaches $M'P$ and we have

$$\lim \left.\frac{\partial g(M, P)}{\partial z_P}\right]_{z\to 0} = \frac{2z_M}{r^3} = \left.\frac{\partial g(M, P)}{\partial z_P}\right]_{z=0}.$$

Therefore

$$\bar{u}(M) = \frac{1}{4\pi}\int_W \frac{2z_M}{r^3}\,\bar{g}(x, y)\,d\sigma_P. \tag{4}$$

We can now show that $\bar{u}(M)$ as given by the above integral is bounded. If ψ is the angle between MP and the downward vertical, we have

$$\cos\psi = \frac{z_M}{r}.$$

Furthermore the projection of $d\sigma_P$ upon a plane perpendicular to MP is

$d\sigma_P \cos\psi$. In fact if we consider the solid angle $d\omega$ subtended upon a unit sphere at M by the element of area $d\sigma_P = dxdy$ at the point P, we have immediately

$$d\omega = \frac{1}{r^2}\, d\sigma_P \cos\psi = \frac{z_M}{r^3}\, d\sigma_P$$

which upon substitution in (4) gives

$$\bar{u}(M) = \frac{1}{2\pi}\int_W \bar{g}(x, y)d\omega.$$

By hypothesis $\bar{g}(x, y)$ is bounded over the x, y plane, which means that $|\bar{g}(x, y)| \leqq K$, where K is a constant. Therefore

$$\bar{u}(M) \leqq \frac{1}{2\pi}\int_W |\bar{g}(x, y)|\, d\omega \leqq \frac{K}{2\pi}\int_W d\omega \leqq K.$$

And further we can show that not only is $\bar{u}(M)$ bounded if $\bar{g}(x, y)$ is bounded, but also that $\bar{u}(M)$ takes on continuously the values $\bar{g}(Q)$ as point M approaches point Q in the x, y plane. If we let P be a variable point in the x, y plane and Q a fixed point, we write

$$\bar{g}(P) = \bar{g}(Q) + h(P),$$

where $\bar{g}(Q)$ is the value of the function $\bar{g}(x, y)$ at Q and where $|h(P)| \leqq \epsilon$ if $\overline{QP} \leqq \delta$, and is bounded $< H$ in W. Consider a circle with center Q and radius δ and call this region about Q σ_δ. Let $\sigma_{\delta'} = W - \sigma_\delta$. Keeping these notations in mind, we write

$$\bar{u}(M) = \frac{1}{2\pi}\int_W \bar{g}(P)d\omega_P = \frac{1}{2\pi}\int_W \bar{g}(Q)d\omega_P + \frac{1}{2\pi}\int_W h(P)d\omega_P$$

$$\therefore \quad \bar{u}(M) - \frac{\bar{g}(Q)}{2\pi}\int_W d\omega_P = \frac{1}{2\pi}\int_{\sigma_\delta} h(P)d\omega_P + \frac{1}{2\pi}\int_{\sigma_{\delta'}} h(P)d\omega_P.$$

Now let M approach Q. We see that

$$\left|\frac{1}{2\pi}\int_{\sigma_\delta} h(P)d\omega_P\right| \leqq \frac{1}{2\pi}\int_{\sigma_\delta} |h(P)|\, d\omega_P \leqq \frac{\epsilon}{2\pi}\cdot 2\pi \leqq \epsilon,$$

since $|h(P)| \leqq \epsilon$ for $\overline{QP} \leqq \delta$. Hence this integral can be made arbitrarily small, independently of M, say $\leqq \eta/2$, by a proper choice of δ. Also we have

$$\left|\frac{1}{2\pi}\int_{\sigma_{\delta'}} h(P)d\omega_P\right| \leqq \frac{H}{2\pi}\int_{\sigma_{\delta'}} d\omega_P \leqq \frac{H}{2\pi}\cdot\epsilon_1,$$

where $|h(P)| < H$, and where ϵ_1 is the solid angle subtended by $\sigma_{\delta'}$ at M. But this becomes arbitrarily small as M approaches Q, and therefore this integral also may be made $\leqq \eta/2$. We have finally

$$|\bar{u}(M) - \bar{g}(Q)| \leqq \eta/2 + \eta/2 \leqq \eta$$

$$\therefore \qquad \lim_{M\to Q} \bar{u}(M) = \bar{g}(Q),$$

since η is chosen arbitrarily small.

4. *A function related to $u(x, y, z)$ which can be extended across $z=0$.* Now recalling the conditions of our theorem that

$$\lim u(x', y', z') = f(x, y)$$

$$\lim \frac{\partial u(x', y', z')}{\partial z'} = f_1(x, y)$$

as (x', y', z') approaches $(x, y, 0)$ in σ, where $f(x, y)$ and $f_1(x, y)$ are continuous in the bounded region σ of the x, y plane and vanish continuously outside of σ, we choose $\bar{g}(x, y)=f(x, y)$ in W, so that $\bar{u}(M)$ approaches $f(Q)$ as point M approaches point Q in the x, y plane. We are now able to set up a function related to $u(x, y, z)$ which can be extended across $z=0$.

We assume that we have a solution $u(x, y, z)$ and define

$$v(M) = u(M) - \bar{u}(M), \tag{5}$$

and since $v(M)$ is the difference of functions harmonic in the upper half space, it is also harmonic in that region. Furthermore $v(x', y', z')$ approaches zero as (x', y', z') approaches $(x, y, 0)$ since

$$\lim \bar{u}(x', y', z') = f(x, y)$$

and also by hypothesis,

$$\lim u(x', y', z') = f(x, y).$$

Also $v(M)$ is bounded in the neighborhood of σ for $z>0$ since it is continuous in a closed region about σ. Therefore by defining $v(x, y, -z) = -v(x, y, z)$, we can extend $v(x, y, z)$ harmonically across the plane $z=0$ in the neighborhood of σ. Moreover $v(x, y, z)$ is analytic on σ since it is harmonic in a region including σ. Hence $\partial v(x, y, z)/\partial z$ is analytic on σ and its values there determine $v(x, y, z)$ uniquely, for there is one and only one solution of Laplace's equation in the neighborhood of σ which with its normal derivatives takes on given analytic values on σ. In this particular case the given value of $v(x, y, 0)$ is identically zero. We may write

$$\left.\frac{\partial v(x, y, z)}{\partial z}\right|_{z=0} = F(x, y),$$

where $F(x, y)$ is a function analytic in σ.

5. *Necessary relation between $f_1(x, y)$ and $f(x, y)$.* From equation (5) for a point M we have

$$
(6)\qquad \left.\frac{\partial v(M)}{\partial z_M}\right|_{z_M>0} = \left.\frac{\partial u(M)}{\partial z_M}\right|_{z_M>0} - \left.\frac{\partial \bar{u}(M)}{\partial z_M}\right|_{z_M>0} = \left.\frac{\partial u(M)}{\partial z_M}\right|_{z_M>0} - \frac{1}{2\pi}\int \frac{\partial}{\partial z_M}\frac{z_M}{r^3} f(x, y)d\sigma_P.
$$

Differentiating under the last integral sign and taking the limit as z_M approaches zero, equation (6) becomes

$$
(7)\quad F(x, y) = f_1(x, y) - \lim_{z_M\to 0}\frac{1}{2\pi}\int_W \frac{(x_P - x_M)^2 + (y_P - y_M)^2 - 2z_M^2}{r^5} f(x, y)d\sigma_P.
$$

When $z_M = 0$ the integral is improper since the denominator approaches zero to a higher order than the numerator, and the fraction becomes infinite like r^3. The integral is moreover not convergent. Before letting z_M approach zero, we make the transformation

$$
\begin{gathered}
x = x_M + r_1 \cos\theta \\
y = y_M + r_1 \sin\theta \\
(x_P - x_M)^2 + (y_P - y_M)^2 = r_1^2,
\end{gathered}
$$

where r_1 and θ are the polar coordinates in the plane referred to (x_M, y_M). Substituting in the integral, we have

$$
(8)\quad \begin{aligned}
I &= \int_W \frac{(x_P - x_M)^2 + (y_P - y_M)^2 - 2z_M^2}{[(x_P - x_M)^2 + (y_P - y_M)^2 + z_M^2]^{5/2}}\cdot f(x, y)d\sigma_P \\
&= \int_0^{2\pi}\int_0^{\infty} f(x_M + r_1\cos\theta, y_M + r_1 \sin\theta)\frac{r_1^2 - 2z_M^2}{(r_1^2 + z_M^2)^{5/2}} r_1 d\theta dr_1 \\
&= \int_0^{2\pi} d\theta \int_0^{\infty} f(x_M + r_1\cos\theta, y_M + r_1\sin\theta)\left[\frac{r_1(r_1^2 + z_M^2)}{(r_1^2 + z_M^2)^{5/2}} - \frac{3r_1 z_M^2}{(r_1^2 + z_M^2)^{5/2}}\right] dr_1.
\end{aligned}
$$

Furthermore we have

$$
\int_0^{r_1}\left[\frac{r_1}{(r_1^2 + z^2)^{3/2}} - 3z^2\frac{r_1}{(r_1^2 + z^2)^{5/2}}\right] dr_1 = -\frac{r_1^2}{(r_1^2 + z^2)^{3/2}},
$$

since

$$
\frac{\partial}{\partial r_1}\left[-\frac{r_1^2}{(r_1^2 + z^2)^{3/2}}\right] = \frac{r_1^3 - 2r_1 z^2}{(r_1^2 + z^2)^{5/2}}.
$$

Therefore if we integrate by parts, we obtain

$$
I = \int_0^{2\pi} d\theta \int_0^{\infty} (f_{10}\cos\theta + f_{01}\sin\theta)\frac{r_1^2}{(r_1^2 + z^2)^{3/2}} dr_1;
$$

in fact, the part outside the integral vanishes, since $f(M, r_1, \theta) = 0$ for $r_1 = \infty$,

and $r_1^2/(r_1^2+z^2)^{3/2}=0$ when $r_1=0$. The f_{10} and f_{01} denote respectively $\partial f(x,y)/\partial x$ and $\partial f(x,y)/\partial y$. But the denominator still approaches zero to too high an order.

We integrate again by parts, first writing

$$\int_0^{r_1}\frac{r_1^2}{(r_1^2+z^2)^{3/2}}dr_1=\int_0^{r_1}\left[\frac{1}{(r_1^2+z^2)^{1/2}}-\frac{z^2}{(r_1^2+z^2)^{3/2}}\right]dr_1$$

$$=\log\left(r_1+\sqrt{r_1^2+z^2}\right)+c-\frac{r_1}{(r_1^2+z^2)^{1/2}}.$$

In this equation, if we let r_1 approach zero, holding z fixed, all terms vanish except $\log (z^2)^{1/2}+C$, so that $C=-\log z$ and we have

$$\int_0^{r_1}\frac{r_1^2}{(r_1^2+z^2)^{3/2}}dr_1=\log\left[\frac{\sqrt{r_1^2+z^2}+r_1}{z}\right]-\frac{r_1}{(r_1^2+z^2)^{1/2}}.$$

Also

$$\frac{d}{dr_1}f(_{10}\cos\theta+f_{01}\sin\theta)=f_{20}\cos^2\theta+2f_{11}\cos\theta\sin\theta+f_{02}\sin^2\theta$$

where

$$f_{20}=\frac{\partial^2 f}{\partial x^2},\qquad f_{11}=\frac{\partial^2 f}{\partial x\partial y},\qquad f_{02}=\frac{\partial^2 f}{\partial y^2}.$$

As before, in the integration by parts, the part outside of the integral vanishes, on account of the continuity of f_{01}, f_{10} and their vanishing outside σ, and there is left

$$I=-\int_0^{2\pi}d\theta\int_0^{\infty}\left\{f_{20}\cos^2\theta+2f_{11}\cos\theta\sin\theta+f_{02}\sin^2\theta\right\}\left\{\log\left[r_1+(r_1^2+z^2)^{1/2}\right]\right.$$

$$\left.-\log z-\frac{r_1}{(r_1^2+z^2)^{1/2}}\right\}dr_1.$$

The integral involving $-\log z$ vanishes, for we have

$$\log z\int_0^{2\pi}d\theta\int_0^{\infty}\left\{f_{20}\cos^2\theta+2f_{11}\cos\theta\sin\theta+f_{02}\sin^2\theta\right\}dr_1$$

$$=\log z\int_0^{2\pi}d\theta\left[f_{10}\cos\theta+f_{01}\sin\theta\right]_0^{\infty}=0,$$

since

$$\int_0^{2\pi}\cos\theta d\theta=0,\qquad\int_0^{2\pi}\sin\theta d\theta=0,\qquad f_{10}\big]_{r_1=\infty}=f_{01}\big]_{r_1=\infty}=0.$$

Thus we have

$$
(9) \qquad \begin{aligned} I = & \int_0^{2\pi} d\theta \int_0^{\infty} \{f_{20} \cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta\} \\ & \cdot \left\{ \frac{r_1}{(r_1^2 + z^2)^{1/2}} - \log\left(r_1 + (r_1^2 + z^2)^{1/2}\right) \right\} dr_1. \end{aligned}
$$

The integral is now convergent and we may write

$$
(10) \quad \lim_{z=0} I = \int_0^{2\pi} d\theta \int_0^{\infty} \{f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta\}\{1 - \log 2r_1\} dr_1,
$$

which reduces in the manner which we have seen to

$$
\lim_{z=0} I = \int_0^{2\pi} d\theta \int_0^{\infty} \{f_{20}\cos^2\theta + 2f_{11}\cos\theta\sin\theta + f_{02}\sin^2\theta\}\{-\log r_1\} dr_1.
$$

Substituting rectangular coordinates, we have

$$
(11) \quad \begin{aligned} \lim_{z=0} I = -\int_W & \left\{ \frac{\dfrac{\partial^2 f}{\partial x^2}(x-x_M)^2 + 2\dfrac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \dfrac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right\} \\ & \cdot \left\{\tfrac{1}{2}\log\left[(x-x_M)^2 + (y-y_M)^2\right]\right\} dx dy = J. \end{aligned}
$$

Therefore we may write equation (7) in the form

$$
(12) \quad \begin{aligned} f_1(x, y) = & F(x, y) \\ & -\frac{1}{4\pi}\int_\sigma \left\{ \frac{\dfrac{\partial^2 f}{\partial x^2}(x-x_M)^2 + 2\dfrac{\partial^2 f}{\partial x \partial y}(x-x_M)(y-y_M) + \dfrac{\partial^2 f}{\partial y^2}(y-y_M)^2}{(x-x_M)^2 + (y-y_M)^2} \right\} \\ & \left\{\log\left[(x-x_M)^2 + (y-y_M)^2\right]\right\} dx dy. \end{aligned}
$$

In fact, by hypothesis, $f(x, y) \equiv 0$ outside of the region σ, and therefore $f_{ij}(x, y)$ vanishes outside of σ and the region of integration becomes the region σ.

The condition on $f(x, y)$ and $f_1(x, y)$ stated in the theorem is therefore necessary.

6. *Sufficiency of the condition on $f(x, y)$ and $f_1(x, y)$.* Suppose that $F(x, y)$ is an arbitrary analytic function and that the condition (2) holds. We show that there is a function $u(x, y, z)$ harmonic in the neighborhood of σ which satisfies the Cauchy conditions (1).

We let $\bar{u}(x, y, z)$ be a function harmonic in the upper half space which takes on continuously the values $f(x, y)$ in W as (x', y', z') approaches $(x, y, 0)$, where $f(x, y)$ is bounded and continuous in W. The function $\bar{u}(x', y', z')$ given by (3), with $\bar{g}(P) = f(x, y)$, in fact satisfies these conditions. Moreover $\partial\bar{u}/\partial z$ takes on continuously the values $J/2\pi$, where J is given by the right-hand member of

(11). And we let $v(x, y, z)$ be the harmonic function which satisfies the analytic Cauchy data

$$v(x, y, 0) = 0$$
$$\frac{\partial v(x, y, 0)}{\partial z} = F(x, y).$$

We define

$$u(x', y', z') = \bar{u}(x', y', z') + v(x', y', z').$$

Letting (x', y', z') approach $(x, y, 0)$, we have

$$\begin{aligned}\lim u(x', y', z') &= \lim \bar{u}(x', y', z') + \lim v(x', y', z') \\ &= f(x, y) + 0 = f(x, y).\end{aligned}$$

Furthermore differentiating $u(x', y', z')$ and letting (x', y', z') approach $(x, y, 0)$ we have

$$\begin{aligned}\lim \frac{\partial u(x', y', z')}{\partial z'} &= \lim \frac{\partial \bar{u}(x', y', z')}{\partial z'} + \lim \frac{\partial v(x', y', z')}{\partial z'} \\ &= J + F(x, y) \\ &= f_1(x, y).\end{aligned}$$

Thus the condition on $f(x, y)$ and $f_1(x, y)$ in the theorem is sufficient.

Appendix.—Proof of extension theorem. If $U(M)$ is harmonic for $z>0$ and takes on continuously the value zero on the bounded region σ of the x, y plane, then $U(M)$ can be extended across the plane $z=0$ so as to be harmonic in the neighborhood of σ, and there is only one such extension.

Take a sphere S with center O in σ and radius less than the distance to any boundary point of σ. We write, for M within S,

$$V(M) = \frac{1}{4\pi a}\int_S \frac{a^2 - \rho^2}{r^3}\bar{U}(P)dS,$$

where P is a point on the surface of the sphere, $\rho = OM$, S denotes integration over the whole surface of the sphere,

$$\begin{cases}\bar{U}(P) = U(P), & z_P > 0 \\ \bar{U}(P) = -U(P'), & z_P < 0,\end{cases}$$

P' being the image of P in the x, y plane. In the following, we let M' similarly be the image of M.

From the definition of the integral we may write

$$V(M) = \lim_{\Delta S\to 0} \sum_{\text{upper half}} \frac{a^2 - \rho^2}{r_i^3}\bar{U}(P_i)\Delta S_i + \lim_{\Delta S\to 0} \sum_{\text{lower half}} \frac{a^2 - \rho^2}{r_i^3}\bar{U}(P_i')\Delta S_i$$

and for each element in the upper half sphere, there corresponds an equal ele-

ment ΔS_i in the lower half sphere. Therefore, letting S_1 denote the surface of the sphere above the x, y plane and S_2 the other half of the sphere, we have for $z_M=0$

$$\begin{aligned} V(M) &= \frac{1}{4\pi a}\int_{S_1} \frac{a^2-\rho^2}{r^3}\bar{U}(P)dS + \frac{1}{4\pi a}\int_{S_2} \frac{a^2-\rho^2}{r^3}\bar{U}(P')dS \\ &= \frac{1}{4\pi a}\int_{S} \frac{a^2-\rho^2}{r^3}U(P)dS - \frac{1}{4\pi a}\int_{S_1} \frac{a^2-\rho^2}{r^3}U(P)dS. \\ &= 0. \end{aligned}$$

Moreover for $z_M\neq 0$ we have $V(M') = -V(M)$ for

$$\begin{aligned} V(M) &= \frac{1}{4\pi a}\int_{S} \frac{a^2-\rho^2}{r^3}\bar{U}(P)dS_P \\ V(M') &= \frac{1}{4\pi a}\int_{S} \frac{a^2-\rho^2}{r'^3}\bar{U}(P)dS_P \\ &= -\frac{1}{4\pi a}\int_{S} \frac{a^2-\rho^2}{r'^3}\bar{U}(P')dS_{P'} = -V(M). \end{aligned}$$

Accordingly the function $V(M)$ is harmonic in the upper hemisphere, is bounded, and takes on continuously in the upper hemisphere and on the portion of σ bounding this hemisphere the same values as $U(M)$. Hence $V(M) \equiv U(M)$ for otherwise the difference $V(M)-U(M)$ would have a positive maximum or a negative minimum inside the hemisphere.

Also $V(M)$ is analytic and harmonic on $z=0$, because $V(M)$ is analytic and harmonic inside S. Hence $V(M)$ provides an analytic and harmonic extension for $U(M)$, namely the extension $U(M')=-U(M)$.

The extension is moreover unique. If there were two functions harmonic in the neighborhood of σ and identical for $z\geqq 0$, they would have to be identical also for $z<0$, since they would both be analytic functions of x, y, z, and their difference would therefore be an analytic function of x, y, z, identically zero for $z\geqq 0$.

ON THE STRAIGHT LINE CONSTRUCTION OF UNICURSAL CUBICS

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Introduction. There are various straight line constructions for cubic curves which depend upon nine arbitrary conditions.[1] The fact that a curve is unicursal is equivalent to one condition[2] and so a cubic with a double point cannot, in

[1] Reye, *Die Geometrie der Lage.* Dritte Abteilung der dritten Vermehrten Auflage, 1892, page 73. Hilton, *Plane Algebraic Curves,* Page 245.

[2] Hilton, loc. cit., page 33.

general, be made to satisfy more than eight arbitrarily chosen geometric conditions. The purpose of this paper is to set up a method for the construction of unicursal cubics in particular and to give the constructions for several sets of given conditions.

If two pencils, S and T, are so related that to one ray of S correspond two rays of T, the locus of the intersections of corresponding rays is a curve of the third order with a double point at T and an ordinary point at S. When a ray of S passes through the center of T the corresponding ray of T is a tangent to the curve at T. Similarly when a ray of T passes through the center of S the corresponding ray of S is tangent at S.[3] Let a fixed point S be chosen inside or outside a given conic c and let the pencil through S be generated by a line through a point R which moves on c. Choose a pencil of the first order with center at any point T, projective to the point row described by R. Since SR has the same position for two different positions of R and a ray of T has one position for each position of R, the conditions of the above theorem are satisfied. If T lies within the range of SR, SR will pass through T twice and T will be a crunode with two real tangents. If T lies on the limiting position of SR, that is, on SR when SR is tangent to c, T will be a cusp with one real tangent. If T is situated outside the range of SR it will be an acnode with no real tangents.

Notation. The letters A, B, C, D, E, S, and T will be used for given points on a cubic to be constructed. The numbers 1 to 9 are used to represent points on a conic c. Such combinations as AB and 27 are used for lines joining A and B, and 2 and 7 respectively. The form (27, 45) is a point which is the intersection of 27 and 45, the form $(27, 45)-(48, AB)$ and $3-(ST, 45)$ will be lines joining given points, and $\{12, 34; 45, 16; 25, 36\}$ is a Pascal line with its three points given. The letter S will be an ordinary point on the cubic and the center of a generating pencil, T will be the double point on the cubic and center of a generating pencil, s will be the tangent to the cubic at S, and t_1 and t_2 the two tangents at T. We shall use $5=7$ to mean 5 coincides with 7, etc.

Theorem I. Let two Pascal hexagrams H_1 and H_2 be superposed upon a given conic c. Let point 5 be a common vertex of the two hexagons and point 6 be a vertex of H_2 only. Let 5 and 6 move on c such that they describe projective point rows on c. Let all other vertices of H_1 and H_2 be fixed. Then Pascal lines of H_2 may be selected which pass through a fixed Pascal point and form with any Pascal line of H_1 a pair of pencils in one to two correspondence, and therefore, a pair which generate a curve of the third order.

Since H_2 has six vertices on c and two of these move, and a Pascal point is determined by only four vertices, and any particular four vertices determine three Pascal points, and through each fixed Pascal point pass four Pascal lines, H_2 has twelve Pascal lines which pass through fixed Pascal points when 5 and 6 move on c. In each of these twelve lines 5 and 6 appear in the notation for two Pascal points only, hence 5 and 6 cannot be adjacent vertices in the particular hexagon which determines one of these lines for at least one of two adjacent

[3] Hilton, loc. cit., page 372.

vertices appears in the notation of each of the three points on a Pascal line. Then 5 and 6 must appear in a form similar to (25, 36) in the notation for the moving Pascal points on the twelve lines above. But since 5 and 6 describe projective point rows and 2 and 3 lie on c, 25 and 36 are projective pencils of the first order and (25, 36) describes a conic. Now since any Pascal line in H_1 describes a pencil of the first order[1] which is projective to the point row described by 5 and hence to the point row described by (25, 36), we have exactly the setup described in the discussion and theorem given in the introduction.

The point rows described by 5 and 6 upon c are paired in involution and the line 56 always passes through a fixed point P.[2] In the following construction problems certain restrictions must be placed upon the arbitrarily chosen conditions. No four points may lie upon a straight line and if three lie upon a straight line a special construction is usually required. Some of these will be given. In case a tangent is not given at a double point the type of double point is unknown and the tangents at that point cannot be constructed by a straight line method.

Problem I. Given a double point and six points arbitrarily chosen, to construct: (a) other points on the curve, (b) the third point of intersection of the cubic and a line determined by two given points, and (c) the tangent to the cubic at any given point.

Construction: Choose as generating pencils the two Pascal lines $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{25, 38; 27, 89; 37, 59\}$ and let P, the center of the involution which determines the projective relation between the point rows described by 5 and 6 as they move on c, lie on the line 79. Then when $5=7$, $6=9$ and vice versa. Now the four positions of S and T which are determined by $5=1$, $5=3$, $5=7$, and $5=9$ are fixed positions since they do not depend upon 5 nor 6. Let $SB=12$, $SC=34$, $TA=27$, $2=(12, 27)$, $3=C$, $CB=38$, $29=2-(TD, 38)$, $39=3-(SA, 27)$, $9=(29, 39)$, $37=3-(SD, 29)$, $7=(27, 37)$, $91=9-(TB, 37)$ and $1=(21, 91)$. Now c is determined through the points 2, 3, 9, 7 and 1 and since the lines 34 and 38 are known the points 4 and 8 may be found by Pascal's theorem. Draw TE which is a Pascal line with vertex 5 unknown. Determine 5 by Pascal's theorem. Draw SE, and using it as the Pascal line S above, and using 5 in the position just determined, find the position of 6. The intersection (56, 79) is P. Now draw a ray of T in any desired direction, locate 5 as was done above, draw $5P$ and locate 6. The corresponding ray of S is now determined and a new point on the curve is constructed. Theorem I shows that the locus is a cubic with a double point at T and an ordinary point at S. When $5=1$ both rays pass through B. When $5=3$ they pass through C. When $5=7$ they pass through A. When $5=9$ they pass through D and when 5 and 6 take their first determined positions they pass through E. The curve therefore satisfies the given conditions.

[1] See the paper by the author in this MONTHLY, vol. 40 (1933), page 251.

[2] Lehmer, *Synthetic Projective Geometry*, page 78.

When $6=2$ the ray of S coincides with 12 which is also SB. Construct the corresponding ray of T and the third point on SB is located. Similarly when $6=4$ the ray of S coincides with 34 and the third point on SC may be constructed.

To construct the tangent at S let TS be a ray of T and construct the corresponding ray of S which is the required tangent.

In the following problems the construction of any point on the cubic, the third intersection of a line and the cubic, and the tangent, s, will be the same as in problem I, therefore we shall assume these constructions made when the conic c is determined.

Problem II. Given a double point and six other points three of which are on a straight line, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of the cubic and a line determined by one of the three given points on a straight line and any other given point, and (*c*) the tangent to the curve at one of the three given points on a straight line.

Construction: Use the same generating pencils as in problem I. Choose S, B, and E to be the given points on a straight line and let this line be 12. In determining P use $6=2$. Otherwise the construction is identical with problem I.

Problem III. Given a double point, five other points and the tangent at one of them, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of the cubic and a line determined by the point with the given tangent and any other given point, and (*c*) the tangent at any given point.

Construction: Let $E=S$ and the construction is identical with problem I with the exception of (*c*). For (*c*), construct one point on the curve and apply problem I to the six known points.

Problem IV. Given a double point and five points three of which are on a straight line and a tangent at one of the three points on the straight line, to construct: (*a*), (*b*), and (*c*) of problem III.

Construction: Choose S, A, and D on the straight line and use the construction of problem I, except that P is determined by the given tangent instead of point E. Then (*c*) may be found as in problem III.

Problem V. Given a double point, five other points three of which are on a straight line and the tangent at one of the three on the straight line, to construct: (*a*) other points on the curve, (*b*) the point of intersection of the given tangent and the cubic.

Construction: Let the generating pencils be $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{25, 34; 57, 38; 74, 28\}$ and choose the following conditions: S, C and D, the points on a straight line, $P=(48, 76)$ when $5=7$, $s=12$, $C=3$, $SC=34$, $TD=28$, $TB=47$, $27=2-(TA, 34)$, and $38=3-(SB, 24)$. The conic c is now determined by the points 2, 3, 4, 7 and 8. When $6=2$ the ray of S is 12, the given tangent, and the corresponding ray of T determines the required point on this tangent.

The (*a*) part of this problem duplicates the (*a*) part of problem IV but since the given conditions of this problem can be determined from any of the given

conditions in this paper, we shall be able by part (*b*) of this problem to construct the intersection of the cubic and a given tangent, or a constructed tangent, for any set of given conditions.

Problem VI. Given any set of given conditions from which the cubic can be constructed by the method of this paper, to construct the tangents at the double point.

Construction: It was stated earlier in this paper that this problem could not be solved by a straight line method. It has a solution, however, by second degree construction which we give here.

It was shown under Theorem I that if S is an ordinary point and center of a generating pencil, each of the moving Pascal points on the Pascal line of S describes a conic. Consider the point (25, 36) on the particular line we have been using. Let T be the double point and ST a ray of S. In second degree construction the intersections of the conic described by (25, 36) and the line ST are known. Each of these intersections determines a ray 25 from which 5 may be found by Pascal's theorem. The corresponding rays of T give the required tangents. If the line intersects the conic in two points there are two real tangents and T is a crunode. If ST is tangent to the conic there is one tangent and T is a cusp. If the line and conic do not intersect there are no real tangents and T is an acnode.

Problem VII. Given a double point, a tangent at the double point and five other points, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of a line determined by any two given points, (*c*) the tangent at any point, and (*d*) the tangent at the double point.

Construction: Let the generating pencils be $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{25, 38; 75, 89; 37, 29\}$ and choose the following conditions: $P=(86, 79)$ when $5=8$ and 6 is determined by the Pascal line SD, $C=3$, $TC=37$, $t_1=29$, $SB=12$, $SC=34$, $BC=38$, $27=2-(TA, 38)$, $39=3-(SA, 27)$ and $8=(TD, 38)$. The conic c is determined by the points 2, 3, 7, 8, and 9. Draw 25 through (29, 38). The point (25, ST) gives one point of intersection of the conic described by (25, 36) and the line ST. Any number of points on this conic are known, hence the other point of intersection can be found by Pascal's theorem. The second tangent at T may then be found as in problem VI. Another method of constructing this tangent will be given in the next problem.

Problem VIII. Given a double point, one tangent at the double point and five points three of which are on a straight line, to construct: (*a*) other points on the curve, (*b*) a tangent at any one of the three given points on a straight line, and (*c*) the second tangent at the double point.

Construction: As generating pencils use $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{78, 34; 28, 35; 24, 57\}$. Let $P=(27, 81)$, S, A, and B lie on 12, $2=(12, TC)$, $ST=34$, $24=2-(TA, SC)$, $28=2-(TA, SD)$, $TD=78$, and 37 be a line through (28, TB) and the intersection of 24 and t_1. The points 2, 4, 8, 3 and 7 determine c. To construct the second tangent at T let $6=4$ and the ray of S becomes ST. The corresponding ray of T is the required tangent. This problem can be used

with any problem where one tangent is given at the double point to construct the other tangent at that point.

Problem IX. Given a double point, a tangent at a double point, four points, and a tangent at one, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of the cubic and a line through the point with the given tangent and any other given point, and (*c*) the second tangent at the double point.

Construction: Let the generating pencils be $S=\{12,\ 34;\ 45,\ 16;\ 25,\ 36\}$ and $T=\{27,\ 38;\ 25,\ 33;\ 37,\ 58\}$ and choose the following conditions: $t_1=27$, $P=(78, 56)$ where 5 is determined by TS used as a ray of T and 6 by s used as a ray of S, $SB=12$, $SC=34$, $TC=38$, $C=3$, $33=3-(12, TB)$, $28=2-(TA, 33)$, and $37=3-(SA, 28)$. The conic c is now determined by the points 2, 3, 7, 8 and the tangent 33. Construct (*c*) by the method of problem VII or by problem VIII.

Problem X. Given a double point, two tangents at the double point, and four points, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of a line joining any two given points, and (*c*) the tangent to the curve at any given point.

Construction: The construction is identical with problem IX excepting that P is determined by the second given tangent at T, then (*a*), (*b*), and (*c*) may be found as in problem I.

Problem XI. Given a double point, two tangents at the double point, three points and a tangent at one, to construct: (*a*) other points on the curve, and (*b*) the third point of intersection of the cubic and a line through the given point with the given tangent and any other given point.

Construction: As generating pencils use $S=\{12,\ 34;\ 45,\ 16;\ 25,\ 36\}$ and $T=\{78, 34; 28, 35; 24, 75\}$. Let $P=(26, 47)$ when $5=2$ and 6 is found by using the Pascal line SB, $SA=12$, $ST=34$, $2=(TB, 12)$ and any line through 2 be 28. Draw 37 through the intersections of s with t_1 and 28 with t_2, and 24 through 2 and the intersection of s with t_1. Let $13=3-(TA, 28)$ and 33 a line through 3 and the intersection of 28 with t_1. The conic c is now determined by the points 2, 3, 4, 1 and the tangent 33.

Problem XII. Given a double point, two tangents at the double point, two points with a tangent at one, and the intersection of the given tangent and the cubic, to construct other points on the curve.

Construction: Use $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{17, 34; 15, 24; 35, 27\}$ as generating pencils and let B lie on s, $P=(24, 37)$, $s=21$, $ST=34$, $TA=17$ and 2 be the intersection of t_1 and 12. Draw 27 through 2 and the intersection of SA with t_2, $13=1-(TB, 27)$, and 24 through 2 and the intersection of 13 and t_2. The points 1, 2, 3, 4 and 7 determine c.

Problem XIII. Given a cusp, tangent at the cusp and four points, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of the cubic and a line joining any two given points, and (*c*) the tangent at any given point.

Construction: This construction is identical with that of problem X if the second tangent at T coincides with the first tangent at that point.

Problem XIV. Given a cusp, tangent at the cusp, three points and a tangent at one, to construct: (*a*) other points on the curve, (*b*) the third point of intersection of the cubic and a line joining the given point with the given tangent and any other given point.

Construction: Let $S=\{12, 34; 45, 16; 25, 36\}$ and $T=\{78, 34; 28, 35; 24, 75\}$ be the generating pencils and choose the following conditions: $ST=34$, $P=(34, 26)$ when $5=2$ and $16=1-(SB, 24)$, $SA=12$, $2=(TB, 12)$. Let 3 be any point on 34, and 33 be any line through 3. Draw 24 through 2 and the intersection of 33 with s, and 28 through 2 and the intersection of 33 with t and let $31=3-(TA, 28)$. The conic c is now determined by the points 1, 2, 3, 4 and the tangent 33.

We should note here that when a cusp is given only five other geometric conditions are necessary to determine the cubic.[1]

Problem XV. Given a cusp, tangent at the cusp, two points and a tangent at one, and the intersection of the tangent and the cubic, to construct other points on the cubic.

Construction: We shall give a new type of solution for this problem. As generating forms we shall use a system of tangents to a conic c_2 projectively related to a pencil of rays of the first order. The double point will be at the center of the first order pencil and if this is outside c_2 it will be a crunode, inside c_2 an acnode, and if it is on c_2 it will be a cusp. The tangents to the cubic at the double point are the rays of the pencil which correspond to the tangents to c_2 which pass through the center of the pencil, and a tangent to c_2 becomes a tangent to the cubic when the corresponding ray of the pencil passes through its point of tangency. Let 1, 2, 3, 4 and 5 be points on a given conic c_1 and let 5 move on that conic, then (25, 41) and (45, 23) are projective point rows on 41 and 23 respectively. Let the system of lines joining corresponding points of these be the tangents to c_2. When $5=3$, 23 becomes a tangent and hence 3 is the point of tangency. Likewise, when $5=1$, 41 becomes a tangent and 1 is its point of tangency. Let 15 be the first order pencil, then 1 is the cusp; 11 is the tangent at the cusp, 4 is a point on the cubic, 23 is tangent to the cubic at 3, and 2 is the intersection of that tangent and the cubic. Now to make the required construction, let the given cusp be 1 and the given tangent at the cusp be 11, let the given point with the given tangent be 3, the given tangent be 23 and the intersection of the given tangent and the cubic be 2, and let the other given point be 4. The conic c_1 is now determined by the points 1, 2, 3, 4 and the tangent 11. Let 5 be any point on c_1. Then the line joining (25, 41) and (45, 23) intersects 15 in a point on the required cubic.

By using combinations of these problems we may construct any number of points on a cubic, the tangent at any given point, the intersection of that tangent and the cubic, the intersection of the cubic and a line joining any two given points, and the tangents at the double points when the cubic satisfies any set of the above given conditions.

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[1] Hilton, loc. cit., page 33.

AN APPLICATION OF NUMBER THEORY TO THE SPLICING OF TELEPHONE CABLES

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Some time ago in connection with the placing of a long telephone cable the writer had occasion to attempt the specification of a splicing scheme designed to minimize the recurrence of same-layer adjacencies among the telephone circuits as they threaded their way through successive lengths of the completed cable. The task, superficially so simple, proved to be one of most intriguing difficulty, and the pursuit of the solution led a confused investigator stumbling into the province of number theory. That speculation upon an art so mundane as that of telephone cable splicing should have led to a proposition in the oldest and most neglected branch of mathematics seemed to be especially worthy of note, for few applications so practical have been found. While the planned splicing of long telephone cables is of no immediate great importance, the present trend toward the use of carrier frequencies in cables suggests that a serious need for such may arise in the future. In the course of the investigation certain small ground apparently was covered for the first time. It was felt, therefore, that the story would be of passing interest alike to the mathematician and to the engineer.

The present standard cables for long distance telephone service are manufactured as a series of concentric layers of conductor units contained within a cylindrical sheath. The conductor units are either pairs or quads of wires. The layers are one unit in thickness, and successive layers either spiral in opposite directions of rotation, or in the same direction but with different pitches. The feature of importance to this discussion is that in an unbroken length of cable any one conductor unit will experience shoulder-to-shoulder adjacency throughout this distance with the two conductor units lying on either side in the same layer, and its experience with these two conductor units will be unique. Cables usually are manufactured in uniform lengths of from 750 to 1000 feet, and a longer cable is made up from a succession of such lengths spliced end-to-end. At each splice point a large number of different splices is possible among conductor units. In general, wire-to-wire splices are not made, and considerable mixing up is achieved. For reasons which need not be given here it is considered desirable from the standpoint of crosstalk control that each telephone circuit experience the minimum amount of same-layer adjacency with every other telephone circuit.

For the purposes of this discussion it will suffice at present to consider the cross-section of a cable as a simple, closed sequence of N consecutively adjacent units. As an example, the array presented by a circular picket fence would be of this character. Each conductor unit in a cable is identifiable, and it will be assumed that each has been "tagged" with one of the numbers $1, 2, 3, 4, \cdots, N$ in such sequence that units bearing consecutive numbers lie adjacent—remembering that unit No. 1 and unit No. N also lie adjacent. While this simple pic-

ture of the cable cross-section is representative truly of only a single layer structure, still the results of a study of it may be fitted to apply to practical cases. Schemes for accomplishing this will suggest themselves to the practical worker, and their discussion here would burden this presentation unduly.

Consider now two consecutive lengths in a completed cable and focus attention upon a conductor unit in one of these. At the splice point this conductor unit may connect to any one of the conductor units in the second length, and the two conductor units which lie alongside the latter in the same layer in the second length may connect to any two of the $N-1$ remaining conductor units in the first length. As an extended conductor unit traverses the completed cable, then, it may experience same-layer adjacency successively with any possible combinations two at a time of the other extended conductor units, and in any order, sequence, or repetition of these as determined by the splicing scheme that is used. Since there can be but $[(N-1)/2]$* totally different combinations two at a time of $N-1$ different objects it is evident that $[(N-1)/2]$ successive cable lengths is the maximum possible number for an extended conductor unit to traverse without incurring repetition of at least one of the same-layer adjacencies that occurred in the first of these lengths.

Any splicing scheme that is devised for practical use must embody the utmost in simplicity. For this reason it is considered highly desirable (1) that the required results be achieved through repetition of the same splicing instruction at consecutive splice points, and (2) that this instruction follow the simplest possible system—e.g., any two adjacent conductor units in one length of cable shall connect to two conductor units having a constant separation in count in the next length. The exposition which follows makes no attempt to solve the general problem, and seeks only to establish the results which can be realized when the above two simplifying restrictions are imposed. At the conclusion is added a description of a minor and acceptable deviation from the second restriction which will enable the practical worker to supplement these results and achieve the maximum possibilities in a number of cases sufficient for his needs. The problem now will be formulated.

$1\to 1$	$1\to 1$	$1\to 1$
$2\to 2$	$2\to 3$	$2\to 4$
$3\to 3$	$3\to 5$	$3\to 7$
$4\to 4$	$4\to 7$	$4\to 10$
$5\to 5$	$5\to 9$	$5\to 2$
$6\to 6$	$6\to 11$	$6\to 5$
$7\to 7$	$7\to 2$	$7\to 8$
$8\to 8$	$8\to 4$	$8\to 11$
$9\to 9$	$9\to 6$	$9\to 3$
$10\to 10$	$10\to 8$	$10\to 6$
$11\to 11$	$11\to 10$	$11\to 9$
Fig. 1	Fig. 2	Fig. 3

* The symbol $[x/y]$ means the greatest integer not greater than x/y.

The three tabulations exhibited in Figs. 1, 2, and 3 show possible ways of splicing two pieces of eleven-unit cable together in systematic fashion. The left-hand columns indicate the consecutively adjacent conductor units in the first or reference piece of cable (remembering that No. 1 and No. 11 are adjacent), and the numbers opposite in the right hand columns indicate the conductor units in the second piece of cable to which splice is made. No importance attaches to the splicing of unit No. 1 to unit No. 1 in each instance. This is simply one of eleven possible "starts," and from the point of view of this discussion there is no preference among these. Note that with Fig. 1 two conductor units which lie adjacent in the first piece of cable connect to conductor units separated by a count of one (adjacent) in the second piece. With Fig. 2 conductor units which lie adjacent in the first piece connect to conductor units separated by a count of two in the second piece. With Fig. 3 conductor units which lie adjacent in the first piece connect to conductor units separated by a count of three in the second piece. Splices made in accordance with the schemes of Figs. 1, 2, or 3 will be described as made with a "spread of one," a "spread of two," or a "spread of three," respectively. It is readily shown that for a spread number s to be applicable to a cable of N units it is necessary and sufficient that s be prime relative to N.

Fig. 4 shows the splicing of six pieces of eleven-unit cable through the successive application of five consecutive identical splices, each with a spread of

$$\begin{array}{r}
1\to 1\to 1\to 1\to 1\to 1\\
2\to 3\to 5\to 9\to 6\to 11\\
3\to 5\to 9\to 6\to 11\to 10\\
4\to 7\to 2\to 3\to 5\to 9\\
5\to 9\to 6\to 11\to 10\to 8\\
6\to 11\to 10\to 8\to 4\to 7\\
7\to 2\to 3\to 5\to 9\to 6\\
8\to 4\to 7\to 2\to 3\to 5\\
9\to 6\to 11\to 10\to 8\to 4\\
10\to 8\to 4\to 7\to 2\to 3\\
11\to 10\to 8\to 4\to 7\to 2
\end{array}$$

Fig. 4

two. Following the "key" of the first and second columns, the succeeding columns are written down immediately. Scrutiny of the sequences of numbers appearing in the several columns reveals at once the fundamental properties of the spread. For a cable of N units these are:

1. Successive applications of a spread of s for n times result in a spread of s^n.
2. A spread of minus s is equivalent effectively to a spread of plus s.
3. A spread of $KN+s$(K is an integer: positive, negative, or zero) is the same effectively as a spread of s.

The problem of achieving the minimum possible recurrence of same-layer adjacencies among conductor units through the application of successive similar

splices in accordance with a simple spread now may be stated formally in the terminology and symbols of number theory. If N, an integer, is the number of conductor units in the cable, and if s, an integer prime to N, is the spread number used, then it is required to find a value for s for which the companion relations

$$s^d \equiv \pm 1 \pmod{N}$$
$$s^b \not\equiv \pm 1 \pmod{N}, \quad b < d$$

determine the largest possible integer d.

From the foregoing introductory discussion it should be noted that values for N less than 5 are of no significance to this problem. In the analysis which follows, therefore, no particular effort has been made to render the general conclusions capable of extension to these extreme and trivial cases.

It is necessary at this point to recall and introduce certain working material. First, there is the established theorem that every positive integer N greater than unity can be represented in one and only one way in the form

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$$

where $p_1, p_2, \cdots, p_t$ are different primes and $\alpha_1, \alpha_2, \cdots, \alpha_t$ are positive integers. Then there is the familiar number theory function $\phi(N)$ which indicates the number of positive integers not greater than N and prime to N.* If p is a prime number and α is a positive integer, then

$$\phi(p^\alpha) = p^{\alpha-1}(p-1);$$

also

$$\phi(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}) = \phi(p_1^{\alpha_1}) \cdot \phi(p_2^{\alpha_2}) \cdots \phi(p_t^{\alpha_t})$$

where $p_1, p_2, \cdots, p_t$ are different primes.
Then there is the λ-function defined in terms of the ϕ-function as follows:

$$\lambda(2^\alpha) = \phi(2^\alpha) \text{ for } \alpha = 0, 1, 2$$

$$\lambda(2^\alpha) = \frac{\phi(2^\alpha)}{2} \text{ for } \alpha > 2$$

$$\lambda(p^\alpha) = \phi(p^\alpha) \text{ for } p \text{ an odd prime}$$

$$\lambda(2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}) = M$$

where M is the least common multiple of

$$\lambda(2^{\alpha_1}), \lambda(p_2^{\alpha_2}), \lambda(p_3^{\alpha_3}), \cdots, \lambda(p_t^{\alpha_t})$$

$2, p_2, p_3, \cdots, p_t$ being different primes.† Finally, it is established that for two

* Euler, *Novi Comm. Ac. Petrop.*, 1760–61, p. 74. Carmichael, *The Theory of Numbers*, John Wiley & Sons, Inc., 1914, pp. 30–32. Dickson, *Introduction to the Theory of Numbers*, Univ. of Chicago Press, 1929, Chap. I.

† Cauchy, Comptes Rendus, Paris, 1841, pp. 824–845. Carmichael, p. 53.

relatively prime integers s and N the value $\lambda(N)$ is the largest possible for the exponent m for which the relations

$$s^m = 1 \pmod N$$
$$s^n \not\equiv 1 \pmod N, \quad n < m,$$

will hold, and that a value for s belonging to this exponent does exist.*

Here it is convenient to consider separately numbers of the two classes—those for which $\lambda(N) = \phi(N)$ and those for which $\lambda(N) < \phi(N)$. For numbers of the first class established theorems may be drawn upon to furnish a complete analysis. For numbers of the second class, however, it will be necessary to extend a bit beyond the ground covered by previous workers, and the steps will be given in considerable detail. This procedure coupled with the inherent complexity will render the treatment for the latter class much less compact and elegant than that for the former.

Case I. $\lambda(N) = \phi(N)$

From the defined relation between the ϕ-function and the λ-function it follows that numbers of the class such that $\lambda(N) = \phi(N)$ are confined to the values

$$1, 2, 4, p^\alpha, \text{ and } 2p^\alpha$$

where p is an odd prime and α is a positive integer.† For a number N of this class it is established that there exists a set of $\phi(\phi(N))$ numbers r, such that

$$r^{\lambda(N)} \equiv 1 \pmod N \quad \text{and} \tag{1}$$

$$r^n \not\equiv 1 \pmod N, \quad n < \lambda(N), \quad \lambda(N) = \phi(N). \tag{2}$$

Such a number is known as a "primitive root" of N.‡ From the properties of the primitive root r of the number N as defined by relations (1), (2) it follows readily that

$$r^{\lambda(N)/2} \equiv -1 \pmod N \tag{3}$$

$$r^n \not\equiv \pm 1 \pmod{N}, \quad 0 < n < \frac{\lambda(N)}{2}, \quad \frac{\lambda(N)}{2} < n < \lambda(N). \tag{4}$$

First there will be considered the companion relations

$$s^d \equiv -1 \pmod N \tag{5}$$

$$s^b \not\equiv \pm 1 \pmod N, \quad b < d \tag{6}$$

and, from comparison with relations (3) and (4), these clearly are satisfied for s a primitive root of N and for $d = \lambda(N)/2$. That no value for d greater than

* Carmichael, p. 54 and pp. 61–63.

† Carmichael, p. 71.

‡ Gauss, *Disquisitiones Arithmeticae*, Art. 52–55. Carmichael, pp. 65–71. Dickson, Sec. 17.

$\lambda(N)/2$ is possible is evident immediately. For let a be any integer prime to N. Then for some exponent k

$$r^k \equiv a \pmod{N} \quad \text{and}$$
$$a^{\lambda(N)/2} \equiv r^{k\lambda(N)/2} \equiv \pm 1 \pmod{N}.$$

Next there will be considered the companion relations

$$s^d \equiv 1 \pmod{N} \tag{7}$$
$$s^b \not\equiv \pm 1 \pmod{N}, \quad b < d. \tag{8}$$

The reasoning just above shows that d cannot be greater than $\lambda(N)/2$. Suppose for the moment that d has this greatest possible value $\lambda(N)/2$. Relations (7), (8) then become

$$s^{\lambda(N)/2} \equiv 1 \pmod{N}$$
$$s^b \not\equiv \pm 1 \pmod{N}, \quad b = 1, 2, 3, \cdots, \lambda(N)/2 - 1.$$

These relations may be written

$$(s^{1/2})^{\lambda(N)} \equiv 1 \pmod{N} \tag{9}$$
$$(s^{1/2})^{2b} \not\equiv \pm 1 \pmod{N}, \quad 2b = 2, 4, 6, \cdots, \lambda(N) - 2. \tag{10}$$

Now relations (9), (10) will be compatible with relations (1), (2), (3), (4) only if $\lambda(N)/2$ is an odd number, for otherwise the restrictions of relation (10) applying to the even numbered exponents from 2 to $\lambda(N)-2$ inclusive would be in conflict with relation (3). For $\lambda(N)/2$ an odd number, then, relations (9), (10), are satisfied for $s^{1/2}$ a primitive root of N. Consequently, with relations (7), (8), $\lambda(N)/2$ is the largest possible value for the exponent d, and a value for s equal to the square of a primitive root of N permits this to be attained.

Case II. $\lambda(N) < \phi(N)$

The inquiry for this case will be divided into four parts. In general $N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}$ where $p_1, p_2, p_3, \cdots, p_t$ are different primes.

(a) First will be considered the case where $p_1, p_2, p_3, \cdots, p_t$ are all odd primes. Then $\lambda(N)$ is the least common multiple of $\lambda(p_1^{\alpha_1}), \lambda(p_2^{\alpha_2}), \lambda(p_3^{\alpha_3}), \cdots, \lambda(p_t^{\alpha_t})$. Suppose now that the highest power of 2 dividing any of the λ's divides $\lambda(p_i^{\alpha_i})$. If this same power of 2 divides more than one of the λ's, arbitrarily select $\lambda(p_i^{\alpha_i})$ as one of them. Then this power of 2 will be exactly that occurring in $\lambda(N)$. Now arbitrarily select p_j as any one of the odd primes other than p_i. Then clearly $\lambda(N)$ will also be the least common multiple of

$$\lambda(p_1^{\alpha_1}), \lambda(p_2^{\alpha_2}), \cdots, \lambda(p_{j-1}^{\alpha_{j-1}}), \frac{\lambda(p_j^{\alpha_j})}{2}, \lambda(p_{j+1}^{\alpha_{j+1}}), \cdots, \lambda(p_t^{\alpha_t}).$$

Now take

$$
\begin{aligned}
r &\equiv g_j^2 \pmod{p_j^{\alpha_j}}, \quad g_j \text{ a primitive root of } p_j^{\alpha_j}, \\
&\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\
&\qquad\qquad k = 1, 2, 3, \cdots, j-1, j+1, \cdots, t.
\end{aligned}
$$

The r thus chosen must be prime to each of the prime factors of N, and hence must be prime to N. Consequently it is known that

$$r^{\lambda(N)} \equiv 1 \pmod{N}.$$

Suppose that m is the smallest exponent for which the congruence

$$r^m \equiv 1 \pmod{N}$$

is true. Then it is noted that the chosen r is such that m must be a multiple of

$$\lambda(p_1^{\alpha_1}),\ \lambda(p_2^{\alpha_2}),\ \cdots,\ \lambda(p_{j-1}^{\alpha_{j-1}}),\ \frac{\lambda(p_{1j}^{\alpha_j})}{2},\ \lambda(p_{j+1}^{\alpha_{j+1}}),\ \cdots,\ \lambda(p_t^{\alpha_t}),$$

and the least multiple common to these is, of course, $\lambda(N)$. Therefore it can be written that

$$
\begin{aligned}
r^{\lambda(N)} &\equiv 1 \pmod{N} \\
r^b &\not\equiv 1 \pmod{N}, \quad b < \lambda(N).
\end{aligned}
$$

Now suppose that for some exponent n less than $\lambda(N)$

$$r^n \equiv -1 \pmod{N}.$$

Then

$$r^{2n} \equiv 1 \pmod{N}$$

and if n is less than $\lambda(N)$, $2n$ is less than $2\lambda(N)$ and can only be equal to $\lambda(N)$. It would necessarily follow then that

$$r^{\lambda(N)/2} \equiv -1 \pmod{N}$$

and it would follow in turn that

$$r^{\lambda(N)/2} \equiv -1 \pmod{p_j^{\alpha_j}}.$$

However, r has been chosen such that

$$r^{\lambda(N)/2} \equiv (g_j^2)^{\lambda(N)/2} \equiv g_j^{\lambda(N)} \equiv 1 \pmod{p_j^{\alpha_j}}.$$

This last relation is incompatible with the one immediately above, and it must be concluded that the assumption

$$r^n \equiv -1 \pmod{N}, \quad n < \lambda(N)$$

is false, and that for the r that has been chosen

$$r^{\lambda(N)} \equiv 1 \pmod{N}$$
$$r^{b} \not\equiv \pm 1 \pmod{N}, \quad b < \lambda(N)$$

and no exponent greater than $\lambda(N)$ is possible.

(b) Next will be considered the case where $p_1 = 2$, $\alpha_1 = 1$, and $p_2, p_3, \cdots, p_t$ are all odd primes. Select p_i as above and take p_j different from 2. Then take

$$\begin{aligned} r &\equiv 1 \pmod{2} \\ &\equiv g_j^2 \pmod{p_j^{\alpha_j}}, \quad g_j \text{ a primitive root of } p_j^{\alpha_j}, \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 2, 3, 4, \cdots, j-1, j+1, \cdots, t, \end{aligned}$$

and the same line of reasoning may be repeated and the same conclusions reached as under part (a) above.

(c) Next will be considered the case where $p_1 = 2$, $\alpha_1 = 2$ and $p_2, p_3, \cdots, p_t$ are all odd primes. Since $\lambda(2^2) = 2$ take p_i different from 2, and for simplicity take p_j as 2. Then take

$$\begin{aligned} r &\equiv 1 \pmod{4} \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 2, 3, 4, \cdots, t \end{aligned}$$

and the same line of reasoning may be repeated and the same conclusions reached as under part (a) above.

(d) Finally will be considered the case where $p_1 = 2$, $\alpha_1 > 2$, and $p_2, p_3, \cdots, p_t$ are all odd primes. Now 5 has the property that

$$5^{\lambda(2^{\alpha_1})} \equiv 1 \pmod{2^{\alpha_1}}$$
$$5^{b} \not\equiv \pm 1 \pmod{2^{\alpha_1}}, \quad \alpha_1 > 2, \quad b < \lambda(2^{\alpha_1}).$$

So by taking

$$\begin{aligned} r &\equiv 5 \pmod{2^{\alpha_1}} \\ &\equiv g_k \pmod{p_k^{\alpha_k}}, \quad g_k \text{ a primitive root of } p_k^{\alpha_k}, \\ &\qquad k = 2, 3, 4, \cdots, t \end{aligned}$$

it is concluded immediately that

$$r^{\lambda(N)} \equiv 1 \pmod{N}$$
$$r^{b} \not\equiv \pm 1 \pmod{N}, \quad b < \lambda(N).$$

The preceding formal analysis for Case I and Case II may be summed up as having established the following general theorem:

If N is a given positive integer and if s is an integer prime to N, then the largest possible exponent d for which the companion congruencial relations

$$s^d \equiv \pm 1 \pmod{N}$$
$$s^b \not\equiv \pm 1 \pmod{N}, \quad b < d$$

will be true is $\lambda(N)/2$ *for numbers such that* $\lambda(N) = \phi(N)$ *and is* $\lambda(N)$ *for numbers such that* $\lambda(N) < \phi(N)$, *and a value for s belonging to this exponent in each instance does exist.*

In order to apply the foregoing results to a practical case Table 1 has been prepared. In the left-hand column appear the numbers 5 to 139, inclusive. In the next column is listed for each number the value of $\lambda(N)/2$ or of $\lambda(N)$, depending upon whether $\lambda(N) = \phi(N)$ or $\lambda(N) < \phi(N)$. In the final column there is listed for each number a suitable value for the spread. There appears to be no advantage of one spread figure over another, and the listing of additional acceptable values is omitted in the interest of economy of space. For the numbers for which $\lambda(N) = \phi(N)$ and for which $\lambda(N)/2$ is odd care has been taken that the listed spread figures are primitive roots, and not the squares of primitive roots which were shown to be equally acceptable. This fact will be recalled later.

It was shown earlier that $[(N-1)/2]$ successive cable lengths would be the maximum possible number for an extended conductor unit to traverse without incurring repetition of at least one of the same-layer adjacencies which occurred in the first of these lengths. On referring to Table 1 it is seen that only for the prime numbers is this maximum attainable. The prime numbers are distinguished by the fact that for them $\lambda(N)/2 = (N-1)/2$, and each has been indicated by an asterisk. The composite numbers are seen to yield quite inferior results in general.

For the benefit of the practical worker there must be described a slight deviation from the second simplifying restriction imposed at the beginning which will permit the maximum possibility to be realized if N is one plus a prime number. This artifice is based upon the fact that for r a primitive root of a number N for which $\lambda(N) = \phi(N)$ and in particular for a prime number N

$$r^{\lambda(N)/2} \equiv -1 \pmod{N}.$$

This means that $\lambda(N)/2$ consecutive splices with a spread r result in a spread of minus one. It is readily shown that this in turn means that there will be two conductor units No. b and No. $b+1$ in the first length of cable which ultimately will be extended to connect respectively to units No. $b+1$ and No. b. In Figure 4 two units No. 6 and No. 7 meet this requirement. To illustrate the use of this artifice it will be supposed that a cable of 12 units is to be spliced. Referring to Figure 4 for guidance, the arrangement shown in Figure 5 is set up readily. The first two columns indicate the splicing assignment, and the succeeding columns are then derived from these. The eleven units $1, 2, \cdots, 5, 6, 8, 9, \cdots, 11, 12$ are assigned exactly in conformity with the scheme of Figure 4, ignoring the break in sequence between No. 6 and No. 8. Unit No. 7 is then simply spliced to itself throughout.

TABLE 1

For each N there is listed the value d and a value s for which the companion relations

$$s^d \equiv \pm 1 \pmod{N} \qquad s^b \not\equiv \pm 1 \pmod{N},\ b<d$$

determine the largest possible integer d.

N	d	s	N	d	s	N	d	s
5*	2	2	50	10	3	95	36	2
6	1	5	51	16	5	96	8	5
7*	3	3	52	12	7	97*	48	5
8	2	3	53*	26	2	98	21	3
9	3	2	54	9	5	99	30	5
10	2	3	55	20	2	100	20	3
11*	5	2	56	6	3	101*	50	2
12	2	5	57	18	5	102	16	5
13*	6	2	58	14	3	103*	51	5
14	3	3	59*	29	2	104	12	7
15	4	2	60	4	7	105	12	2
16	4	3	61*	30	2	106	26	3
17*	8	3	62	15	3	107*	53	2
18	3	5	63	6	2	108	18	5
19*	9	2	64	16	3	109*	54	6
20	4	3	65	12	3	110	20	3
21	6	2	66	10	5	111	36	2
22	5	7	67*	33	2	112	12	3
23*	11	5	68	16	3	113*	56	3
24	2	5	69	22	2	114	18	5
25	10	2	70	12	3	115	44	2
26	6	7	71*	35	7	116	28	3
27	9	2	72	6	5	117	12	2
28	6	5	73*	36	11	118	29	11
29*	14	2	74	18	5	119	48	3
30	4	7	75	20	2	120	4	7
31*	15	3	76	18	21	121	55	2
32	8	3	77	30	2	122	30	7
33	10	5	78	12	7	123	40	7
34	8	3	79*	39	3	124	30	7
35	12	2	80	4	3	125	50	2
36	6	5	81	27	2	126	6	11
37*	18	2	82	20	7	127*	63	3
38	9	3	83*	41	2	128	32	3
39	12	2	84	6	5	129	42	14
40	4	3	85	16	3	130	12	3
41*	20	6	86	21	3	131*	65	2
42	6	11	87	28	2	132	10	5
43*	21	3	88	10	3	133	18	2
44	10	3	89*	44	3	134	33	7
45	12	2	90	12	7	135	36	2
46	11	5	91	12	2	136	16	3
47*	23	6	92	22	3	137*	68	3
48	4	5	93	30	13	138	22	7
49	21	5	94	23	5	139*	69	2

The asterisk indicates a prime number

$$
\begin{array}{l}
1\rightarrow\ 1\rightarrow\ 1\rightarrow\ 1\rightarrow\ 1\rightarrow\ 1\\
2\rightarrow\ 3\rightarrow\ 5\rightarrow 10\rightarrow\ 6\rightarrow 12\\
3\rightarrow\ 5\rightarrow 10\rightarrow\ 6\rightarrow 12\rightarrow 11\\
4\rightarrow\ 8\rightarrow\ 2\rightarrow\ 3\rightarrow\ 5\rightarrow 10\\
5\rightarrow 10\rightarrow\ 6\rightarrow 12\rightarrow 11\rightarrow\ 9\\
6\rightarrow 12\rightarrow 11\rightarrow\ 9\rightarrow\ 4\rightarrow\ 8\\
7\rightarrow\ 7\rightarrow\ 7\rightarrow\ 7\rightarrow\ 7\rightarrow\ 7\\
8\rightarrow\ 2\rightarrow\ 3\rightarrow\ 5\rightarrow 10\rightarrow\ 6\\
9\rightarrow\ 4\rightarrow\ 8\rightarrow\ 2\rightarrow\ 3\rightarrow\ 5\\
10\rightarrow\ 6\rightarrow 12\rightarrow 11\rightarrow\ 9\rightarrow\ 4\\
11\rightarrow\ 9\rightarrow\ 4\rightarrow\ 8\rightarrow\ 2\rightarrow\ 3\\
12\rightarrow 11\rightarrow\ 9\rightarrow\ 4\rightarrow\ 8\rightarrow\ 2
\end{array}
$$

Fig. 5

Undoubtedly there are other equally acceptable artifices for extending further the practical scope of the simple results. The prime numbers and the prime numbers plus one constitute nearly fifty percent of all numbers in the range in which the practical worker is likely to be interested, however, and when it is borne in mind that normally he has latitude in his choice of N it is seen that the material here presented is adequate for his needs.

The writer is indebted to Dr. D. H. Lehmer for pertinent suggestions. The entire treatment for the case of numbers for which $\lambda(N) < \phi(N)$ follows a line of attack suggested by Mr. Marshall Hall, and but for his helpful interest this presentation would have been lacking in formal completeness.

ON THE UNRESTRICTED PARTITIONS OF A POSITIVE INTEGER

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A partition of a positive integer, n, is a representation of n as a sum of positive integers less than or equal to n. Two partitions of n which differ only in the order of the terms in the sum are said to be the same. For example the five partitions of 4 are: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4.

The foundation for the development of the theory of partitions was laid by Euler who discussed the problem of finding the number of ways a given positive integer, n, can be divided into a given number of distinct positive integral parts and also the case where the parts need not be distinct. If the number of parts be m, then the solutions are the coefficients of x^n in the expansions of

$$\frac{x^{m(m+1)/2}}{D} \text{ and } \frac{x^m}{D}$$

respectively, where

$$D = (1 - x)(1 - x^2) \cdots (1 - x^m).$$

The unrestricted partitions of n are given by a somewhat similar expression.

Following along these lines numerous investigators have contributed to the theory. The most important of these are Sylvester,* MacMahon, and Glaisher.

I propose in the following discussion not to treat the problem through the properties of symmetric functions but to develop the solution directly.

The first step of the solution is the development of a reduction system. Calling a number contained in the partitions a unit, say j, we observe that in the partitions of n (denoted by P_n) we have the unit j contained at least P_{n-j} times. But in this expression, P_{n-j}, we can say definitely only that it gives the first approximation of the number of units j. However, we see as before that in P_{n-j} we have the unit j contained at least P_{n-2j} times.

Now we must see what is the last expression of such a formulation of the total number of units of one kind, j. On inspection of P_k where $j<k<2j$ it is obvious that, as before, the number of units j contained in P_k is P_{k-j} for, since $(k-j)<j$, P_{k-j} contains no units j. But should $k=j$ we see that P_j contains the unit j once and once only. Therefore we are led to the definition of P_0 by the extension $1=P_{j-j}=P_0$.

By the preceeding analysis we have for the total number of units of one kind in the partitions of n, indicated by S_j^n, the following expression

$$S_j^n = P_{n-j} + P_{n-2j} + \cdots + P_k \tag{1}$$

where $k=n-tj$ and t is the greatest integer less than or equal to n/j.

To illustrate this procedure we take $n=5$ and seek the number of 2's contained in P_5. Now for partitions of 5 containing 2's we have $2+3$, $2+2+1$, $2+1+1+1$, and subtracting 2 from each we have 3, $2+1$, and $1+1+1$ which constitute the partitions of 3. So we have a number of 2's equal to P_{5-2} or P_3 as our first approximation. But P_3 also contains one 2 in $2+1$ and subtracting 2 we have a further number of 2's equal to P_{3-2} or P_1. So the number of 2's in P_5 is

$$S_2^5 = P_{5-2} + P_{5-2\cdot 2} = P_3 + P_1.$$

Should we add together all the units of magnitude equal to j we would get

$$jS_j^n = jP_{n-j} + jP_{n-2j} + \cdots + jP_k. \tag{2}$$

But in the partitions of n, P_n, if we should add together all the units of all magnitudes which are present, that is where $1 \leqq j \leqq n$, we should get a sum equal to nP_n since in each partition the sum is by definition equal to n. Therefore we have the following reduction system

$$P_n = \frac{1}{n}\sum_{j=1}^{n}\sum_{t=1}^{m_j} jP_{n-tj}\,, \tag{3}$$

where m_j is the greatest integer less than or equal to n/j. Written out the equation presents the following form

* Sylvester, Coll. Math. Papers, vol. II, pp. 90–99; MacMahon, Proc. London Math. Soc., vol. 28, 1896–7, pp. 5–32; Glaisher, Quar. Jour. Math., vol. 40, 1909, pp. 57–143.

$$P_n = \frac{1}{n}\left[\begin{array}{l} P_{n-1} + 2P_{n-2} + 3P_{n-3} + \cdots + m_1P_{n-m_1} \\ + P_{n-2} + 2P_{n-4} + 3P_{n-6} + \cdots + m_2P_{n-2m_2} \\ + P_{n-3} + 2P_{n-6} + 3P_{n-9} + \cdots + m_3P_{n-3m_3} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ + P_0 \end{array}\right]$$

From equation (3) we have, putting $tj = r, j = r/t$,

(4) $$P_n = \frac{1}{n}\sum_{r=1}^{n}\sum_{t}\frac{r}{t}P_{n-r}.$$

The coefficient of P_{n-r} equals $\sum r/t$ for all possible divisors t of r. Thus we may define a new function H_r by writing $H_r = \sum r/t = \sum t$ where the sum is over all divisors t of r. Thus, for example,

$$H_4 = 4(1 + \tfrac{1}{2} + \tfrac{1}{4}) = 7.$$

Using this function we may write

(5) $$P_n = \frac{1}{n}(H_1P_{n-1} + H_2P_{n-2} + \cdots + H_nP_0).$$

We see therefore that $P_n = f(H_1, H_2, \cdots H_n)$.

Now we define θ as an operator on H_r by the equation

(6) $$\theta H_r = H_{r+1}$$

then

(7) $$\theta P_{n-1} = \frac{1}{(n-1)}(H_2P_{n-2} + H_3P_{n-3} + \cdots + H_nP_0).$$

Therefore

$$nP_n - (n-1)\theta P_{n-1} = H_1P_{n-1}$$

or

$$P_n = \frac{\{H_1 + (n-1)\theta\}}{n}P_{n-1}$$

and continuing the reduction of P_{n-1} by the same process we get

(8) $$P_n = \frac{\{H_1 + (n-1)\theta\}\{H_1 + (n-2)\theta\}\cdots(H_1 + \theta)H_1}{n!}.$$

It can be seen quite clearly that also

(9) $$P_n = \frac{(H_1 + \theta)(H_1 + 2\theta)(H_1 + 3\theta)\cdots\{H_1 + (n-1)\theta\}H_1}{n!}.$$

QUESTIONS, DISCUSSIONS AND NOTES

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE CLASSIFICATION AND GENERAL SOLUTION OF CERTAIN DIOPHANTINE PROBLEMS WHICH INVOLVE SPECIAL SYSTEMS OF EQUATIONS OF THE SECOND DEGREE

By H. A. SIMMONS, Northwestern University

1. *Introduction.* By *Diophantine equation* we mean an algebraic equation in integral unknowns with integral coefficients. In this paper we consider a system of second-degree Diophantine equations without square terms and we associate with the equations certain inequalities. By fixing one particular variable in our system of equations, we obtain a linear system. Then we apply to this system the known theory for linear systems, namely: *if a system of linear Diophantine equations, say m linearly independent equations in n unknowns, $m<n$, has a solution, it can be found, and the n unknowns can be expressed linearly in terms of a known solution and $n-m$ parameters in such a way as to give without repetition every solution of the system and not to give any set of integers that does not constitute a solution of the system.** Finally, the solutions which we seek are those of the system which satisfy the associated inequalities.

We first solve in §2 below a *special problem* which is only a slight modification of a problem that Professor Gilman† devised for the purpose of this paper. In §3 we treat a general problem that includes the *special problem* and has only a finite number of solutions. In §4 we make a slight modification of the problem of §3 and obtain a problem which has infinitely many solutions if it has one. In §5, we state Theorem 1, which is an analytical summary of the results in §§3, 4. In §6, we state Theorem 2, which includes Theorem 1. In §7, we interpret geometrically our separation of the problems that we consider into two classes, those with a *finite*, and those with an *infinite*, number of solutions.

2. *Special problem.* Three herdsmen of different ages‡ but with a common anniversary decide to celebrate the anniversary by setting their sons up in the cattle business. Accordingly each son receives a number of cattle equal to his father's age in years. The oldest herdsman distributes half of his herd; the next oldest has two less sons than the oldest and distributes 150 less cattle; the youngest has four less sons than the oldest and distributes 260 less cattle.§

* Cf., for example, E. Cahen, *Théorie des nombres,* (1914), p. 110–118.

† Precisely his problem is treated in §4.

‡ It is not necessary to assume that the herdsmen are of different ages. Exactly two of them or all three of them might have the same age. In §§2, 3, 4 we assume the herdsmen to be of different ages, but in §6 we generalize as far as we can.

§ The numbers 150, 260 were taken so that the ages in years of the herdsmen and the numbers of their sons could be 60, 50, 40 and 5, 3, 1, respectively.

How many sons has each herdsman and how many animals does each son receive?

Restrictions. In dealing with the *special problem* and other similar problems (Cf. §3), one may exclude certain solutions that do not accord with *human longevity* and *reproductive capacity** by laying down restrictions of the following types: the youngest (oldest) herdsman must be at least a (at most b) years old and no herdsman may have more than c sons,† where a, b, c are certain positive integers. In the *special problem* it turns out that we only need to make two of these types of restrictions, namely those concerning the values of a and c. We shall require that the youngest herdsman be at least 35 years old and that no herdsman have more than 20 sons, and we shall call the resulting problem the *restricted problem.* We shall list *all solutions* of the *special problem* and we shall mark with a star such of these solutions as are not solutions of the *restricted problem.*

Solution of the special problem. Let A, B, C be the three herdsmen in decreasing order of ages, and let x, y, z be the ages in years of A, B, C, respectively, on the anniversary in question, so that $x>y>z$. Let n and N be respectively the number of sons and the number of cattle that A has. Then from the statement of the problem, we have

$$(N/2 - 150)/y = N/2x - 2, \quad (N/2 - 260)/z = N/2x - 4;$$

or, using the fact that $N=2nx$,

$$x = y + 2(75 - y)/n, \quad x = z + 4(65 - z)/n. \tag{1}$$

In discussing equations (1), we shall consider separately the cases where (i) n is odd, (ii) n is even and not a multiple of 4, and (iii) n is a multiple of 4. We assume that C has at least one son, so that $n \geqq 5$.

Case (i): *n is odd.* Here we let $(75-y)/n=s$ and $(65-z)/n=t$, where s and t are integers. Substituting these expressions in (1), we get

$$x = 75 + (2 - n)s = 65 + (4 - n)t, \quad y = 75 - ns, \quad z = 65 - nt. \tag{2}$$

Since n is odd, $n-2$ and $n-4$ are relatively prime. Hence the general solution of the equation connecting s and t, namely $(n-2)s-(n-4)t=10$, is

$$s = 5 + (n - 4)u, \quad t = 5 + (n - 2)u,$$

where u is an arbitrary integer (as it is in the sequel except where explicit values are assigned to it). Therefore the general solution of (2), and of (1) in case (i), is

$$\begin{gathered} x = 85 - 5n - (n - 2)(n - 4)u, \qquad y = 75 - 5n - n(n - 4)u, \\ z = 65 - 5n - n(n - 2)u. \end{gathered} \tag{3}$$

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* Professor Gilman used this language in connection with his special problem.

† In some problems of types that we consider, in order to obtain solutions which accord with experience, it might be necessary to require that the age in years of each herdsman exceed the number of his sons by at least a given number, say 18.

In the special problem $x>y>z>0$; therefore $n<13$[Cf. (3)]. In case (i), we therefore only need to examine the cases $n=5, 7, 9, 11$. Doing this, we find: if $n=5$, $(x, y, z)=(60, 50, 40)$, $(57, 45, 25)^*$, $(54, 40, 10)^*$; if $n=7$, $(x, y, z)=(50, 40, 30)^*$; if $n=9$, $(x, y, z)=(40, 30, 20)^*$; if $n=11$, $(x, y, z)=(30, 20, 10)^*$.

Case (ii): *n is even and not a multiple of* 4. In equations (1), we now make the substitutions $2(75-y)/n=s$, $2(65-z)/n=t$. Hence

$$x = 75 - (n/2 - 1)s = 65 - (n/2 - 2)t,\ y = 75 - ns/2,\ z = 65 - nt/2. \tag{4}$$

The general solution of the equation connecting s and t, namely

$$(n/2 - 1)s - (n/2 - 2)t = 10,$$

is

$$s = 10 + (n/2 - 2)u,\ t = 10 + (n/2 - 1)u.$$

Hence the general solution of (4), and of (1) in case (ii), is

$$\begin{aligned} x &= 75 - (n/2 - 1)[10 + (n/2 - 2)u], \\ y &= 75 - (n/2)[10 + (n/2 - 2)u], \\ z &= 65 - (n/2)[10 + (n/2 - 1)u]. \end{aligned} \tag{5}$$

Since $z>0$, $u\geqq 0$ implies that $n<13$ [Cf. (5)]; if $u<0$, $y>z$ implies that $n<(-20/u)$. Hence in case (ii) the special problem has no solution in which $n>20$. Consequently, we need only to examine the cases where $n=6, 10, 14, 18$. Doing this, we find: if $n=6$, $(x, y, z)=(61, 54, 53)$, $(59, 51, 47)$, $(57, 48, 41)$, $(55, 45, 35)$, $(53, 42, 29)^*$, $(51, 39, 23)^*$, $(49, 36, 17)^*$, $(47, 33, 11)^*$, $(45, 30, 5)^*$; if $n=10$, $(x, y, z)=(47, 40, 35)$, $(35, 25, 15)^*$; if $n=14$, $(x, y, z)=(45, 40, 37)$; if $n=18$, $(x, y, z)=(51, 48, 47)$.

Case (iii): *n is a multiple of* 4. In equations (1), we make the substitutions $2(75-y)/n=s$, $4(65-z)/n=t$. Hence

$$x = 75 - (n/2 - 1)s = 65 - (n/4 - 1)t,\ \ y = 75 - ns/2,\ z = 65 - nt/4. \tag{6}$$

The general solution of the equation connecting s and t, namely

$$(n/2 - 1)s - (n/4 - 1)t = 10,$$

is

$$s = 10 + (n/4 - 1)u,\quad t = 20 + (n/2 - 1)u.$$

Consequently the general solution of (6), and of (1) in case (iii), is

$$\begin{aligned} x &= 75 - (n/2 - 1)[10 + (n/4 - 1)u], \\ y &= 75 - (n/2)[10 + (n/4 - 1)u], \\ z &= 65 - (n/4)[20 + (n/2 - 1)u]. \end{aligned} \tag{7}$$

If $u\geqq 0$, $n<13$ [Cf. (7)]; if $u<0$, $y>z$ implies that $n<(-40/u)$, so that $n<40$. Hence we only need to examine here the cases $n=8, 12, 16, 20, 24, 28, 32, 36$.

Doing this, we find: if $n=8$, $(x, y, z)=(57, 51, 49)$, $(54, 47, 43)$, $(51, 43, 37)$, $(48, 39, 31)$*, $(45, 35, 25)$*, $(42, 31, 19)$*, $(39, 27, 13)$*, $(36, 23, 7)$*, $(33, 19, 1)$*; if $n=12$, $(x, y, z)=(55, 51, 50)$, $(45, 39, 35)$, $(35, 27, 20)$*, $(25, 15, 5)$*; if $n=16$, $(x, y, z)=(47, 43, 41)$, $(26, 19, 13)$*; if $n=20$, $(x, y, z)=(21, 15, 10)$*; if $n=24$, $(x, y, z)=(20, 15, 11)$*; if $n=28$, $(x, y, z)=(23, 19, 16)$*; if $n=32$, $(x, y, z)=(30, 27, 25)$*; if $n=36$, $(x, y, z)=(41, 39, 38)$*.

The requirement that $z \geqq 35$ may be taken to account for all of the above stars except the last one; that $n \leqq 20$ requires the last five stars.

3. *Generalization of the special problem.* M herdsmen $A_i(i=1, \cdots, m \geqq 2)$ of different ages but with a common anniversary decide to celebrate the anniversary by setting their sons up in the cattle business. Accordingly each son receives a number of cattle equal to his father's age in years. A_i is older than A_{i+1}. A_1 distributes half of his herd; $A_j(j=2, \cdots, m)$ has e_j less sons than has A_1, $0<e_2<e_3< \cdots <e_m$, and distributes e_jk_j less cattle* than does A_1. How many sons has each herdsman and how many animals does each son receive?

Assuming that the youngest herdsman, A_m, has at least one son, we shall prove that for every admissible set of values of the e_j, k_j this problem, like the *special problem*, has a finite number of solutions. From these solutions one may select, by means of a few restrictions of types that were mentioned in §2, all solutions of our problem that accord with experience.

Let x_i be the age in years of A_i on the anniversary in question, so that $x_1>x_2> \cdots >x_m$, and let n and N be respectively the number of sons and the number of cattle that A_1 has. Then from the statement of the problem, we have

$$(N/2 - e_jk_j)/x_j = N/2x_1 - e_j;$$

or, using the fact that $N=2nx_1$,

$$x_1 = x_j + e_j(k_j - x_j)/n. \tag{8}$$

With a, b equal to any two positive integers, let $D(a, b)$ stand for the greatest common divisor of a and b. Let $D(n, e_j)=g_j$ and write $e_j=E_jg_j$. Then from (8), we have

$$x_1 = x_j + E_j(k_j - x_j)/(n/g_j). \tag{8a}$$

Further, let $(k_j-x_j)/(n/g_j)=s_j$. Substituting these s_j's in (8a) and expressing the x_i in terms of the new parameters, we get

$$x_1 = k_j - (n/g_j - E_j)s_j, \quad x_j = k_j - ns_j/g_j. \tag{9}$$

Equations (9) and the inequalities $x_1>x_j$ imply that $s_j>0$; then (9) and the inequality $x_m>0$ imply that *not more than a finite number of admissible values of n can yield a solution of our problem.* According to the theorem stated in §1, equations (9) either have no solution (in which case our problem has a finite number of solutions) or they have a one-parameter family of solutions (Cf. the

* From facts previously given in the problem, $0<k_j$ and $e_pk_p<e_qk_q$ if p, q, with $p<q$ are admissible values of j.

next paragraph) among which there is, as we shall prove, only a finite number of solutions of our problem.

The solvability of (9) depends on that of the following system of $m-2$ equations in $m-1$ unknowns

$$(10)\quad k_p - (n/g_p - E_p)s_p = k_{p+1} - (n/g_{p+1} - E_{p+1})s_{p+1},\ p = 2, \cdots, m-1.$$

The rank of the matrix of the coefficients of this system is obviously $m-2$ since we assume that $n > e_j$ (the youngest herdsman having at least one son). Consequently the matrix of the coefficients of equations (10) and the augmented matrix of those equations have the same rank. Hence (9) has a solution if, and only if, the greatest common divisor of all $(m-2)$-rowed determinants of the former of these matrices equals that of the latter.* When this condition is fulfilled (10), and therefore (9), has a one-parameter family of solutions by the theorem stated in §1.

To make the last proof mentioned in the second paragraph above, we need to note certain facts about the coefficients of the parameter, u, in the general solution of (9), which we now suppose existent. This solution may be obtained by finding the general solution of an equation in s_2, s_3 of the form $as_2 - bs_3 = c$, where a, b, c are integers and a, b have the same sign; then doing the same for a similar equation in s_4 and the parameter that was used in expressing the general solution of $as_2 - bs_3 = c$; then continuing similarly with a new equation of the same form in s_5 and the parameter in terms of which s_4 was expressed; etc., finally expressing all of the parameters s_j in terms of one parameter. Since the coefficients a, b in each such equation are opposite in sign, it follows that the general solution of (9) is of the form

$$(11)\qquad x_1 = k_j - (n/g_j - E_j)(s_{j0} + s_{j1}u),\ x_j = k_j - (n/g_j)(s_{j0} + s_{j1}u)$$

where the s_{j0} form a particular solution of system (10) and the s_{j1} are positive integers that satisfy the equations which result when the k's of (10) are all set equal to zero; these s_{j1} are defined by

$$(12)\ s_{j1} = (n/g_2 - E_2)(n/g_3 - E_3)\cdots$$
$$(n/g_{j-1} - E_{j-1})(n/g_{j+1} - E_{j+1})\cdots(n/g_m - E_m)/d$$

where $d = d_{23}d_{234}\cdots d_{23\cdots m}$, with $d_{23} = D(n/g_2 - E_2,\ n/g_3 - E_3)$ and

$$d_{23\cdots j} = D[(n/g_2 - E_2)(n/g_3 - E_3)\cdots$$
$$(n/g_{j-1} - E_{j-1})/d_{23}d_{234}\cdots d_{23\cdots(j-1)},\ (n/g_j - E_j)],\ j = 4, \cdots, m.$$

The coefficient of u in the expression for each of the x's is negative. Therefore there is only a finite number of values of $u \geqq 0$ for which $x_i > 0$. Further, from (11) and (12) one readily finds that the coefficient of u in the expression for x_i is numerically less than that in the expression for $x_{i+1} (i = 1, \cdots, m-1)$. Therefore there is only a finite number of negative values of u for which $x_1 > x_2 > \cdots$

* Cf. E. Cahen, l.c., p. 174.

$>x_m$. Hence *only a finite number of values of u can yield a solution of our problem.* This fact and the statement just below equations (9) insure that our problem has only a finite number of solutions.

If in the general solution of (9) the coefficient of u in the expression for x_i had turned out to be larger numerically than that of u in the expression for x_{i+1}, with all of these coefficients of the same sign, our problem would have had infinitely many solutions (on the assumption that it had one). We next state a problem, somewhat like the problem of this section, for which the coefficients of u in the equations analogous to (11) have the property just stated.

4. *A modification of the problem of* §3 *that has infinitely many solutions if it has one.* M herdsmen A_i ($i=1, \cdots, m\geqq 2$) celebrate as in the problem of §3 and each son receives a number of cattle equal to his father's age in years; A_i is older than A_{i+1}; A_1, who owns the largest herd,* distributes half of his herd; A_j ($j=2, \cdots, m$) has e_j more sons than has A_1, $0<e_2<e_3< \cdots <e_m$, and distributes e_jk_j more† cattle than does A_1, where k_j is an integer. How many sons has each herdsman and how many cattle does each son receive?

Using the notation of §3, we find that the analogs of (8), (9), (11) are here the equations (13), (14), (15) below, respectively.

$$x_1 = x_j + e_j(x_j - k_j)/n, \tag{13}$$

$$x_1 = k_j + (n/g_j + E_j)s_j, \qquad x_j = k_j + ns_j/g_j,\ s_j \equiv (x_j - k_j)/(n/g_j), \tag{14}$$

$$x_1 = k_j + (n/g_j + E_j)(s_{j0} + s_{j1}u), \quad x_j = k_j + (n/g_j)(s_{j0} + s_{j1}u), \tag{15}$$

where the s_{j0} are a particular solution of the following analog of (10)

$$(n/g_p + E_p)s_p - (n/g_{p+1} + E_{p+1})s_{p+1} = k_{p+1} - k_p, \ (p = 2, \cdots, m-1), \tag{16}$$

and the s_{j1} are obtained from (12) by replacing $-E_p$ in this equation (d included) by E_p ($p=2, \cdots, j-1, j+1, \cdots, m$).

The problem of this section has no solution if $D(n/g_p+E_p,\ n/g_{p+1}+E_{p+1}) = d_{p,p+1}$ fails to divide $k_{p+1}-k_p$; it has a solution, and indeed an infinite number of solutions [Cf. (15) and the expressions for the s_{j1} defined in the last paragraph above], if $d_{p,p+1}=1$ ($p=2, \cdots, m-1$). As to the general criterion for the existence of a solution of (16), and therefore of (14), we merely refer to the paragraph that includes (10).

Example. By taking $m=3$, $e_2=2$, $e_3=4$, $k_2=21$, $k_3=15$ in the problem of this section, we obtain the problem of Professor Gilman which was referred to in §1. Its general solution can be obtained by treating, as in §2, precisely the cases (i), (ii), (iii) of §2. Doing this, we find that the analogs here of equations (3), (5), (7) are (17), (18), (19), respectively, below:

* The problem obtained by omitting the assumption that A_1 owns the largest herd has of course at least as many solutions as the problem of this section.

† We might say "less" instead of "more". The point is this: in previous notation the number of cattle distributed by A_j is $N/2+e_jk_j$, and we wish to say here that the number distributed by A_j is as just stated rather than $N/2-e_jk_j$ since we write in this section certain equations that we call analogous to certain equations of §3.

(17) $x_1 = 27 + 3n + (n + 2)(n + 4)u, \quad x_2 = 21 + 3n + n(n + 4)u,$
$x_3 = 15 + 3n + n(n + 2)u;$

(18) $x_1 = 27 + 3n + (n/2 + 1)(n/2 + 2)u, \quad x_2 = 21 + 3n + (n/2)(n/2 + 2)u,$
$x_3 = 15 + 3n + (n/2)(n/2 + 1)u;$

(19) $x_1 = 27 + 3n + (n/2 + 1)(n/4 + 1)u, \quad x_2 = 21 + 3n + (n/2)(n/4 + 1)u,$
$x_3 = 15 + 3n + (n/4)(n/2 + 1)u.$

Inspection of the coefficients of u in (17), (18), (19) shows that this problem has infinitely many solutions. However, to conform to the possibilities of experience, the problem should be limited by such restrictions as were mentioned in §2.

5. *Analytical summary of previous results.* We define an inequality to be Diophantine if the symbols used in it are integers.

THEOREM 1. *The mixed system of Diophantine equations (in n and x_1' $x_2, \cdots, x_m$) and inequalities*

$$S: \quad \begin{array}{l} n(x_1 - x_j) + e_j x_j + f_j = 0, \ (j = 2, \cdots, m), \ [\text{Cf. } (8)], \\ x_1 > x_2 > \cdots > x_m, \ 0 < e_2 < e_3 < \cdots < e_m < n, \end{array}$$

where each f_j is a given integral multiple of e_j, has a finite number (possibly zero) of, solutions. The system S' obtained by merely changing the sign of e_jx_j in S, if solvable, has infinitely many solutions. System S is solvable if $D(n/g_p - E_p, n/g_{p+1} - E_{p+1}) = 1$, $(p = 2, \cdots, m-1)$; system S', if $D(n/g_p + E_p, n/g_{p+1} + E_{p+1}) = 1$. The general criterion for the existence of a solution of S (S') can be stated in language of a type that was used just below (10).

6. *A generalization of the results of preceding sections.* We state without proof a general theorem which can be obtained by the methods of §§3, 4.

THEOREM 2. *Consider the mixed system of Diophantine equations and inequalities*

$$T: \quad \begin{array}{l} a_1(n)x_1 - a_j(n)x_j + b_j(n) = 0, \ j = 2, \cdots, m, \\ x_1 \geqq x_2 \geqq \cdots \geqq x_m > 0, \ n > 0, \ a_1(n) \geqq a_2(n) \geqq \cdots \geqq a_m(n), \end{array}$$

where the $a_i(n)$, $i = 1, \cdots, m$, $b_j(n)$ are functions of n that take only positive integral values for positive integral values of n and the inequality sign holds between at least one pair of the a's (for every positive integer n). Suppose that T is solvable when n is assigned a particular positive integral value, n_0, and let T_0 be the system gotten by setting $n = n_0$ in T. System T_0 has only a finite number of solutions. If also there is only a finite number of values of n for which T has a solution (as was the case in §3 where $a_1(n) = n$, $a_j(n) = n - e_j$, $j = 2, \cdots, m$, and $b_j(n) = -e_jk_j$ with e_j equal to a positive integer $< n$), then T, considered as a system with n and the x's as unknowns, has only a finite number of solutions. The equations of T surely have a solution if $D[a_p(n), a_{p+1}(n)] = 1$ for $p = 1, \cdots, m-1$, *though T*

itself may be inconsistent. Let T' be the system that is obtained from T by merely reversing the inequality signs between the a's. Then if n_0 is a value of n for which T' is solvable, T' has infinitely many solutions associated with n_0 (as is the case in certain examples of §4, where $a_1(n)=n$, $a_j(n)=n+e_j$, $j=2, \cdots, m$, and $b_j(n)=e_jk_j$ with e_j equal to a positive integer). If $D[a_p(n_0), a_{p+1}(n_0)]=1$ for every integer p of the set $p=1, \cdots, m$, and if there is in this set an integer q such that $a_q(n_0)<a_{q+1}(n_0)$, then T' has infinitely many solutions. As to the general criterion for the solvability of $T(T')$, we again refer to the paragraph above that includes (10).

7. *Geometrical interpretations.* For a fixed value of n, say $n=n_0$, the equations

$$\begin{gathered} a_1(n)x_1 - a_j(n)x_j + b_j(n) = 0, \; j = 2, \cdots, m, \\ a_1 \geqq a_2 \geqq \cdots \geqq a_m > 0, \; n > 0, \end{gathered} \tag{20}$$

where the a's and b's are functions of n that take only integral values for integral values of n and the inequality sign holds between at least one pair of the a's (for every positive integer n), define a line L in the m-space of the x's. Let p be one of the numbers $1, \cdots, m$ such that $a_p>a_{p+1}$. The projection of L on the plane of the variables x_p, x_{p+1} is a line L' which has the equation

$$a_p(n_0)x_p - a_{p+1}(n_0)x_{p+1} + b_{p+1}(n_0) - b_p(n_0) = 0, \; a_p(n_0) > a_{p+1}(n_0).$$

Since the slope of L' when x_{p+1} is regarded as the independent variable and x_p as the dependent is $a_{p+1}(n_0)/a_p(n_0)$, which is positive and less than unity, L' contains only a finite number (possibly zero) of points with integral coordinates such that $x_p \geqq x_{p+1}$. Consequently there is only a finite number of points of L [solutions of (20)] for which $x_1 \geqq x_2 \geqq \cdots \geqq x_m$.

If we modify the problem just discussed merely by reversing the inequality signs between the a's, with $a_i>0$ $(i=1, \cdots, m)$ the slope of the line L' will be greater than unity. Hence, in this case, there is an infinity of solutions if there is one.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of the World. Translated into English by Andrew Motte in 1729. The translation revised, and supplied with an historical and explanatory appendix, by Florian Cajori. Berkeley, University of California Press, 1934. xxxvi+680 pages. $10.00.

We have here the last work from the pen of Professor Florian Cajori (1859–1930) late professor emeritus of the history of mathematics in the University of

California. He had long felt that if the out-moded technical terms and infelicitous expressions in the complete English translation of the *Principia* most readily available, were appropriately changed, and if explanatory, historical, and bibliographic commentary were added, one of the greatest works ever penned would be in much more serviceable form for students and scholars. It is now evident that Professor Cajori was entirely correct in his belief, and it is to be hoped that his admirable work may soon find a place in many libraries of America and Europe. Had Professor Cajori himself been seeing the work through the press he would doubtless have made many changes in, and additions to, his manuscript, and never have allowed the work to appear without an index, and that a detailed one.

The main basis of the new English translation was that of Andrew Motte published (not "translated," as the title page incorrectly states) in 1729. The title-page also refers to the popular sketch "System of the world" as "translated by Andrew Motte in 1729," but the anonymous English translation, as well as the original Latin edition, were already published in 1728; when the translation was made is unknown. In 1819 Davis attibuted the English translation to Motte, and Cajori adduces another good reason for suspecting that the translator was Motte. Other reasons could be stated to make this much more certain. Whether or not this popular sketch in Latin of the "System of the world" is the one which Newton originally wrote as book III of the *Principia*, but later replaced by an elaborate mathematical treatment, is unknown. Hence the title page here makes another claim which can not be substantiated.

The contents of the work under review are as follows: (a) an English translation of the fine Ode dedicated to Newton by Edmond Halley which appeared in unmutilated form in the first (1687) and third (1726) editions of the *Principia;* (b) translations of Newton's prefaces to the three editions, the last one being published just a year before his death at the age of 85; (c) a translation of Cotes's preface to the second edition; (d) a portrait frontispiece from the original india-ink drawing made about 1691 and to be seen at Magdalene College, Cambridge; (e) facsimile of one of the title-pages of the first edition of the *Principia;** (f) the translation of the *Principia* (p. 1–626); (g) "An historical and explanatory appendix" by F. Cajori (p. 627–680).

About one fifth of this "appendix" deals with the matter in p. i–xxxv. In revising Motte's translation assistance was derived from Thorpe's English translation of book I and Wolfers's German translation. The nature of the other notes may be suggested by reference to a few of them. In connection with various propositions by Newton "on the geometry of conics," are given references to discussions of Newton's treatment by Milne (1927), Kötter (1901), Chasles (1837), Graham (1890), and C. Taylor (1880). An error made by Newton in Book I, lemma xxviii, "on oval figures," is explained, and a reference to a paper of Brougham and Routh (1855) is given. A bibliographic note on Kepler's problem (p. 647–648) articulates Newton's method of solving the transcendental

* (a)–(e) fill p. i–xxxv.

equation $x - e \sin x = z$, where e and z are known. Newton's determination of the attraction of a solid sphere on an external point is more fully explained by the aid of modern analysis by A. N. Kriloff in a paper of 1925 to which reference is given. A reference is also given to Pierpont's paper in the Bulletin of the American Mathematical Society, vol. 35 (1929), where it is shown that in elliptic space two spheres attract each other by a force varying inversely as the square of the distance between their centers. There is a long note (p. 639–640) on "absolute motion and absolute time" ending with the consideration of relativity theories. Book II, prop. VII, scholium, naturally demands comment on "Newton and Leibniz on the invention of the calculus" (p. 655–666). A note on "surface of least resistance" (pp. 657–661, Book II, prop. XXXIV, scholium) shows the connection with calculus of variation discussions. Book III, lemma V, prop. XL suggested a bibliographic note on "Newton's formulas of interpolation" (p. 667).

The press-work on the volume is of high order and the binding is very attractive, both in keeping, not only with the noble work enshrined within its covers, but also with Florian Cajori's geniality and idealism, roses for December in many a memory's garden.

R. C. ARCHIBALD

Elements of Coordinate Geometry. By J. M. Child. London, The Macmillan Company, 1933. xii+468 pages. $3.25.

This book is intended to be a "Coordinate Geometry" that is not merely a "Conics." The author's endeavor is to show that coordinate geometry is a powerful analytical weapon of attack, and that most of the necessary rules are to be found in an exhaustive study of the straight line and the circle. The book is not supposed to be a treatise, but an elementary textbook. The student is assumed to be much more mature than our college freshmen, with a good knowledge of higher algebra, trigonometry, and some acquaintance with the calculus.

A chapter of plotting of statistics leads to a discussion of the linear equation. Mr. Child defines the gradient of a line as the tangent of the acute angle which the line makes with the positive x-axis, and hence has no ambiguity in the sign of the perpendicular from a point to a line. Simultaneous equations are discussed, and much stress laid on the exact meaning of $L+kL'=0$. There is no separate chapter on the circle, but the concept is used almost from the start.

Part II deals with oblique and polar coordinates, and change of axes. Tangents and gradients are considered as limits from an arithmetical standpoint. In the eighty-odd pages on the conics, there is much unusual work for this stage, as envelopes, centers of curvature, evolutes, orthogonal projection, etc., as well as cross-ratios, poles and polars.

Part III treats points and lines at infinity, foci, confocals, and systems of conics.

Part IV considers general homogeneous coordinates and includes a chapter on miscellaneous topics.

There are some five hundred examples. Many of the proofs are long and

arduous, involving much manipulation, deliberately made straightforward rather than elegant.

There are several printers' slips, none of them serious. Part I is not labelled as such. According to the implications of the statement on p. 13, the numbers e and π could not be laid off on the complete scale of real numbers. On p. 43, line 4 from bottom, $+$ should be $\div$. On p. 55, the fractions in line 10 should be inverted. On p. 57 the phrase "the figure most convenient" seems awkward, and the semi-colon two lines above it should be a comma. On p. 70, and later on in the chapters on conics, technical terms, such as focus, etc., are introduced without warning the student. The term parameter is used on p. 74 before its definition on p. 81. The points marked on the graph on p. 83 are not those given in the accompanying table of values. A lemma concerning the sum of the projections of the segments of a broken line is needed in the proof of the theorem on p. 84. The drawing on p. 156 is not properly shaded. Definite cross-references in Chapters XII–XIV would be helpful. The punctuation near the end of Section 126 is faulty.

One who mastered the contents of this book would have a knowledge of analytic geometry exceeding that of many of our graduate students. The text makes stimulating reading for an experienced teacher, and might well be in reference collections. I doubt that it could be used as a text in analytic geometry in this country, unless, perhaps, in an honors course. My chief criticism is directed against the use of the phrase "elementary textbook."

C. A. RUPP

Mathematics Essential for Elementary Statistics. By Helen M. Walker. New York, Henry Holt and Company, 1934. xiv+246 pages. $1.50.

The author states in the preface that this small volume was written primarily for the adult layman who has forgotten all his elementary mathematics and now finds this fact a handicap to the study of elementary statistics. There is no pretense in the text to give a logical development of the subjects of algebra or analytical geometry, but rather to give in simple form those topics needed for a course in elementary statistics.

The text is divided into two parts. Part I consists of a rather elementary list of topics from arithmetic and elementary algebra. Many of these, although elementary, are not readily accessible to the student of elementary statistics; e.g., number of places to be retained in computation with approximate numbers, short cuts in computations, etc. Part II presents a more thorough and difficult discussion of the subject. It is designed to furnish the necessary mathematical background for the study of any statistical text which does not require the calculus as a prerequisite. It is here that the author discusses simply and forcibly the elementary notions of variable, constant, function, parameter, symmetry and homogeneity of algebraic expressions, the normal curve, etc.

The book is well written in simple language which will be easily understood by anyone possessing very little previous knowledge of mathematics. On the

other hand, the reviewer feels that a sufficient amount of material has been included to enable one to carry on successfully the study of elementary statistics. The illustrative examples are well chosen and the problems following each topic are carefully graded.

A. W. RICHESON

Mathematics of Finance. By C. N. Hulvey. New York, The Macmillan Company, 1934. x+306 pp. $3.00.

Mathematics of Finance. Second Edition. By L. L. Smail. New York, McGraw-Hill Book Company, 1934. xiv+273 pp. $2.75.

These two texts lend themselves to joint review, although two different aspects of the subject are considered. Hulvey develops the subject more by computation, while Smail works on a purely mathematical basis. Hulvey states that his text is "intended to be a happy medium between theoretical mathematics and practical training in the solution of finance problems."

Hulvey's book presupposes at least one semester of college algebra before proceeding with the study of mathematics of finance; consequently he does not include a review of the usual subjects from algebra. Otherwise the topics discussed are of the standard type. He suggests that one semester of three or four semester hours be devoted to algebra, while the remainder of the year be used for the study of mathematics of finance.

Hulvey's book treats the usual subjects of simple and compound interest, annuities, amortization, sinking funds, depreciation, bond valuation, building and loan associations, life annuities, and life insurance. The text is supplied with the usual sets of tables to eight decimal places, which are adequate for most practical purposes. One valuable feature in the treatment of the subject by Hulvey, is the introductory paragraphs at the beginning of each topic. This gives the student a valuable insight into the subjects of bonds, depreciation, etc.

Hulvey's book as a whole is well organized and well written. The illustrative examples are carefully chosen and the problem sets are well graded. There is a long list of problems at the end of the book which are exceptionally well chosen and should be sufficient for the problem work of the course. On the other hand, the derivation of some of the formulas seems to be unnecessarily long and drawn out. An example of this is found on page 94 in the derivation of the formula for the present value of an annuity.

The book contains a few errors of minor importance. In several problems the rate of interest has been omitted, while on pages 70 and 71 the decimal point has been placed in the wrong position, thus multiplying the rate by 10.

My chief criticism is the multiplicity of formulas and the large number of illustrative problems. It is my opinion that it is unnecessary to use an illustrative example for every special case, since this practice in many cases will tend to confuse the student rather than clarify the subject.

The present edition of Smail's *Mathematics of Finance* is a revision of the

edition published in 1925 and reviewed in this MONTHLY in May 1926. The revised edition discusses the same topics as the old, but in a slightly different order. Several of the chapters have been combined, while the supplementary topics, of a more or less theoretical nature, have been placed at the end of the text matter. The list of problems has been revised and rewritten and the summary of formulas at the end of each chapter has been omitted entirely. The topics discussed are the usual ones and are almost identical with those discussed in the Hulvey book.

The two books lend themselves to comparison very readily. The subject matter differs in that Smail includes a review of logarithms, the binomial theorem, progressions, and the theory of probability, whereas Hulvey's does not. In cases where a review of some topics from algebra is necessary, the inclusion of these topics may be desirable.

Smail treats the subject more from a purely mathematical point of view than Hulvey. In many cases the derivation of the formulas by Smail is more logically and scientifically done than by Hulvey. Smail has attempted to reduce the number of formulas used and the amount of illustrative material to a minimum. This is in contrast to the Hulvey book, with its large number of formulas and wealth of illustrative material. In this manner Smail's book gives the impression of being the more concisely and scientifically written.

On the other hand the introductory paragraphs before the beginning of each topic in Hulvey's book will give the student a better background from which to proceed to the various topics than does Smail's book. Although the tables in the two books are almost identical, it is my opinion that those in Hulvey's will suit the needs of the students somewhat better than those in Smail's.

On the whole, both texts are well written and practically free of errors. Certainly they should be well received by both students and instructors.

A. W. RICHESON

A First Course in College Mathematics. By W. E. Anderson. New York, Harper and Brothers, 1933. x+326 and 92 pages. $2.75.

One difficulty in selecting a text for a so-called unified course in mathematics is that no two people can agree on the topics to be included. This volume by Anderson is not a mere survey course, but covers in detail all the topics usually treated in freshman trigonometry and analytic geometry, and includes some college algebra as well. It might very well be used by engineering students in preparation for the calculus.

The order of topics, though logical, is unusual. Chapter II, for example, treats of functions, coordinates, graphs, trigonometric functions, identities, radian measure and polar coordinates. There are four chapters on solid analytics but no spherical trigonometry. Included is a good set of five-place tables.

Aside from the order of topics, there is nothing very unusual about the author's treatment of trigonometry and analytic geometry. The explanations

in the text are careful and complete, and the selection of problems seems very good. Answers are given to odd numbered problems. The topics from algebra include a review of elementary algebra, permutations, probability, progressions, the binomial theorem, and theory of equations. There are a few pages of calculus. Logarithms are introduced in the first chapter, which is otherwise a review of algebra.

The reviewer has not found many errors. The author's views of how extraneous roots are introduced when clearing an equation of fractions, as found on pages 16 and 144, are not very orthodox, and the illustrative example on page 144 is definitely wrong.

In conclusion it may be stated that this seems to be a very usable text for a unified one year course designed to replace the ordinary freshman trigonometry and analytics, and cover about the same ground. It also contains a goodly amount of algebra not ordinarily found in textbooks on trigonometry and analytic geometry.

ORRIN FRINK, JR.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES
1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and in the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Iowa State College

The officers for the academic year 1933–1934 were: R. H. Cook, Director; O. M. Swain, Vice Director; Frances Hanson, Secretary; C. C. Hurd, Treasurer; C. P. Wells, Librarian; Dr. D. L. Holl, Faculty Adviser; Dr. E. R. Smith, Head of the Department of Mathematics, Permanent Secretary.

The chief activity of our chapter was the sponsoring of a mathematics club. A large number of Junior College students appeared on the programs.

The winner of the Pi Mu Epsilon prize for the highest scholastic average in the first two years was William J. Stolp.

The meetings and programs were as follows:

November 1, 1933: "Arithmetical puzzles" by George Downing and Robert Peck; "Mt. Wilson Observatory" by Professor Gertrude A. Herr.

November 12, 1933: House party at the home of Dr. and Mrs. E. R. Smith, followed by a play, "Alice in wonderland," as well as games and a "Freshman quiz."

December 7, 1933: "Flatland" by M. J. Kirby; "Euclid and his modern rivals" by Miss Wright; "Lantern exhibit of rare mathematica" by Dr. J. S. Turner.

January 10, 1934: "A geometrical problem" by H. P. Hoberger; "Quadrature of the parabola" by V. C. Mandia; "An arithmetic problem of minimum digits" by Janet Lewis; "Polar planimeter" by R. H. Cook; "Development of hyperbolic functions" by L. F. Robertson.

February 8, 1934: Business meeting and election. Thirteen new members were elected. "Envelope of the Simpson line of a triangle" by Dr. J. V. McKelvey.

February 14, 1934: "Definition and application of probability" by R. Anderson and J. Kerrigan; "Binomial theorem as applied to probability" by Harriett Wilson; "Binomial theorem and genetics" by Dr. A. E. Brandt.

March 14, 1934: "Poles and polars" by M. Kooker; "Epicycloids" by Edgar Timm; "Improper integrals" by Miss Coykendall; "Nine-point circle" by Max Richardson.

April 4, 1934: "Dynamics of a general string pendulum" by J. Gustafson; "Mathematical wrinkles" by Miss Crook; "Geometry in a German Private School" by Julius Allen.

April 26, 1934: Initiation and banquet.

May 15, 1934: "Graphical methods for the cubic and biquadratic" by G. Oswood; "Development of π" by Winnifred McBeath; "Some geometric properties of conics" by R. Dewey; "A problem in differentials" by Ida Younkin.

FRANCES HANSON, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the Case School of Applied Science

The club was organized in 1930, and has for its purpose the furthering of interest in the study of mathematics. It meets about once every three weeks for a talk on some mathematical subject by either a professor or a student. The club has at present twenty-seven members.

The following is a list of talks presented at regular meetings during the past year:

October 12, 1933: "Some applications of complex numbers to geometry" by Dr. M. G. Boyce, Western Reserve University.

November 1, 1933: "The Pell equation" by Mr. C. Goffman.

November 22, 1933: "Hyperbolic functions and their applications to electrical transmission" by Mr. A. W. Brooke.

December 15, 1933: "The theory of probability" by Mr. D. M. Cameron.

January 10, 1934: "A curve for the volume of any cylinder" by Mr. R. J. Stava.

February 21, 1934: "Magic squares" by Mr. R. S. LaGanke.

March 7, 1934: "Some mathematics involved in the computation of orbits" by Mr. R. J. Zavesky.

March 21, 1934: "Integration" by Mr. B. C. Getchell.

April 18, 1934: "Infinite products" by Mr. W. A. Rense.

May 11, 1934: "Higher mathematics" by Professor O. L. Dustheimer, Baldwin-Wallace College.

D. M. CAMERON, *Secretary*

The Mathematics Club of the University of Michigan

The Mathematics Club of the University of Michigan is composed regularly of the members of the teaching staff in the Department of Mathematics. Graduate students in the department are invited to attend and occasionally to present papers on the results of their research. The primary purpose of the club is to promote research in mathematics.

At the October meeting the following officers were elected: Professor J. A. Nyswander, President; Professor C. C. Craig, Secretary-Treasurer. The program committee elected by the club for the year consisted of Professor R. V. Churchill and Professor W. D. Baten.

Regular meetings were held and papers were presented as follows:

October 10, 1933: "Some developments in polygenic function theory"—the retiring president's address—by Professor V. C. Poor.

November 14, 1933: "Graphical methods for graduating weighted data" by Professor T. R. Running.

December 12, 1933: "Product integrals and applications" by Professor G. Y. Rainich.

January 16, 1934: "Some aspects of E. H. Moore's General Analysis" by Professor T. H. Hildebrandt.

February 13, 1934: "Geometrical aspects of linear transformations" by Dr. T. E. Raiford; "The asymptotic expansion of analytic functions" by Mr. Franklin C. Smith.

March 13, 1934: "Planetary motion" by Professor C. J. Coe; "Stieltjes mean integrals" by Mr. H. S. Kaltenborn.

April 17, 1934: "A note on L-spaces" by Dr. Ben Dushnik; "Triangular symmetry" by Professor N. H. Anning.

May 8, 1934: "Additions in arithmetic, 1500–1700, to the sources of Cajori's 'History of mathematical notations' and Tropfke's 'Geschichte der Elementar-Mathematik'" by Sister Leontius; "The integration of interval functions" by Mr. B. C. Getchell; "The frequency function of xy" by Professor C. C. Craig.

July 26, 1934: "Recent trends in the theory of probability" by Professor A. H. Copeland.

The club also had dinner meetings for its members, wives, and guests on November 21, 1933 and April 30, 1934.

C. C. CRAIG, *Secretary*

The Undergraduate Mathematics Club of the University of Michigan

Any student interested in mathematics may become a member by simply coming to a meeting and asking to join.

The meetings and programs were as follows:

October 12, 1933: Election of officers. The officers elected were:
Edward Campbell, President; Irene Hall, Secretary.

October 31, 1933: "The mathematics of patterns" by Professor Gaudsmit.

January 9, 1934: "Construction" by Philip Jones.

May 8, 1934: "Angles and sines" by Professor N. H. Anning.

May 22, 1934: Election of officers for the academic year 1934–1935.

IRENE HALL, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 136. *Proposed by V. Thébault, Le Mans, France.*

With the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, used once each, form two numbers whose product is a maximum. Form two other numbers in the same way whose product is a minimum. None of the numbers is to start with a zero.

E 137. *Proposed by M. J. Turner, Ball State Teachers College.*

Four lines, concurrent at M, are cut by a transversal in the points A, P, Q and B, in that order, with angles AMP and QMB equal. Prove that

$$MA:MB::(MP\cdot AQ):(MQ\cdot BP).$$

E 138. *Proposed by W. B. Campbell, Judson College, Rangoon, Burma.*

An ellipse of fixed area A and variable eccentricity e is rotated about its major axis, generating an ellipsoid of surface area S. Express S as a function of e. Determine the value of e for which S is maximum. Explain what happens when e approaches its limiting values, 0 and 1.

E 139. *Proposed by Raphael Robinson, University of California at Berkeley.*

By folding a rectangular sheet of paper three times, six superposed congruent triangles are obtained. Show that the ratio of the length and width of the rectangle is either $3:1$ or $\sqrt{3}:1$.

E 140. *Proposed by Maud Willey, Gulfport, Mississippi.*

In the following example in long division, six instead of ten was used as a number base. (Thus $2\times3=10$, $2\times4=12$, $4\times5=32$, $5\times5=41$, etc.) Then each digit was replaced by a code letter. Reconstruct the problem and show that the solution is unique.

```
a b ) c d e f ( e d
      c c d
      -----
        e d f
        e a e
        -----
            d
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E 141. *Proposed by W. P. Udinski, University of Texas.*

Show that in every tetrahedron there must be at least one vertex at which each of the face angles is acute.

SOLUTIONS

E 106. *Proposed by W. B. Campbell, Judson College, Rangoon, Burma.*

The minute and hour hands of a clock are indistinguishable in appearance. At what times does this make the time reading indeterminate?

Solution by W. E. Buker, Leetsdale High School, Pa.

Starting with the hands together at noon, let the hour hand traverse $1/x$ of the circle. During the same time the minute hand will have traversed $12/x$ of the circle. Let $p=[12/x]$ be the greatest integer in $12/x$, so that

$$12/x = p + q/x. \tag{1}$$

Now if the time is indeterminate, the positions of the hands will be interchangeable, and we could have the hour hand at a distance from 12 o'clock representing q/x of a revolution. In this case, the minute hand would have traversed $12q/x$ of the circle, and if $r=[12q/x]$, then

$$12q/x = r + 1/x, \tag{2}$$

If we now eliminate q between equations (1) and (2), there results

$$x = 143/(r + 12p) \tag{3}$$

where r and p are integers less than twelve. Hence $(r+12p)$ may take any integral value from 0 through 143.

We note that, if $(r+12p)$ is zero or a multiple of 13, the hands are together, and the time is not indeterminate. After these apparent solutions are rejected, there remain 132 actual solutions. The time is thus indeterminate at 12:05 5/143 o'clock and at intervals of 5 5/143 minutes thereafter, except as noted, when the hands are together.

Solved also by M. L. Constable, Hansraj Gupta, Elmer Latshaw, A. J. Lewis, S. Lines, F. L. Manning, W. R. Ransom, E. P. Starke, W. J. Thome, J. E. Trevor, C. W. Trigg, Simon Vatriquant, Maud Willey and the proposer.

E 107. *Proposed by J. B. Coleman, University of South Carolina.*

A straight line cuts two concentric circles in the points A, B, C and D in that order. AE and BF are parallel chords, one in each circle. CG is perpendicular to BF at G, and DH is perpendicular to AE at H. Prove that $GF=HE$.

Solution by Roy MacKay, Eastern New Mexico Junior College.

Let DH intersect BF at P, and let $m=AB=CD$. Then in the right triangle GPH,

$$GH^2 = GP^2 + PH^2 = m^2 \sin^2 A + m^2 \sin^2 D = m^2,$$

since angles A and D are complementary. Therefore $GH=m$, and the trapezoid $ABGH$ is isosceles. ($ABGH$ is not a parallelogram because $BG<AH$.)

Since AE and BF are parallel chords in two concentric circles, they have a common perpendicular bisector; hence the trapezoid $ABFE$ is also isosceles. Consequently $EFGH$ is a parallelogram and $GF=HE$.

Solved also by W. E. Buker, W. B. Clarke, Hansraj Gupta, L. M. Kelly, Morris Lieblich, Wm. Salkind, E. P. Starke, C. W. Trigg, M. J. Turner and Simon Vatriquant.

E 108. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N.Y.*

Show how to construct a triangle when the orthocenter, the incenter and one vertex are given.

No solution has been received for this problem.

E 109. *Proposed by F. L. Manning, Ursinus College.*

Sum the infinite series,

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \cdots.$$

Solution by J. D. Hill, Brown University.

Consider the slightly more general series, $\sum_{n=0}^{\infty}(n+k)^2/n!$, where k may be any complex number. The ratio test shows that this series converges. To find its sum, we expand

$$S_k = \sum_{n=0}^{\infty}(n+k)^2/n! = \sum_{n=0}^{\infty}n^2/n! + 2k\sum_{n=0}^{\infty}n/n! + k^2\sum_{n=0}^{\infty}1/n!.$$

Now

$$\sum_{n=0}^{\infty}n^2/n! = \sum_{n=1}^{\infty}n/(n-1)! = \sum_{n=0}^{\infty}(n+1)/n! = \sum_{n=0}^{\infty}n/n! + \sum_{n=0}^{\infty}1/n!$$

and

$$\sum_{n=0}^{\infty}n/n! = \sum_{n=1}^{\infty}1/(n-1)! = \sum_{n=0}^{\infty}1/n! = e.$$

Consequently, $S_k=(k^2+2k+2)e$, and if $k=1$, then

$$S_1 = 1^2/0! + 2^2/1! + 3^2/2! + 4^2/3! + \cdots = 5e.$$

Mr. D. W. Hall points out that this is a special case of the more general problem of summing $\sum_{n=0}^{\infty}(n+1)^2x^n/n!$, which was given in the 1933 Joint Associate Examinations for Admission to the Actuarial Society of America and the American Institute of Actuaries, the solution of which was later published by the Society in their *Problems and Solutions*.

Solved also by C. E. Buell, Fred Discepoli, Churchill Eisenhart, Wm. Salkind and E. P. Starke.

E 110. *Proposed by Robert McLaughlin, Ursinus College.*

A circular tower one hundred feet in diameter stands in a level field. A cow is tied by a rope a hundred feet long, the other end of which is fastened to a point at the base of the tower. Over what area may the cow graze?

Solution by M. J. Turner, Ball State Teachers College.

The area desired consists of a semicircle of radius 100 feet, with its bounding diameter tangent to the tower at the fixed end of the rope, and of two congruent and symmetrically placed involutes generated by the free end of the rope as it unwinds from around the base of the tower until it is straight. Since the length of the rope equals the diameter of the tower, the central angle within the tower, from the arc of which the rope unwinds, runs from zero to two radians. If this angle be denoted by θ, the area of this involute is given by

$$\tfrac{1}{2}\int_0^2 (50\theta)^2 d\theta = 3,333.33 + \text{ square feet.}$$

The area of the semicircle is obviously $5{,}000\pi$, or 15,707.96+square feet. Consequently, the total area available for grazing is

$$3,333.33 + 3,333.34 + 15,707.96 = 22,374.63 \text{ square feet.}$$

Solved also by W. E. Buker (two solutions), A. H. Katz, E. T. Krach, W. J. Thome, Simon Vatriquant, Maud Willey and the proposer.

E 111. *Proposed by W. R. Ransom, Tufts College.*

It is sometimes said that the differential equation $(D^2-1)y=0$ has the "general solution" $y=a\ \sinh\ (x+b)$. Show how this so-called general solution is related to the particular solution $y=ce^{-x}$.

Solution by E. P. Starke, Rutgers University.

In the general solution, if we replace a by $-2ce^b$, there results

$$y = a \sinh (x + b) = a(e^{x+b} - e^{-x-b})/2 = - ce^{2b+x} + ce^{-x}$$

and if we now let b increase negatively without limit, the limiting value for y is the particular solution sought, $y=ce^{-x}$.

Solved also by Simon Vatriquant.

E 112. *Proposed by Harry Langman, Cooper Union, New York.*

In a "round-robin" tournament, every entrant plays one round against each of the other entrants. If no one plays more than one round a day, how may the entries be paired so as to complete the contest in the fewest number of rounds?

Solution by E. P. Starke, Rutgers University.

With $2k$ players, k pairs a day may play, while with $2k-1$ players, only $k-1$ pairs a day. But ${}_{2k}C_2=k(2k-1)$ and ${}_{2k-1}C_2=(k-1)\ (2k-1)$. Hence it will take at least $2k-1$ days in either case. It will appear that no more days are needed.

If a solution is known for $2k$ players, a solution for $2k-1$ players may be readily derived from it. Assume that the solution for $2k$ players is written out in a rectangular array, with the k pairs who play on any one day in successive columns of one row, and with each of the $2k-1$ rows representing the playing schedule for one of the $2k-1$ plays. If we now disqualify one player, the other players may follow the same schedule, save only that the opponent of the disqualified player on any day will miss playing on that one day of the tournament, so that each player will miss just one day of play, and each qualified player will play every other qualified player.

If a solution is known for $2k$ players, a solution for $4k$ players may be readily derived from it. Let one way of pairing the $4k$ players be $a_1a_2,\ a_3a_4, \cdots a_{2k-1}a_{2k},\ b_1b_2,\ b_3b_4, \cdots b_{2k-1}b_{2k}$. Put into the first $2k-1$ rows of the first k columns the pairs corresponding to the known solution for the $2k$ players $a_1,\ a_2, \cdots a_{2k}$. Fill in the last k columns of these same $2k-1$ rows with the corresponding known solutions for the $2k$ players $b_1,\ b_2, \cdots b_{2k}$. Into the $2k$th row put $a_1b_1,\ a_2b_2, \cdots a_{2k}b_{2k}$. Derive the remaining $2k-1$ rows from the $2k$th row by a cyclic permutation of the b's, keeping the a's in the same columns. The resulting rectangular array contains $4k-1$ rows with $2k$ pairs in each row, and no pair appears twice in the table. Consequently every player plays just once against each other player.

If a solution is known for k players, the preceding paragraph shows how to derive a solution for $2k$ players, provided k is even. But if k is odd, proceed as

follows. Let one way of pairing the $2k$ players be $a_1a_2, a_3a_4, \cdots a_kb_1, b_2b_3, \cdots b_{k-1}b_k$. Now we found that the solution for an odd number of players forced each of the players to sit out just one day of the tournament. We may then fill in the first k rows of the first column with $a_1b_1, a_2b_2, \cdots a_kb_k$, fill in the next $(k-1)/2$ columns of each row with that row of the solution for k a's which omits the a in the first column of that row, and fill in the last $(k-1)/2$ columns of each row with that row of the solution for k b's which omits the b in the first column of that row. The remaining $k-1$ rows are now filled in by writing the pairs in the first column with the b's permuted cyclicly in the $k-1$ possible ways. Since no pair has been duplicated and since the number of pairs in our table is the maximum possible, no possible pair has been omitted, and our arrangement constitutes a solution for $2k$ players when k is odd.

We may thus start from the trivial case of a single pair of players, who play off all possible games in a single contest on a single day, and by successive doublings and deductions of one, build up in succession solutions for four, three, six, five, eight, seven, etc. cases as follows:

(2)	(4)		(3)		(6)			(5)			(8)				(7)			
12	12	34	12	3	14	23	56	14	23	5	12	34	56	78	12	34	56	7
	13	24	13	2	25	13	46	13	25	4	13	24	57	68	13	24	57	6
	14	23	23	1	36	12	45	12	45	3	14	23	58	67	14	23	67	5
					15	26	34	15	34	2	15	26	37	48	15	26	37	4
					16	24	35	24	35	1	16	27	38	45	16	27	45	3
											17	28	35	46	17	35	46	2
											18	25	36	47	25	36	47	1

Solved also by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo., All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3721. *Proposed by Albert Whiteman, Philadelphia, Pa.*

Prove by Fermat's method of infinite descent that an odd prime p of the form $3n+2$ has the quadratic non-residue -3.

3722. *Proposed by E. P. Starke, Rutgers University.*

Let the points z of the complex plane correspond to those of the sphere by stereographic projection. (See Townsend, *Functions of a Complex Variable*, pp. 184–190.) Find the transformation on z which results from an arbitrary rotation of the sphere.

3723. *Proposed by William Hoover, Columbus, Ohio.*

The envelope of all paraboloids with respect to which a given tetrahedron is self-conjugate is the set of seven planes each of which bisects three edges of the tetrahedron.

3724. *Proposed by R. E. Gaines, University of Richmond.*

At any point A of a conic a normal chord AB is drawn. The chord BC is drawn perpendicular to AB. Tangents at B and C meet the tangent at A in P and P'. Secants PQR and $P'Q'R'$ are drawn, and the lines AQ and AR meet the lines BR' and BQ' in O and O'. Prove that OO' is parallel to PP'.

3725. *Proposed by G. W. Petrie III, Pittsburgh, Pa.*

A storekeeper is interested in purchasing a balance and set of weights which will enable him to measure any weight in ounces up to and including 205 lbs., using weights in both pans when necessary. One company agrees to sell him such an outfit for \$100. Another company proposes to charge \$10 for the balance, scale pans, etc., and \$.90 for each weight plus one cent an ounce for the total amount of metal involved in making the weights. Which company should receive the storekeeper's order?

SOLUTIONS

3647. [1933, 610] *Proposed by H. Grossman, New York.*

Prove that

$$\sum_{n=0}^{\infty} \frac{(-4\pi^2r^2)^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-4\pi^2r^2)^n}{(2n+2)!} = 0,$$

where r is any integer.

Solution by T. L. Smith, Carnegie Institute of Technology.

The first series may be written in the form

$$\frac{1}{2\pi r} \sum_{n=0}^{\infty} \frac{(-1)^n(2\pi r)^{2n+1}}{(2n+1)!} = \frac{1}{2\pi r} \sin 2\pi r, \quad r \neq 0,$$

while the second series similarly becomes

$$\frac{1}{4\pi^2r^2} \sum_{n=0}^{\infty} \frac{(-1)^n(2\pi r)^{2n+2}}{(2n+2)!} = \frac{1-\cos 2\pi r}{4\pi^2r^2}, \quad r \neq 0.$$

Thus each series vanishes for any integer r, except $r=0$; and the first series also vanishes for any half-integer. Neither series is defined for $r=0$.

Solved also by M. F. Becker, J. A. Bullard, M. Charosh, F. G. Dressel, Harry Langman, Roy McKay, W. P. Udinski and F. Underwood.

3650 [1933, 610]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given in a plane any five points which are known to lie upon an ellipse;

construct by ruler and compasses the principal axes of the ellipse. Show how the construction may fail if the points are chosen arbitrarily.

Solution by Otto J. Ramler, The Catholic University of America.

Let the given points taken in a definite order around the ellipse be A, B, C, D, E. Draw a line through E parallel to AB and by a well known method of projective geometry, determine the other point of intersection, F, of the ellipse with this line. Locate another point G similarly so that EG is parallel to AC. The line m joining the midpoints of AB and EF, meets the line n joining the midpoints of AC and EG at O, the center of the ellipse. Another familiar construction of projective geometry, enables us to find the two points P_1Q_1 in which a line through O parallel to AB meets the ellipse. Let m meet the ellipse in P_1' and Q_1'. Then P_1Q_1 and $P_1'Q_1'$ are conjugate diameters. In a similar manner determine at least two additional pairs of conjugate diameters, P_2Q_2, $P_2'Q_2'$ and P_3Q_3, $P_3'Q_3'$. Three pairs of conjugate diameters establish two projectively related pencils on the center O. Consider in addition the diameters OP_1'', OP_2'', OP_3'', $\cdots$ which can be drawn perpendicular to OP_1', OP_2', OP_3', $\cdots$ respectively. Then OP_i and OP_i'' are projectively related. The self corresponding rays of these two projectively related pencils give the directions of the principal axes, and the intersections of these principal rays with the ellipse determine their lengths.

If the restriction be removed that the given points shall lie on an ellipse, that part of the above construction which determines the lengths of the principal axes will fail because the ellipse is the only conic which has real and finite intersections with each of its principal diameters.

Solved also by J. W. Clawson.

Editorial Note. The construction given by Clawson used Pascal's theorem to determine the tangents a, b, c, d, e at the given points A, B, C, D, E of the ellipse. Then the tangents a', b' parallel to a, b were constructed by Brianchon's theorem. The center O of the ellipse is easily determined from the parallelogram a, b, a', b'. Two pairs of conjugate diameters OA, OA'; OB, OB' can be determined in direction; and the directions of the principal axes can be found from the involution of conjugate diameters. For the lengths of the principal axes reference was made to the construction in Hatton's *Principles of Projective Geometry*, p. 244.

The proposer referred to a construction in *Lehrbuch der Darstellenden Geometrie*, Rohn and Papperitz, vol. 1, 4th ed., p. 25. Since this construction is quite interesting, a brief sketch of it will be given with slight variations from the text. An affine transformation of the plane is determined by a fixed straight line a in the plane (the axis of invariant points) and a pair of points A', A of the plane, neither on a, A being, say, the transform of A'. The transform of any point B' not on $A'A$ is constructed by finding K, the intersection of $A'B'$ with a. The parallel through B' to $A'A$ cuts KA in B, the transform of B'. The transform of any point on $A'A$ can then be determined in the same way using B' and B in

determined by the transformation. To construct the principal axes draw a circle with center on a passing through O and O'. If this circle cuts a in X and Y, then OX and OY are the axes of the ellipse being the transforms of $O'X$ and $O'Y$. The transforms of the four points where the latter cut the circle give the four vertices of the ellipse; and this completes the construction.

The affine transformation may be used with advantage in combination with the methods of the two solutions. Suppose that as in Clawson's solution the tangents at A and B have been determined and also the diameter BOM, where O is the center. Let the tangents at B and M cut the tangent a at A in B_1 and M_1, and let BM cut a in V. Construct the point B' so that $B_1B'=B_1A$ and is perpendicular to $B'V$. Let the perpendicular to a at A cut $B'V$ in O', and let M' be the foot of the perpendicular from M_1 on $B'V$. Then $O'A=O'B'=O'M'$ follows from the fact that B_1, A, M_1, V form an harmonic set. With a as the axis the circle with center O' and radius $O'A$ transforms into the ellipse with B, O, M as the transforms of B', O', M'. The principal axes are then easily constructed in length and position.

Instead of having the projecting lines $A'A$ parallel, i.e., passing through the fixed point F at infinity, we may take F as a finite point in the plane. In the general case the transformation is sometimes called an *homology*. See Cremona's *Elements of Projective Geometry*, 3rd ed., pp. 14–19. These transformations are important in descriptive geometry, since, if the plane of the original figure is turned about the axis a to a new position, the transformed figure is the spacial projection of the original figure in its new position by a cylindrical or conical projection.

3652 [1933, 610]. *Proposed by A. F. Stevenson, University of Toronto.*

The practice of certain cigarette manufacturers of supplying playing cards, poker hands, etc. with their cigarette packets, and of offering various articles in exchange for complete set of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards, and that these cards are distributed among the packets at random (the number of packets available being infinite), what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

I. *Solution by C. Eisenhart, Princeton, N.J.*

If the probability of an event occurring in a single trial is p, then *on the average* it *will* occur in $1/p$ trials (that is, at least once on the average).

In the problem given, the probability of getting some card in the first pack is 52/52, in other words one is certain to get some card. For the second pack of cigarettes, the probability of getting a card not gotten in the first pack is 51/52. In similar manner the probability of getting a card which is not one of the two already obtained is 50/52 for a single trial, and so forth.

Applying the theorem stated above the number of trials necessary *on the average* to get a second (different) card will be the reciprocal of 51/52, namely, 52/51; likewise after two different cards have been obtained *on the average*

a third card, which is different from these, will be obtained in 52/50 trials, etc.

Therefore, on the average the entire set of 52 cards will be obtained in $52[1/52+1/51+\cdots+1/3+1/2+1]$ trials. Hence the average number of trials needed will be

$$52\sum_{n=1}^{52} 1/n.$$

This summation may be carried out directly, and we know that it will not have an integral value (see this MONTHLY, January, 1934, page 48); or its value may be determined from

$$\sum_{1}^{n}\frac{1}{n} = .57721566 + \log_e n + \frac{1}{2n} - \frac{1}{12n(n+1)} - \frac{1}{12n(n+1)(n+2)} - \frac{19}{20n(n+1)(n+2)(n+3)} \cdots$$

which gives the average number of trials in this case to be 235.976, and since we must buy an integral number of cigarette packets the number will be 236.

II. *Partial Solution by the Proposer*

We shall obtain a formula for the average minimum number for the general case in which there are r cards in a complete set, though we shall not carry out the somewhat laborious numerical computation for $r=52$.

Consider a random selection of $n(\geqq r)$ packets, and let us first calculate p_n, the probability of obtaining *at least one* complete set of cards in this selection. Let the different cards of a set be designated by the numbers $1, 2, \cdots, r$; then the probability of finding n_1 one's, n_2 two's, $\cdots$, n_r r's in this selection is

$$\frac{1}{r^n}\,\frac{n!}{n_1!n_2!\cdots n_r!} = \frac{1}{r^n}A_{n_1\cdots n_n}, \text{ say,}$$

where $A_{n_1\cdots n_r}$ is the coefficient of $x_1^{n_1}\cdots x_r^{n_r}$ in $X=(x_1+\cdots+x_r)^n$. Hence $p_n=r^{-n}\sum A_{n_1\cdots n_r}$ the summation being taken over all $n_1, \cdots, n_r$ subject to $n_1+\cdots+n_r=n$, $n_1>0, \cdots, n_r>0$. Now the sum of all the coefficients in X is r^n, and the sum of all those not containing x_1 is $(r-1)^n$; hence the sum of the coefficients of the terms containing x_1 is $r^n-(r-1)^n$. Similarly, the sum of the coefficients of the terms containing both x_1 and x_2 is

$$r^n - (r-1)^n - [(r-1)^n - (r-2)^n] = r^n - 2(r-1)^n + (r-2)^n.$$

Proceeding successively in this way, or by induction, we find*

$$\sum A_{n_1\cdots n_r} = r^n - r(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^{r-1}$$

* It is quite possible that this result may be well-known from the theory of probability or otherwise; I do not recall having seen it before.

whence

$$(1)\qquad p_n = 1 - r\left(1 - \frac{1}{r}\right)^n + \binom{r}{2}\left(1 - \frac{2}{r}\right)^n - \cdots + (-1)^{r-1}r\left(\frac{1}{r}\right)^n.$$

The probability of *just* completing the set of r cards by a purchase of n packets is $p_n - p_{n-1}$, so that the required average minimum number is

$$(2)\qquad \begin{aligned} N_r &= \sum_{n=r}^{\infty} n(p_n - p_{n-1}) = \sum_{n=r}^{\infty}[(n-1)(1-p_{n-1}) - n(1-p_n) + (1-p_n)], \\ &= r - 1 + \sum_{n=r}^{\infty}(1 - p_{n-1}) = r + \sum_{n=r}^{\infty}(1-p_n), \end{aligned}$$

since $p_{r-1}=0$ and $\lim_{n\to\infty} n(1-p_n)=0$. Substituting (1) in (2) and performing the summation for each term, we easily find

$$N_r = r\left[1 + r\left(1 - \frac{1}{r}\right)^r - \frac{1}{2}\binom{r}{2}\left(1 - \frac{2}{r}\right)^r + \cdots + \frac{(-1)^{r-2}}{r-1}r\left(\frac{1}{r}\right)^r\right].$$

It does not appear possible to simplify this result further, and it is not in a very practical form for numerical computation for large values of r, since a large number of terms in the series would have to be taken and the large binomial coefficients render the accurate calculation of these terms difficult. For $r=52$, the calculation, though quite feasible, would involve a fair amount of labor, and it has not been thought worth while to embark on it; possibly some adept at computation might care to do so. For $r=1, 2, 3, 4, 5$, the approximate values of N_r are respectively 1, 3, 5.5, 8.3, 11.4.

III. *Solution by E. G. Olds, Carnegie Institute of Technology*

Let us assume the general case of $k+1$ cards. In any series of purchases, let X_i represent any card distinct from those previously obtained ($i=1, 2, 3, \cdots, k+1$), and let Y_{ij} represent the jth duplicate of any one of the distinct cards, $X_1, X_2, \cdots, X_i$ ($j=1, 2, 3, \cdots \infty$). Then any series of purchases, which results in success in obtaining a complete set of $k+1$ cards, provides the following cards:

$$(1)\qquad X_1, Y_{11}, Y_{12}, \cdots, Y_{1\alpha_1}, X_2, Y_{21}, Y_{22}, \cdots, Y_{2\alpha_2}, \cdots \quad X_k, Y_{k1}, Y_{k2}, \cdots, Y_{k\alpha_k}, X_{k+1}$$

where, for example, Y_{2j} is a duplicate of either X_1 or X_2 and any series of Y's may be absent. The number of cards in (1) is, apparently, $k+1+n$ where $\sum_{i=1}^{k}\alpha_i = n$, the α's being non-negative integers.

Now, the probabilities for the individual cards are given by

$$(2)\qquad p\{X_i\} = \frac{k+2-i}{k+1}$$

and

$$(3) \qquad p\{Y_{ij}\} = \frac{i}{k+1}.$$

Also, the probability of success on the $(k+1)$th purchase (where success has not occurred previously), is given by

$$(4)\ p_r = \frac{k+2-1}{k+1}\cdot\left(\frac{1}{k+1}\right)^{\alpha_1}\cdot\frac{k+2-2}{k+1}\left(\frac{2}{k+1}\right)^{\alpha_2}\cdots\frac{k+2-k}{k+1}\left(\frac{k}{k+1}\right)^{\alpha_k}\frac{1}{k+1}$$

where $\sum_{i=1}^{k}\alpha_i = r$. This may be expressed more compactly as

$$(5) \qquad p_r = \frac{(k+1)!}{(k+1)^{k+1}}\,\frac{1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots k^{\alpha_k}}{(k+1)^r}.$$

In order to determine the average minimum number of packets to be purchased to obtain a complete set, it is necessary to evaluate $E(k+1+r)$, where

$$(6) \qquad E(k+1+r) = \sum_{r=0}^{\infty}(k+1+r)p_r.$$

Let us choose, as a generating function,

$$(7) \qquad f(z) = \sum_{r=0}^{\infty} z^r p_r.$$

Using (5), we find

$$(8) \qquad f(z) = \frac{(k+1)!}{(k+1)^{k+1}}\sum_{r=0}^{\infty}\left(\frac{z}{k+1}\right)^{\alpha_1}\left(\frac{2z}{k+1}\right)^{\alpha_2}\cdots\left(\frac{kz}{k+1}\right)^{\alpha_k}.$$

Equation (8) may be replaced by

$$(9) \qquad f(z) = \frac{(k+1)!}{(k+1)^{k+1}}\sum_{\alpha_1=0}^{\infty}\left(\frac{z}{k+1}\right)^{\alpha_1}\sum_{\alpha_2=0}^{\infty}\left(\frac{2z}{k+1}\right)^{\alpha_2}\cdots\sum_{\alpha_k=0}^{\infty}\left(\frac{kz}{k+1}\right)^{\alpha_k}.$$

We then readily get

$$(10) \qquad f(z) = \frac{(k+1)!}{(k+1)^{k+1}}\cdot\frac{k+1}{k+1-z}\cdot\frac{k+1}{k+1-2z}\cdots\frac{k+1}{k+1-kz}.$$

Re-writing more compactly, after simplification

$$(11) \qquad f(z) = \frac{k!}{(k+1-z)(k+1-2z)\cdots(k+1-kz)}.$$

The solution of the problem, along with a check on (5), may now be obtained by using (7) and (11). We have at once

$$f(1) = \sum_{r=0}^{\infty} p_r = \frac{k!}{k!} = 1 \tag{12}$$

and

$$f'(1) = \sum_{r=0}^{\infty} r p_r = \frac{1}{k} + \frac{2}{k-1} + \cdots + \frac{k-1}{2} + \frac{k}{1}. \tag{13}$$

From (6), is obtained

$$E(k+1+r) = (k+1)\sum_{r=0}^{\infty} p_r + \sum_{r=0}^{\infty} r p_r = k+1+\left(\frac{1}{k} + \frac{2}{k-1} + \cdots + \frac{k}{1}\right). \tag{14}$$

The expression on the right of (14) is unwieldy for large values of k but can be roughly approximated by using the well-known inequality

$$(k+1)\log_e k - k < \frac{1}{k} + \frac{2}{k-1} + \cdots + \frac{k}{1} < (k+1)\log_e k + 1, \text{for } k > 1. \tag{15}$$

The second member of this inequality is approximately equal to half the sum of the first and third members. The error in this approximation is less than half the difference of the third and first members of the inequality. Making use of this fact and of equation (14), we may write,

$$E(k+1+r) = k+1+(k+1)\log_e k + \frac{1-k}{2} \pm s\left(\frac{1+k}{2}\right), \tag{16}$$

$$\text{where } 0 \leqq s < 1.$$

For the proposed problem, $k+1=52$, and the exact value, as given by (14), is 236. The value given by (16) is $231.5 \pm 26s$ $(0 \leqq s < 1)$.

Solved also by L. M. Bauer, J. D. Leith, F. L. Manning, and B. D. Roberts.

Editorial Note. The two methods of solution of this problem leading to the same result give it considerable interest in the theory of probability. Bauer's solution was practically the same as I; and he gave the value

$$52\sum_{n=1}^{52} 1/n = 235.9782854364,$$

where the sum was obtained from Glover's Tables I. The solution by Manning used the argument in Whitworth's *Choice and Chance*, 4th ed. p. 200, which shows that the average number of packages that should be bought to get any one of i remaining cards, needed to complete a set, is $52/i$. So that this solution is also similar to I.

Roberts's solution states that the probability of obtaining a complete set of cards in a purchase of 52 packets is $52!/52^{52}$, and that the smallest integer n such that ${}_nC_{52}$ is greater than or equal to the reciprocal of the above probability

gives the solution, $n=80$. Leith used the same inequality but with ${}_nC_{52}$ replaced by $2{}_nC_{52}$. His result was also $n=80$.

The solution by the proposer, although incomplete, is interesting, since it puts the solution in a form which may be more conveniently handled by the methods of finite differences. Since his result for p_n in (1) is obtained by r successive differences, we have at once

$$r^n p_n = \Delta^r 0^n,$$

which is the number of ways of obtaining the r cards in n purchases. Then the number of ways of just completing a set of r cards by n purchases is $r\Delta^{r-1}0^{n-1}$. Hence

$$N_r = \sum_{n=r}^{\infty} \frac{nr}{r^n}\Delta^{r-1}0^{n-1} = \Delta^{r-1}\sum_{n=r}^{\infty} n\left(\frac{x}{r}\right)^{n-1}\Big]_{x=0},$$
$$= \frac{1}{r^{r-2}}\Delta^{r-1}\frac{x^{r-1}[r^2-(r-1)x]}{(x-r)^2}\Big]_{x=0}.$$

The fraction in x gives by division an integral part of degree $r-2$, which yields zero after differencing $r-1$ times. The fractional part is easily found by two synthetic divisions by $x-r$; and it turns out to be $r^r/(x-r)^2$. Hence we have

$$N_r = r^2\Delta^{r-1}(x-r)^{-2}\Big]_{x=0}.$$

This may be evaluated as follows:

$$\Delta^{r-1}(x-r)^{-2} = \Delta^{r-1}(x-r)^{(-1)}(x-r)^{(-1)}.$$
$$= \sum_{i=0}^{r-1} {}_{r-1}C_i\Delta^i(x-r+r-1-i)^{(-1)}\Delta^{r-1-i}(x-r)^{(-1)},$$
$$= (-1)^{r-1}\sum_{i=0}^{r-1}\frac{(r-1)!}{(x-r)(x-r+1)\cdots(x-1)(x-i-1)}.$$

Therefore

$$N_r = r\sum_{i=0}^{r-1}\frac{1}{i+1},$$

which is the result in I.

The probability of just completing the set of r cards by n packages as given above, when equated to the expression given by the proposer, gives the identity

$$\Delta^r 0^n = r[\Delta^r 0^{n-1} + \Delta^{r-1}0^{n-1}].$$

This formula may be used to compute the values of $\Delta^n 0^{n+1}$ and $\Delta^n 0^{n+2}$ given in the note on problem 3625 [1934, 455].

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Science News Letter reports the discovery of a new mathematical theory of relativity. This theory was presented to the United Provinces Academy of Sciences by Sir Shah Sulaiman, chief of justice of the High Court of Allahabad. Its equations reduce to those of Newton as a first approximation and those of Einstein as a second approximation. Its assumptions coincide very nearly with those of Newton, rather than those of Einstein.

Professor J. W. Harrelson, Chairman of the Department of Mathematics at the State College branch of the University of North Carolina, has been elected Dean of Administration, in place of Dr. E. C. Brooks, resigned. Professor H. A. Fisher succeeds Professor Harrelson as Head of the Mathematics Department.

Dr. J. M. Clarkson, of Cornell University, has been appointed to an assistant professorship at the North Carolina State College, Raleigh.

Dr. J. G. Estes of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the North Carolina State College, Raleigh.

Dr. T. H. Hildebrandt has been appointed chairman of the department of mathematics at the University of Michigan. He succeeds Professor J. W. Glover, who resigned to devote more time to research problems and investigations dealing with federal government plans for economic security for the individual. Professor Glover still remains on the faculty as "James Olney Distinguished Professor."

Dr. Marston Morse, professor of mathematics at Harvard University, has accepted a call to a professorship of mathematics at the Institute for Advanced Study at Princeton, New Jersey. The staff of the Schocl of Mathematics now consists of the following members: Drs. Albert Einstein, Oswald Veblen, J. W. Alexander, John von Neumann, Herman Weyl and Marston Morse.

H. V. Park of the Chapel Hill branch of the University of North Carolina has been appointed to an instructorship at that University.

Dr. Hans Rademacher, formerly professor of mathematics at the University of Breslau, has joined the staff of the department of mathematics at the University of Pennsylvania for one year under a joint grant from the Emergency Committee in Aid of Displaced German Scholars and the Rockefeller Foundation.

Professor M. D. Earle of Furman University died September 13, 1934. He was a charter member of the Mathematical Association.

Professor J. A. McLaughlin of St. Bonaventure College died November 17, 1934.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN.
SOUTHEASTERN, Decatur, Ga., March.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE NINETEENTH ANNUAL MEETING OF THE ASSOCIATION

The nineteenth annual meeting of the Mathematical Association of America was held at Pittsburgh, Pennsylvania, from Friday to Tuesday, December 28, 1934, to January 1, 1935, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Three hundred fifty-two were in attendance at the meetings, including the following two hundred thirty-one members of the Association:

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RICHARD BRAUER, Institute for Advanced Study
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LILLIAN O. BROWN, Hood College
H. E. BUCHANAN, Tulane University
W. E. BUKER, High School, Leetsdale, Pa.
C. T. BUMER, Kenyon College
R. S. BURINGTON, Case School of Applied Science
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The opening session and reception of the A.A.A.S. occurred on Thursday evening. The address of the retiring president, Professor H. N. Russell, on "The atmospheres of the planets," was given Monday evening, general lectures oc-

curring each afternoon and evening of the week. The Council of the A.A.A.S. met each morning, the Association being represented by Professor C. S. Atchison and Secretary Cairns. The Council elected Professor T. H. Hildebrandt vice-president and chairman of Section A for the year 1935 and Professor E. B. Stouffer member of the Section committee.

The greater part of the mathematicians stayed at the Hotel Webster Hall, which was within easy reach of the new Mellon Institute, Carnegie Hall, and the lecture halls of the Carnegie Institute of Technology where the mathematics sessions were held. Registration headquarters for mathematicians were very conveniently located in the hotel, and the hotel lobby and parlors served excellently as social rooms. A large number of mathematicians were present at a very pleasant function Thursday afternoon, a tea given by the ladies of the departments of mathematics of the Carnegie Institute of Technology and the University of Pittsburgh. Another very enjoyable entertainment afforded those who remained over Sunday was an evening of social recreation in recognition of New Year's Eve, high in the Cathedral of Learning of the University of Pittsburgh. The services of the large and effective committee on local arrangements, under the chairmanship of Professor J. S. Taylor, were recognized at the joint session on Monday morning in a resolution offered by Professor R. E. Langer on behalf of the visitors, and seconded by Professor J. O. Hassler on behalf of the National Council of Teachers of Mathematics. This sincere expression of appreciation and thanks to the authorities of the inviting institutions and to the local committee, whose work was so largely responsible for the success of the meetings, was adopted unanimously by rising vote.

Two hundred eighty-two mathematicians and guests attended the annual dinner Saturday evening in the Georgian Room of the Hotel Webster Hall. We were highly honored through the unexpected presence of Doctor Albert Einstein of the Institute for Advanced Study. The toastmaster, Professor Dunham Jackson, referred appropriately to the presence of Doctor Einstein and introduced the three speakers. Professor Lefschetz, the president-elect of the Society, spoke of the value of our national meetings which serve to bring us together in order to confer with one another as to what we are doing and what our young men are doing. Professor Birkhoff voiced our gratification and pleasure arising from the excellent arrangements for the meetings and spoke of the fine opportunity for visiting in the Orient in his recent trip, mentioning several mathematicians and philosophers whom he met then. He gave a general invitation for all to come to the tercentenary celebration of Harvard University in September 1936. Professor Dines expressed the pleasure of the teachers in Pittsburgh in having the mathematicians present in these meetings and in having the National Council there for the first time, and emphasized the great contribution which the presence of the men in mathematical research makes toward an extension of research in the Pittsburgh region.

The American Mathematical Society held sessions Thursday morning and afternoon, also Friday and Saturday mornings, for the reading of short papers.

The annual business meeting was held Friday afternoon at which time announcement was made of the election of Professor Solomon Lefschetz as president for 1935–1936. This was followed by the retiring presidential address by Professor A. B. Coble on "The geometry of the Weddle manifold W_p," and by an address by Professor Marston Morse on "Uniform instability and dynamical discontinua." An outstanding event of the Society program was the Josiah Willard Gibbs lecture by Doctor Albert Einstein on "An elementary proof of the theorem concerning the equivalence of mass and energy." By the express desire of the lecturer the address was given in a hall accommodating only 450 persons, and as a consequence the distribution of tickets was made with great discrimination, being confined for the most part to the mathematicians and in part to members of the American Physical Society. This address will appear in the Bulletin of the American Mathematical Society.

Several conferences were held, as at the Cambridge meetings, by the officers and committee members of the National Council and the Mathematical Association. These conferences considered further developments for cooperative action with respect to studying and strengthening the status of mathematics in secondary education. It is being realized more and more that such cooperation is of the highest value. Many persons expressed their judgment that the meetings of the two organizations in conjunction resulted in the mutual strengthening of the programs.

The mathematicians participated in a meeting of the Econometric Society held Friday evening with the general topic, "The nature and limitations of statistical proof." Four papers were presented: "What is a proof?" by Professor E. B. Wilson; "What do time-series correlation coefficients show?" by Professor C. F. Roos; "Statistical proofs of periodicity in economic series" by Professor H. T. Davis; and "Practical difficulties in proving statistical relationships" by Dr. Max Sasuly.

The National Council of Teachers of Mathematics met by special arrangement at the Pittsburgh meetings, with a view to ascertaining by actual experiment the relative advantages of affiliating with the A.A.A.S. A very satisfactory number were in attendance from the United States at large and from the Pittsburgh region in particular. Aside from the joint session mentioned below, the National Council held a session Friday evening at which Professor W. D. Reeve of Teachers College presented a proposal for mathematics in the United States, which would depend upon suitable financial support. He added that teachers should understand the strategic position of mathematics, as shown by the Tree of Knowledge at the Century of Progress, and should be able intelligently to present this to others. He spoke further of the immediate need of educating the teachers of mathematics and the public as well in a true reform of mathematics which should affect it from the elementary grades up to the field of adult education. He spoke further of the preparation of a suitable syllabus for all these school years with an accompanying manual for teachers, and described not merely the topics which should be omitted, but the character of the subject

matter and the methods of presentation which should be included. Following this, Dr. Elizabeth B. Cowley directed a symposium on methods of making mathematics interesting, in which eight teachers of the Pittsburgh schools gave specific presentation of expedients which they themselves employed. At the session on Saturday morning, under the general topic "Mathematical concepts of value to high school teachers," Professor H. W. Brinkmann spoke on "Concepts in geometry" and Professor C. C. MacDuffee on "Different kinds of equality." At the conclusion of this program, Professor W. D. Cairns gave a short address introducing and explaining a display of English textbooks and examination papers which he had collected while visiting English "public schools" on his recent trip abroad.

The Mathematical Association held a joint session with the National Council of Teachers of Mathematics on Saturday afternoon, a joint session with Section A and the American Mathematical Society on Monday morning, and two sessions on Monday afternoon and Tuesday morning. President Dresden presided at the first joint session, President Hassler of the National Council introducing the speakers. Professor Rietz presided at the joint session on Monday morning, Professor Hurwitz at the session Monday afternoon and President-elect Curtiss at the final session. The Association is indebted to the program committee, under the chairmanship of Professor Dines, for the organization of a valuable program. This follows, together with abstracts of some of the papers, numbered in accordance with their place on the program.

Joint Session of the Association with the National Council of Teachers of Mathematics

"The need for a reorientation of mathematics in the secondary schools."

1. From the viewpoint of modern educational theory, by Professor P. W. Hutson, University of Pittsburgh.

2. From the viewpoint of the university teacher of mathematics, by Professor W. L. Hart, University of Minnesota.

3. From the viewpoint of the high school teacher, by Doctor M. L. Hartung, University High School, Madison, Wisconsin.

These papers are to be published in the *Mathematics Teacher*. The discussion which followed made it evident that Professor Hutson represented much the traditional point of view of the educationists, quite different from that of the teachers of mathematics, and brought out the desirability of more personal contacts and deliberations between these two groups.

1. Two major shifts in educational theory during the past thirty-five years have posed the need for adjustment in the organization and teaching of mathematics. One of these changes is the tendency to center education about the pupil instead of the subject, thus giving rise to the concepts of the psychological rather than the logical order, present rather than deferred values, education as life

rather than preparation for life. The other change is the emphasis on direct and specific values as the outcomes of teaching in contrast with the older theory of general or disciplinary values. Thorndike's investigation of the growth in general mental power resulting from the various school subjects carries great weight in establishing the view that no subject can claim preeminence in this regard.

Mathematics has made some progress in adjustment, but primarily on the junior high school level. Authorities in the field, writing a quarter century ago, said that but little of algebra and geometry had practical value. Progressive secondary schools which have been freed from the necessity of meeting college entrance requirements so that they can experiment with integrated curriculums, are doing very little to draw mathematics other than arithmetic into their integrated curriculums. This is significant, because the effort at integration is an attempt to help pupils have experiences which are more real and meaningful to them than those to be had from the school subjects, and evidently no need is felt for algebra or geometry in such experiences.

2. Professor Hart first gave a sketch of the position of mathematics in the college curriculum, to provide a setting for his later remarks. He asserted that today, in the intellectual word, mathematics is in an unusually strong position not only because of its cultural values and its well known utility in the so-called mathematical sciences, but also because of growing tendencies toward mathematical methodology in all natural sciences, most of the social sciences and in other fields. He then discussed certain aspects of the present unfavorable situation of mathematics in the secondary field. He advanced the opinion that psychological research is weakening the position of those who formerly have attacked secondary mathematics on the basis of debatable theories about the ease of transfer of training. Also, he emphasized, as a major problem, the presence of large numbers of students from the lower end of the intelligence scale in the classes in secondary mathematics.

In light of this background, Professor Hart made two major suggestions. First, he expressed the opinion that conditions justify an offensive, rather than a defensive, attitude on the part of mathematics in the secondary field. In this connection he urged that experimentation in secondary mathematics should not be carried on hysterically but should be based on the assumption that the content and methodology of the field is thoroughly sound, when well-prepared instructors are teaching reasonably intelligent students under decent class room conditions. This viewpoint entails emphasis on increased mathematical training for teachers of secondary mathematics, and militant opposition to the practice of employing miscellaneous teachers from outside fields to perform much of the mathematical teaching. As a second recommendation, Professor Hart suggested that a placement system should be employed to separate high school students into two (or more) categories which would be given differentiated mathematical treatment. For *all* students of the best category, he recommended at least two and one-half units of mathematics, with the viewpoint that secondary mathe-

matics by itself is culturally and practically useful regardless of later college contacts. He suggested that students of the lower category should study a concrete variety of mathematics in the ninth grade, and perhaps also in the tenth grade, where any algebraic methods and logical procedures involved are pitched on a low plane and where recollections of the classical courses in secondary mathematics are not permitted to interfere with the development of that course which is most appropriate for the students concerned.

3. Attacks on mathematics have increased during the last thirty years in spite of curricular reforms initiated by E. H. Moore, the National Committee, and others. Better achievement through improved classroom methods is an urgent need at present. Systematic diagnostic testing and reteaching, attention to individual differences, and maintenance programs are means to this end. Types of reorientation also needed are the popularization of mathematics, closer articulation of secondary schools and colleges, and a broader concept of testing programs. Evidence is accumulating that prominent theories of learning and transfer of training may be invalid, and that much of the curriculum reorganization based on them is premature. Closer cooperation of educationists and mathematicians is needed to work out a program satisfactory to both.

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

1. "Mathematics and science" by Professor C. N. Moore, University of Cincinnati, retiring vice-president of Section A.

2. "A program for mathematics" by Professor Arnold Dresden, Swarthmore College, retiring president of the Association.

1. The address by Professor Moore appeared in *Science* for January 11, 1935.

2. The address by Professor Dresden will appear in full in an early number of the Monthly.

First Session of the Association

A symposium (of non-advanced character) on "Equipotential loci of Green's function."

1. "Some geometric properties of lemniscates and of equipotential curves of Green's function" by Professor J. L. Walsh, Harvard University.

2. "The location of the roots of the derivative of a polynomial" by Professor Morris Marden, University of Wisconsin at Milwaukee.

3. "Extensions to three dimensions" by Professor J. J. Gergen, University of Rochester.

The address by Professor Walsh was published in the Monthly for January, 1935; and the papers by Professor Marden and Professor Gergen will appear in later issues of the Monthly.

SECOND SESSION OF THE ASSOCIATION

1. "The cubic equation and the geometry of the triangle" by Professor A. A. BENNETT, Brown University.

2. "Linear systems of algebraic surfaces" by Professor T. R. HOLLCROFT, Wells College.

3. "Productive scholarship in the undergraduate college" by Professor R. L. JEFFERY, Acadia University.

1. It is here shown that the study of the literal algebraic cubic polynomial in one nonhomogeneous variable leads naturally to a type of analytic geometry particularly fitted for the analytic study of the geometry of the triangle. The reduced cubic, "Cardan's formulas," the "canonical linear function," etc., are interpreted directly in the geometry. While the methods are essentially analogous to those used by H. A. Dobell (this MONTHLY, vol. 39 (1932), p. 71) and by Morley and Morley: *Inversive Geometry*, the study is extended to imaginary points of the geometry, particularly to a figure of six circles and twenty-two points including the circumcircle and three Apollonian circles. Unlike the treatment in the references cited, the field is here abstract, and the cubic remains unrestricted. The "conjugates" here used while enjoying properties analogous to those of the complex conjugate of algebra are "subic conjugates" and involve essentially reciprocals. The talk was illustrated by a sequence of charts.

2. A linear system of algebraic surfaces is defined by the equation

$$\sum \lambda_i f_i = 0, \ i = 1, 2, \cdots, \alpha,$$

wherein the $f_i = 0$ are algebraic surfaces of order n with given basis elements and

$$2 \leqq \alpha \leqq \ (n+1)(n+2)(n+3) - q,$$

q being the number of conditions determining the basis elements. For $i = 2, 3, 4$, the respective names pencil, net, web of surfaces are used.

The title of this address might well apply to the discussion of linear systems of algebraic surfaces with any number of parameters. However, except for certain special cases, very few properties of such general systems are known. This address was limited, therefore, to the discussion in some detail in three-space, of those linear systems of algebraic surfaces all of whose properties are now known namely, the pencil, net and web. Also, since the formulas defining the properties of these systems of surfaces with assigned basis points or curves are very long and involved, for the sake of brevity, only the formulas applying to systems of surfaces without assigned basis elements are given here.

Since the details given in this talk have been published previously, only a synopsis with references to these publications will be given here.

The characteristics of a pencil of algebraic surfaces in three-space are readily obtained by setting $r = 3$ in the formulas of a paper dealing with pencils of hypersurfaces.*

* T. R. Hollcroft, Pencils of hypersurfaces, American Journal of Mathematics, Vol. 53 (1931), pp. 929–936.

The properties of a net of surfaces are obtained by establishing a (1, 1) correspondence between the surfaces of the net and the lines of a plane. The net contains a double infinity of pencils of surfaces. It also contains six non-linear systems of surfaces, two singly infinite and four finite, whose properties are defined by the characteristics of the branch-point curve of the transformation.

The properties of a net of hypersurfaces of order n in i dimensions have been obtained in a former paper* by the above method. For $i=3$, these formulas reduce to those belonging to a net of surfaces in three-space.

Special webs of surfaces are associated with certain geometric transformations, e.g., homoloidal webs with Cremona transformations. In general, a (l, k) involution in space is associated with a web of surfaces with n^3-k basis points or their equivalent. For $k \leqq n(n+1)(5n-11)/6+3$, these basis points can not all be independent, so that webs associated with such transformations are special webs. In space involutions, except for $n=2$, treated up to this time, k is less than this limit.

Another type of a special web is the web of first polars of an algebraic surface.

In the case of general webs of algebraic surfaces, webs of quadrics have been extensively studied,† but nothing had been done prior to 1932 for surfaces of order $n>2$, except to find the order of the jacobian.

The properties of the general web of surfaces are obtained by establishing a (1, 1) correspondence between the suraces of the web and the planes of three-space. The web contains the following linear systems of surfaces: a quadruple infinity of pencils of surfaces, a triple infinity of nets of surfaces, a triple infinity of nets of space curves which are complete intersections of pairs of surfaces of the web. These nets of surfaces and nets of space curves have the same set of jacobian curves which form a web of curves on the jacobian surface of the web of surfaces.

The web of surfaces contains the following non-linear systems, of surfaces; two doubly infinite systems, four singly infinite systems and six finite systems. The characteristics of the web and of its non-linear systems are associated with the characteristics of the branch-point surface belonging to the involution of order n^3 defined by the above (1, 1) correspondence.

A detailed treatment and the characteristics of a general web of algebraic surfaces without basis elements are to be found in a paper recently published in the Transactions.‡

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* T. R. Hollcroft, Nets of manifolds in i dimensions, Annali di Matematica, Ser. IV, Vol. 5 (1927–28), pp. 261–267.

† Pascal, Repertorium der höheren Mathematik, Vol. II_2 (1922) pp. 629–631.

Encyklopädie der Mathematischen Wissenschaften, Vol. III_2, pp. 250–254.

Virgil Snyder and F. R. Sharpe, Space involutions defined by a web of quadrics, Transactions American Math. Soc., Vol. 19 (1918), pp. 275–290.

‡ T. R. Hollcroft, The general web of algebraic surfaces of order n and the involution defined by it, Transactions of the American Math. Soc., Vol. 35 (1933), pp. 855–868.

Since the publication of this paper, these results have been extended to webs with basis points and, in a paper presented to the Mathematical Society, December 27, 1934, to webs with basis curves.

3. Professor Jeffery's paper will appear in an early number of the MONTHLY.

MEETINGS OF THE BOARD OF TRUSTEES

Ten members of the out-going and of the incoming Board were present at the Pittsburgh meetings.

The following thirty-eight persons and one institution were elected to membership on applications duly certified:

To Individual Membership

A. A. ALBERT, Ph.D. (Chicago) Asst. Prof., Univ. of Chicago, Chicago, Ill.

Sister ANN ELIZABETH, Ph.D. (Wisconsin) Head of Dept., St. Mary Coll., Leavenworth, Kans.

KATHRYN ASCHENBRENNER, A.B. (Iowa) Instr., Wright Branch, Chicago City Colleges, Chicago, Ill.

S. LOUISE BEASLEY, B.S. (Springfield, Mo., State Teachers Coll.) Teacher, High School, Sullivan, Mo.

VIRGINIA I. BENNETT, A.B. (New River State Coll.) Teacher, Midway Jr. High School, Belle, W. Va.

E. O. BOX, M.S. (Southern Methodist) Asst. Prof., State Teachers Coll., Commerce, Texas

M. C. BROWN, A.M. (Kentucky) Asst. Prof., Univ. of Kentucky, Lexington, Ky.

R. C. BULLOCK, Ph.D. (Chicago) Head of Dept., Arkansas Poly. Coll., Russellville, Ark.

T. C. CARSON, A.M. (Duke) Prof., State Teachers Coll., Johnson City, Tenn.

J. A. DAUM, B.S. (Creighton) Grad. Asst. in Chem., Creighton Univ., Omaha, Nebr.

F. L. DENNIS, A.M. (Cornell) Instr., Ursinus Coll., Collegeville, Pa.

R. L. ECHOLS, Ph.D. (Virginia) Student, Inst. for Advanced Study, Princeton, N. J.

H. F. FEHR, A.M. (Lehigh) Instr., State Teachers Coll., Montclair, N. J.

H. H. FERNS, Ph.D. (Toronto) Asst. Prof., Univ. of Saskatchewan, Saskatoon, Sask.

W. W. FLEXNER, Ph.D. (Princeton) Asst. Prof., Cornell Univ., Ithaca, N.Y .

J. J. GERGEN, Ph.D. (Rice) Asst. Prof., Univ. of Rochester, Rochester, N. Y.

L. O. HANSEN, A.M. (Colorado) Instr., Union Coll., Lincoln, Nebr.

E. E. HEIMANN, A.M. (Texas) Asso. Prof., State Teachers Coll., Ada, Okla.

M. R. HESTENES, Ph.D. (Chicago) Instr., Harvard Univ., Cambridge, Mass.

R. O. HUTCHINSON, Ph.D. (Chicago) Prof., Tennessee Poly. Inst., Cookeville, Tenn.

L. W. JOHNSON, A.M. (Oklahoma) Asso. Prof., State Teachers Coll., Edmond, Okla.

MIKIWO KOBAYASHI, A.B. (Tohoku Imp. Univ.) Prof., Furistu-Koto-gakko, Tokyo, Japan

H. R. LEIFER, A.B. (Pittsburgh) Grad. student, Univ. of Pittsburgh, Pittsburgh, Pa.

Sister THOMAS MARIE MALONEY, A.B. (Emmanuel Coll) Instr., Trinity Coll., Washington, D.C.

Sister MARY ANGELA MARGRAF, A.M. (St. Louis Univ.) Prof., Ursuline Coll., Cleveland, Ohio

C. J. MCGEE, M.S. (Catholic Univ.) Instr., Univ. of Dayton, Dayton, Ohio

Sister MARIE GERTRUDE MCNEIL, M.S. (Notre Dame) Prof., Seton Hill Coll., Greensburg, Pa.

J. S. MORREL, Ph.D. (Illinois) Asst. Prof., Vanderbilt Univ., Nashville, Tenn.

A. J. O'LEARY, A.B. (St. Anselm's Coll.) Grad. student, Catholic Univ., Washington, D.C.

G. W. PETRIE, III., B.S. (Carnegie Inst. of Tech.) Grad. Asst., Univ. of Pittsburgh, Pittsburgh, Pa.

D. H. Porter, A.M. (Indiana) Asst. Prof. and Registrar, Marion Coll., Marion, Ind.
Elizabeth Renwick, A.M. (Indiana) Instr., Grove City Coll., Grove City, Pa.
C. A. Richmond, B.S. (Pomona) Tyngsboro, Mass.
Norma K. Stelford, A.M. (Northwestern) Asst. Prof., State Teachers Coll., DeKalb, Ill.
W. B. Storm, A.M. (Chicago) Asso. Prof., State Teachers Coll., DeKalb, Ill.
G. C. Watson, A.M. (Virginia) Service Fellow, Univ. of Virginia, University, Va.
K. W. Wegner, Ph.D. (Wisconsin) Head of Dept., Math. and Physics, Whitworth Coll., Spokane, Wash.
J. W. Wrench, Jr., A.M. (Buffalo) Grad. Asst., Univ. of Buffalo, Buffalo, N.Y.

To Institutional Membership

Nazareth College, Rochester, N.Y.

The financial report of the Secretary-Treasurer for the year 1934 was presented, approved by Professor Slaught for the Finance Committee; and this was accepted, subject to inspection by a sub-committee. Professors Brink and Rietz later examined the report and the evidences of assets and found them to be satisfactory. The Finance Committee was authorized to transfer approximately $2000 to the General Endowment Fund.

It was voted to hold the meeting of the Association in December 1935 at St. Louis in affiliation with the American Association, to reappoint Professors Atchison and Cairns as representatives of the Association on the Council of the American Association for the year 1935, and to appoint Professor J. O. Hassler to the Board of Trustees to fill the vacancy caused by the election of Professor Curtiss as president, the term to expire in January 1937.

The following were appointed associate editors of the Monthly for the year 1935, as nominated by Professor Carver:

W. F. Cheney	R. E. Gilman	H. L. Olson
N. A. Court	R. A. Johnson	R. E. Sanger
Otto Dunkel	B. W. Jones	D. E. Smith
B. F. Finkel	J. R. Musselman	J. H. Weaver
T. C. Fry		F. W. Weida

It was voted to nominate Professor W. R. Longley as representative of the Association on the National Research Council for a three-year term from July 1, 1935, in succession to Professor H. L. Rietz.

Professor Moulton presented to the Board a report on (a) the unemployment of doctors of philosophy in mathematics, and (b) the training of teachers. The report was accepted; part (b) will be published later in this Monthly and part (a) appears elsewhere in this issue.

Professor Everett for the committee on tests and surveys of college mathematics presented a report with reference to the cooperation of the Association with the Committee on Educational Testing of the American Council on Education in the construction of tests in first and second year college mathematics.

The report was accepted and the committee discharged with an expression of the thanks of the Board, plans for the cooperation being given to the new administration for further development.

Annual Business Meeting

The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He reported also the deaths of the following members:

C. L. Arnold, Professor of mathematics emeritus, Ohio State University. (November 8, 1934)
O. J. Bond, Professor of mathematics, The Citadel, Charleston, S.C. (October 1, 1933)
M. D. Earle, Professor of mathematics, Furman University. (September 13, 1934)
W. H. Echols, Professor of mathematics, University of Virginia. (September 25, 1934)
F. W. Hanawalt, Professor of mathematics and astronomy, College of Puget Sound. (November, 1933)
R. B. Harkness, Jr., Chemical engineer, Everett, Mass. (December 17, 1933)
A. S. Hathaway, Professor of mathematics, retired, Boerne, Texas. (March 11, 1934)
E. N. Johnson, Professor of mathematics, Butler University. (April 24, 1934)
C. D. Killebrew, Professor of mathematics, Alabama Polytechnic Institute. (March 9, 1934)
J. K. Long, Instructor in mathematics, Purdue University. (December 30, 1933)
E. A. Lyman, Professor of mathematics, Michigan State Normal College (October 9, 1934)
Sister M. Cecelia Mangold, Professor of mathematics, Trinity College, Washington, D.C. (February 9, 1934)
J. A. McLaughlin, Dean and Professor of mathematics, St. Bonaventure's College. (November 17, 1934)
Thomas Muir, Late Superintendent-General of Education, Cape Colony, South Africa. (March 21, 1934)
D. A. Murray, Professor of applied mathemetics, McGill University. (October 19, 1934)
Arthur Ranum, Professor of mathematics, Cornell University. (February 28, 1934)
Paul Saurel, Professor of mathematics, retired, College of the City of New York. (January 21, 1934)
Martha L. Smith, Dean of women, and Professor of mathematics, Virginia Union University, Richmond, Va. (January 25, 1934)
V. B. Teach, Associate Professor of mathematics, Armour Institute of Technology. (September 8, 1934)
R. A. Wells, Professor of mathematics, Park College. (October 8, 1934)

The result of the election of officers for 1935 was as follows:

President for 1935–36: D. R. Curtiss, Northwestern University

Vice-Presidents: L. L. Dines, Carnegie Institute of Technology; A. J. Kempner, University of Colorado.

Additional members of the Board of Trustees, to serve until January 1938: H. E. Buchanan, Tulane University; Arnold Dresden, Swarthmore College; E. R. Hedrick, University of California at Los Angeles; F. D. Murnaghan, Johns Hopkins University.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 12, 1934

RECEIPTS

Balance Dec. 12, 1933		$7,443.36
1932 indiv. dues	139.20	
1933 indiv. dues	633.49	
1933 instit. dues	19.30	
1933 subscriptions	12.65	
1934 indiv. dues	6,225.36	
1934 instit. dues	741.80	
1934 subscriptions	862.95	
Initiation fees	242.00	
Advertising	343.00	
Rec'd from reprints	158.47	
Sale copies of MONTHLY	268.61	
Sale copies of *Register*	3.50	
Sale copies of Catalog	1.50	
Sale First Carus Mon.	16.25	
Sale Second Carus Mon.	15.00	
Sale Third Carus Mon.	20.00	
Sale Fourth Carus Mon.	15.00	
Sale Fifth Carus Mon.	613.06	
Sale Rhind Papyrus	331.98	
Sale Archibald's Outline of Hist. of Math.	74.25	
Royalty on Assn. pin	6.25	
Received *Annals* subscriptions	5.00	
Int. Oberlin Savgs. Bk.	59.17	
Int. Peoples Bkg. Co.	48.89	
Int. Cleveland Trust Co.	72.79	
Int. Hardy Fund	120.00	
Int. certif. of deposit	1.84	
Int. Genl. End. Fund	677.65	
Int. Carus Fund	131.25	
Int. Chace Fund	238.75	
Int. Chauvenet Fund	35.00	
Payment from restricted Carus Fd.	99.40	
Payment from restricted Chace Fd.	4.40	
Transferred from Carus certif. of deposit	101.40	12,339.16
Total 1934 receipts to date		19,782.52
Total expenditures		11,901.30
Balance to end of 1934 business		7,881.22
Received on 1935 business		607.68
Book balance Dec. 12, 1934		$8,488.90

EXPENDITURES

Publisher's bills (Nov.'33–Oct. '34)		$ 4,249.21
Reprints		161.69
President's office		17.00
Manager's office		28.90
Editor-in-chief's office		627.68
Committee on membership		57.70
Expense Com. on Training and Utilization		84.94
Expense Com. on Place of Math.		49.56
Expense *Register*		386.20
Secretary-Treasurer's office:		
Postage	376.50	
Bond	11.26	
Safety deposit	4.40	
Office supplies	87.95	
Express, tel., etc.	64.50	
Clerical work	1,994.18	
Printing	272.25	
Library expense	40.80	
Paid copies MONTHLY	22.45	
Cambridge meeting	119.56	
Williamstown meeting	143.02	
Typewriter	55.05	
Bank tax	3.80	3,195.72
Annals subvention		375.00
Conference with Natl. Council		11.75
Expense sections from init. fees		337.27
Paid B. F. Finkel int. Hardy Fd.		120.00
Ins. back copies of MONTHLY		17.70
Transferred to Chace Fd.		380.96
Forwarded *Annals* subscriptions		5.00
Paid *Annals* subscriptions		10.00
Sust. memb. in Amer. Math. Soc.		100.00
Expense acct. Carus Mon. Fd.		72.37
Honorarium Fifth Carus Mon.		300.00
Printing Fifth Carus Mon.		1,192.15
Refund subscription		4.50
Printing Archibald's Outline		116.00
Total expenditures		11,901.30
Checking account		572.11
Oberlin Savgs. Bk. acct. unrestricted		1,965.70
Oberlin Savgs. Bk. acct. restricted		1,332.80
Peoples Banking Co. acct.		1,684.93
Cleveland Trust Co. Savgs. acct.		2,870.79
Certif. of deposit, unrestricted		62.57
Bank balance Dec. 12, 1934		$8,488.90

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance Dec. 12, 1933		$ 6,459.36
Receipts: Sales	$ 679.31	
Interest	210.63	889.94
		$7,349.30
Expenditures: On account of sections	72.37	
Printing Fifth Carus Monograph	1,192.15	1,264.52
		$6,084.78
Certificate of deposit	$2,628.45	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Pacific Power & Light Co. 5% gold bond, market value	770.00	
$3\frac{1}{8}$% U.S. Treasury Bond of 1946–49	1,000.00	
Cash in bank, restricted, certificate of participation	795.20	6,193.65
Less amount due to "current funds"		108.87
Balance Dec. 12, 1934		$6,084.78

Arnold Buffum Chace Fund

Balance Dec. 12, 1933		$5,410.33
Receipts: Sale Papyrus	$ 331.98	
Interest	244.72	576.70
		$5,987.03
Iowa Elec. Light & Power Co. 5% Bond	$1,000.00	
Western United Gas and Elec. Co. Bonds	2,370.00	
$3\frac{1}{8}$% U.S. Treasury Bond of 1946–49	1,000.00	
Certificates of deposit	123.40	
Certificate of deposit, Northern Trust Co., Chicago	883.30	
Cash in bank, restricted, certificate of participation	35.20	
Cash in bank, unrestricted	575.13	
Balance December 12, 1934		$5,987.03

Chauvenet Prize Fund

Balance Dec. 12, 1933		$ 554.38
Interest		35.00
		$ 589.38
Iowa Elec. Light & Power Co. 5% Bond	$ 500.00	
Cash in bank, unrestricted	89.38	
Balance Dec. 12, 1934		$ 589.38

Life Membership Fund

Liability on life memberships Dec. 12, 1933	$ 703.69
To be transferred to current funds, surplus	41.98
Liability on life memberships as of Jan. 1, 1935	$ 661.71

GENERAL ENDOWMENT FUND

Balance Dec. 12, 1933		$12,585.00
$3\frac{1}{4}\%$ U.S. Treasury Bonds of 1944–46	$1,000.00	
$4\frac{1}{4}$–$3\frac{1}{4}\%$ U.S. Treasury Bond of 1943–45	1,000.00	
Land Trust Certificate	700.00	
Cleveland Trust Investment Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Northwestern Electric Co. Bonds	3,000.00	
Texas Power and Light Co. 5% Bonds, market value	885.00	
Iowa Elec. Light & Power Co., 5% Bonds	3,000.00	
Balance Dec. 12, 1934		$12,585.00

Of the funds on hand, indicated in the first division of this financial report, $575.13 belongs to the Arnold Buffum Chace Fund, $89.38 belongs to the Chauvenet Prize Fund, and $661.71 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1935, while $108.87 is due the general treasury from the Carus Monograph Fund. Aside from these amounts, the various funds of the Association are carried in the form shown in the inventories under the exhibit above.

When the accounts were closed Dec. 12, 1934, there remained on the total business for 1934 the following items:

BILLS RECEIVABLE

1934 individual dues	$200.00
Advertising	50.00
Due from Carus Mon. Fund	108.87
	$358.87

BILLS PAYABLE

Publisher's bills (Nov.–Dec. '34)	$ 950.00
President's office	30.00
Manager's office	30.00
Editor-in-chief's office	70.00
Secretary-Treasurer's office	265.00
Chace Fund	575.13
Chauvenet Prize Fund	89.38
Life Membership Fund	661.71
Init. fees due to sections	800.00
	$3,471.22

If to the balance on 1934 business shown in the report, $7,881.22, there be added the bills receivable, $358.87, and there be subtracted the estimated bills payable, $3,471.22, there results an estimated final balance on 1934 business of approximately $4,770.00, which represents the accumulated surplus in current funds. Since the annual meeting the Finance Committee has bought two $500 U.S. Treasury bonds and, under the advice of investment authorities in Chicago and Cleveland, is changing certain of its utility investments into government bonds.

W. D. CAIRNS, *Secretary-Treasurer*

THE EIGHTEENTH ANNUAL MEETING OF THE KENTUCKY SECTION

The eighteenth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, May 12, 1934. Professor L. W. Cohen, chairman of the Section, presided at both the morning and afternoon sessions. The attendance was sixty-five including the following twenty-four members of the Association: N. B. Allison, J. G. Black, P. P. Boyd, L. W. Cohen, J. M. Davis, H. H. Downing, Arnold Dresden, L. A. Fair, A. R. Fehn, Georgia M. Haswell, Charles Hatfield, W. R. Hutcherson, C. G. Latimer, Elizabeth LeStourgeon, W. L. Miser, W. L. Moore, Sister Charles Mary Morrison, Mabel I. Nowlan, R. S. Park, Sallie E. Pence, D. W. Pugsley, Guy Stevenson, Sister M. Domitilla Thuener, and H. A. Wright.

Chairman L. W. Cohen, who represented the Section at a meeting of a committee of the Kentucky Academy of Science to discuss the question of the affiliation of the two organizations, gave a report. After a full discussion the Kentucky Section decided to affiliate with the Kentucky Academy of Science with certain reservations.

It was voted to authorize the Secretary to express the Section's gratitude and appreciation to the national organization for sending President Arnold Dresden to us. Professor Miser of Vanderbilt University made a few remarks appealing to this Section for help to organize the members in Tennessee. The question of a fall meeting was discussed, when our Tennessee neighbors should be invited to attend and to take part in the program. This was left to the discretion of the chairman of the Section. Officers were elected for the year 1934–1935 as follows: Chairman, W. R. Hutcherson, Berea College; Secretary, A. R. Fehn, Centre College.

The following papers were presented:

1. "On certain formulas in trigonometry and analytic geometry" by Professor C. G. Latimer, University of Kentucky.

2. "Mathematics restored" by Professor H. A. Wright, Transylvania College.

3. "Poncelet's quadratic transformation" by Professor P. P. Boyd, University of Kentucky.

4. "The center of ocular rotation in the horizontal plane" by Professor R. S. Park, Eastern State Teachers College.

5. "Mathematics in German schools and universities" by Professor Richard Brauer, University of Kentucky.

6. "The logical foundations of mathematics" by Professor Arnold Dresden, Swarthmore College.

Abstracts of these papers follow, numbered in accordance with their place on the program:

1. Professor Latimer showed that the addition formulas of trigonometry and the rotation formulas of analytic geometry are essentially the same. These proofs seem to have certain pedagogical advantages.

2. Professor Wright indicated how teachers of mathematics in colleges and universities might help to restore mathematics to the place it deserves in high schools, colleges and universities. Every professor of mathematics should be an earnest and enthusiastic advocate of mathematics as a means of developing ways of thinking which are valuable in character building; the cultural and transfer values should be stressed as well as the so-called practical and factual aspects. In addition teachers of mathematics as individuals and as groups should give serious consideration to curriculum building and the training of teachers.

3. Dean Boyd presented two "first editions" from the University of Kentucky library, one, Newton's *Optics*, containing in his "Enumeration of cubic curves" his interesting "organic" generation of a quadratic transformation by means of two rotating angles; and the other, Poncelet's classic work of 1820 on the "*Properties of projective figures*." Poncelet's derivation of a quadratic involution was then described. He begins with a system of coaxal circles and by projective properties arrives at the theorems, first, that the polars of a point with respect to a pencil of conics are concurrent, and, second, that in the point relationship thus established the image of a line is a conic.

4. This paper dealt with the determination of the space and body centrodes of the human eye produced in its rotation in the horizontal plane. In it Professor Park discussed the instrument used in making the measurements upon the eye and the mathematical handling of the data which led to the conclusion that the center of ocular rotation was not fixed, as had long been supposed, but moved along a curve a variable distance, from 1.066 mm. to 1.653 mm. to 0.893 mm., nasalward from the visual axis and always on a line perpendicular to that axis. Also a variable distance, from 14.732 mm. to 13.918 mm. to 12.950 mm. from the corneal vertex as the angle of the visual axis took the positions from 39° to 4° nasalward and from 4° nasalward to 38° templeward, respectively.

5. A survey of the German educational system in mathematics was given by Professor Brauer. In particular, he described the training of high school teachers; this varies considerably from the American system. In Germany the high school teacher has had to study mathematics for four years at a university, and then spend two years in practical training at some high school. Results of this system were related.

6. Professor Dresden discussed some of the fundamental problems of the foundations of mathematics and presented points of view of various mathematicians who have dealt with the subject.

A. R. FEHN, *Secretary*

THE UNEMPLOYMENT SITUATION FOR PH.D.'S IN MATHEMATICS

By E. J. MOULTON, Northwestern University

The problem of finding suitable employment for advanced students of mathematics during the depression years has caused much concern among mathematicians and the Mathematical Association of America appointed a Commission last year to study the training and utilization of such students. One of the activities of the Commission has been to determine as accurately as possible the present situation with regard to the employment of Doctors of Philosophy in mathematics. This note will give briefly some of the information obtained.

Last winter a questionnaire was sent to 50 leading universities in America asking for information concerning persons who already held the doctorate or probably would secure it during 1934 and who were seeking positions for 1934–35. Nearly all of the universities replied, and 120 persons were named who were seeking positions. There were 60 other men and women who received the doctorate in mathematics during 1934. Many of these 180 people held positions, some of which might be considered permanent but others were certainly temporary.

On October first a second questionnaire was sent out to determine the status at that date of the 180 persons under consideration. Replies were received concerning 149 of them.

The answers to this second questionnaire show that on October first, 1934, there were 14 of these 149 persons holding the doctorate in mathematics who were unemployed. Others, however, had accepted positions which obviously were makeshifts. For example, 5 had work which was in no way related to their special training, and 18 held assistantships which gave them employment at low salaries. The other 112 had obtained positions of a nature more or less satisfactory to them. Of these, 12 were in government service or in business where their mathematical training was a direct asset; 12 had been granted fellowships which allowed them to continue their studies; 88 had secured teaching positions (21 in universities, 53 in colleges, 2 in normal schools, 2 in junior colleges, and 10 in high schools or academies).

The situation might be roughly summarized by stating that there were about 40 or 50 Doctors of Philosophy in mathematics who had not, on October first, found employment reasonably satisfactory to them.

It should be added that the unemployment situation would appear worse than this from the point of view of the graduate schools. In many cases young men who were recently granted the doctorate by a university were retained as instructors, where in former years the instructorships would have been awarded to graduate students working for the doctorate. The effect of this has been to replace graduate assistants by Ph.D. instructors and to reduce the number of openings for graduate students before they have obtained the degree. The burden of unemployment is thus partially passed on to the graduate student

(opinions may differ as to whether this result is for the good of mathematical education).

As to the outlook for the future, we may call attention to the following facts.

In the first place, the output of Ph.D.'s in mathematics in America is shown by the following figures: For successive five-year periods beginning in 1910–15 the average number of doctorates in mathematics conferred annually has been 24, 25, 26, 45, 75. The boom began in 1925, and there has been a considerable increase since that time, but the rate of increase has fallen decidedly in the last few years. It is interesting to note in passing that there have been as many doctorates conferred since 1925 as there were altogether prior to that time.

In the second place, we note the probable future demand for persons holding the doctorate. A fairly recent survey of 1098 colleges of the United States and Canada, including junior colleges and degree-granting normal schools, showed that they employed 3488 mathematics teachers, of whom 937 held the doctorate and 2551 did not. Under normal conditions we would expect over 100 replacements annually of teachers of mathematics in these institutions due to retirements, resignations, and deaths. Accordingly, we may expect a demand from them for approximately 100 highly trained teachers of mathematics annually, and presumably persons holding the doctorate will be given preference where other qualifications are approximately equal. We should recognize, however, that some of these institutions may regard other preparation and qualifications more important than research training, and choose, for instance, a man holding a master's degree who has had special teacher training and experience in preference to a young Ph.D. without such training or experience. Moreover, it is probable that in the case of junior colleges not infrequently teachers of mathematics from neighboring high schools will be chosen to fill vacancies.

The demand will therefore presumably exceed the present rate of supply of Ph.D.'s in the near future, if the number of mathematics teachers does not decrease. This appears more certain in view of the fact that a large number of the persons working for the doctorate are among the 2551 already employed, so that new positions are not required for them.

The increase in registration in our colleges this year and the conditions stated above lead us to be reasonably optimistic concerning the employment situation for the near future. The salaries which the colleges will be able to pay are, however, very uncertain, and it may be that many of the Ph.D.'s will find positions in secondary schools more attractive than college positions. On this account, it would be wise for candidates for the doctorate to note particular requirements for teachers in the secondary field.

The Mathematical Association of America is attempting to assist advanced students to obtain positions, this work being carried on at present under the direction of the writer of this report.

A PROPERTY OF CYCLIC SUBSTITUTIONS OF EVEN DEGREE

By ABRAHAM SINKOV, Washington, D.C.

Given a cyclic substitution s on n letters $a_1, a_2, \cdots, a_n$, the interval between any two of the elements a_i and a_j will be defined as the power (mod n) of s in which a_j immediately follows a_i. Thus, in the normal cyclic substitution $a_1a_2 \cdots a_n$, the interval between a_i and a_j is $j-i$ while the interval between a_j and a_i is $n-(j-i)$ or $i-j$.

The following problem is now proposed: If any two random cyclic substitutions of these n letters are assigned, is it possible to find a pair of letters which shall be separated by the same interval in both substitutions? Or, to put it differently, is it possible to find two cyclic substitutions of n letters such that no two letters are separated by the same interval in both substitutions?

In studying the answer to this problem, one may without any loss of generality suppose one of the two substitutions to be fixed, e.g.

$$a_1a_2a_3 \cdots a_n. \tag{1}$$

Let the other be represented by

$$a_ja_ka_l \cdots a_pa_q. \tag{2}$$

If now the interval α between two letters of (2) should be the same (mod n) as the difference between their subscripts, then the letters will be separated by the same interval in both (1) and (2). Suppose that the successive differences of the subscripts in (2) are denoted by x_i thus:

$$\begin{aligned} k - j &= x_1 \\ l - k &= x_2 \\ &\cdots\cdots \\ q - p &= x_{n-1}. \end{aligned}$$

Then a necessary and sufficient condition that the interval between each two letters in (2) be different from the interval between the same two letters in (1) is

$$\sum_{j=i}^{i+\alpha-1} x_j \not\equiv \alpha \pmod{n},$$

where $\alpha = 1, 2, \cdots, n-1$ and $i+\alpha-1 < n$.

It is possible to introduce a second condition, which expresses the fact that no two letters in (2) are identical, i.e., that no two subscripts yield a difference zero (mod n). This condition is

$$\sum_{j=i}^{i+\alpha-1} x_j \not\equiv 0 \pmod{n},$$

for all positive integers i, α such that $i+\alpha-1 < n$.

The original problem may now be stated in the following form: Is it possible to write down an ordered sequence of $n-1$ positive integers less than n (not necessarily distinct), without satisfying one of the conditions

$$\sum_{j=i}^{i+\alpha-1} x_j \equiv 0, \alpha \pmod{n}?$$

It will be supposed in what follows that $n-2$ such numbers have been obtained, and that it remains only to add a final number to the set. Let the numbers already obtained be denoted by

$$x_1, x_2, x_3, \cdots, x_{n-2}.$$

If, in the equations of condition

$$\sum_{j=i}^{i+\alpha-1} x_j \equiv 0 \pmod{n},$$

i takes on the values $1, 2, \cdots, n-1$ in turn, it results that x_{n-1} may not take any of the following values

$$\begin{aligned} -(x_1 + x_2 + \cdots + x_{n-2}) &\equiv \beta_1, \\ -(x_2 + x_3 + \cdots + x_{n-2}) &\equiv \beta_2, \\ \cdots\cdots\cdots\cdots\cdots & \\ -x_{n-2} &\equiv \beta_{n-2}, \\ 0 &\equiv \beta_{n-1}. \end{aligned}$$

A similar process applied to the conditions

$$\sum_{j=i}^{i+\alpha-1} x_j \equiv \alpha \pmod{n}$$

requires x_{n-1} to be different from all the numbers

$$\beta_1 - 1, \beta_2 - 2, \cdots, \beta_{n-1} - (n-1).$$

If all these numbers are written in the following tabular array:

$$\begin{array}{lllll} \beta_1, & \beta_2, & \beta_3, & \cdots, & \beta_{n-1} \\ \beta_1 - 1, & \beta_2 - 2, & \beta_3 - 3, & \cdots, & \beta_{n-1} - (n-1) \end{array}$$

it results that x_{n-1} must not be equal (mod n) to any one of the above numbers. Should it be possible to show that the above array contains (mod n) every number from 0 to $n-1$ inclusive, it will follow that two sequences are impossible without at least one repetition of the type considered.

All $n-1$ numbers on the upper row are different. For, if

$$\beta_\mu = \beta_\nu \qquad (\nu > \mu)$$

then

$$\beta_\nu - \beta_\mu = \sum_{j=\mu}^{\nu-1} x_j \equiv 0 \pmod{n}.$$

Similarly all the numbers on the lower row are different. For, if

$$\beta_\mu - \mu = \beta_\nu - \nu$$

$$\beta_\nu - \beta_\mu = \sum_{j=\mu}^{\nu-1} x_j \equiv \nu - \mu \pmod{n}.$$

Hence, it only remains to prove that the lower row is not a permutation of the upper one.

Suppose it were. Then $\beta_1 - 1$ would be equal to some number on the upper row, say β_s, and $\beta_s - s$ of the lower row could be written $\beta_1 - (1+s)$. This number in turn would equal to some β_t whence $\beta_t - t$ could be written $\beta_1 - (1+s+t)$. Continuing in this fashion, one must finally arrive at a number in the lower row which is equal to β_1, e.g.

$$\beta_1 \equiv \beta_1 - (1 + s + t + \cdots + u) \pmod{n}.$$

In order for this equation to hold, it is necessary and sufficient to have

$$1 + s + t + \cdots + u \equiv 0 \pmod{n}.$$

The closed chain of values $\beta_1, \beta_s, \beta_t, \cdots, \beta_u$ which has been set up in this way may include all the β_i. If it does not, a second chain can be set up in the same way and a similar condition

$$v + w + \cdots + y \equiv 0 \pmod{n}$$

will be obtained as a result. Continuing in this fashion until all the β_i have been exhausted and adding together the various equations of condition, there results the following congruence:

$$1 + 2 + 3 + \cdots + (n-1) \equiv 0 \pmod{n};$$

$$n(n-1)/2 \equiv 0 \pmod{n}.$$

This condition is possible only if n is odd.

The result which has thus been obtained may be stated as follows:

THEOREM 1. *It is not possible to obtain two permutations of the same $2m$ letters, without having at least one pair of letters separated by the same interval in both permutations.*

The same theorem can be written down quite differently from the standpoint of Number Theory, viz.

THEOREM 2. *Given any ordered sequence of $2m-1$ positive integers each less than $2m$, then at least one of the conditions*

$$\sum_{j=i}^{i+\alpha-1} x_j \equiv 0, \alpha \pmod{2m}.$$

must be satisfied.

The fact that $n(n-1)/2$ is a multiple of n for $n=2m+1$ does not permit the same statement for odd numbers. In fact if s represents any permutation of $2m+1$ letters, s^p (p prime to $2m+1$) will have no repetitions with s. There are also other exceptions than these for every odd $n>5$.

The theorems just obtained may now be used to derive some results in abstract group theory. If s and t are two cyclic substitutions on the same $2m$ letters, there will be at least one pair of letters, say a_i and a_j having the same interval in both. Let this interval be x. Then s^x and t^x will both contain the sequence

$$\cdots a_i a_j \cdots$$

and $s^x t^{-x}$ will leave the letter a_i unchanged. Hence, $s^x t^{-x}$ is of degree less than $2m$. More generally, if s is compared with t^p (p prime to $2m$) the same remarks will hold.

THEOREM 3. *If s and t are any two substitutions cyclic in the same $2m$ letters, and if further, p is prime to $2m$, then there exists at least one number $x<2m$ for which $s^x t^{-px}$ is of degree less than $2m$.*

Suppose now that the above-mentioned substitutions s and t are non-commutative. Then in the group G generated by s and t, consider the set of operators $s^\alpha t^{-p\alpha}$. Denote this set by A.

$$A: \begin{cases} st^{-p} \\ s^2t^{-2p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ s^{2m-1}t^{-(2m-1)p}. \end{cases}$$

The subgroup H generated by A is invariant in G. For,

$$\begin{aligned} s^{-1}(s^\alpha t^{-p\alpha})s &= \left[s^{\alpha-1}t^{-(\alpha-1)p}\right]\left[s^{-1}t^p\right]^{-1} \\ t^{-1}(s^\alpha t^{-p\alpha})t &= \left[s^{-\nu}t^{p\nu}\right]^{-1}\left[s^{\alpha-\nu}t^{-(\alpha-\nu)p}\right] \end{aligned}$$

where ν is defined by the congruence $p\nu\equiv 1 \pmod{2m}$.

Suppose next that the number x, which corresponds to this particular p, is prime to $2m$. Then the set A can be rewritten in the following form:

$$A: \begin{cases} s^x t^{-px} \\ s^{2x}t^{-2px} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ s^{(2m-1)x}t^{-(2m-1)px}. \end{cases}$$

It is now possible to replace this set A by an equivalent set ξ, in the sense that ξ will generate the same group H. This new set is obtained by forming the complete set of conjugates of $s^x t^{-px}$ under s, as follows:

$$\xi_{i+1} = s^{ix}(s^x t^{-px}) s^{-ix} = (s^{(i+1)x} t^{-(i+1)px})(s^{ix} t^{-ipx})^{-1}.$$

The relations between A and ξ are identical with those obtained in a paper entitled *Families of Groups Generated by Two Operators of the Same Order.**

Now, theorem 3 indicates that $s^x t^{-px} = \xi_1$ is of degree less than $2m$. The method of formation of the set ξ requires all the remaining ξ_i to be of the same degree as ξ_1. Hence,

THEOREM 4. *If two substitutions s and t, cyclic in the same $2m$ letters generate a non-cyclic group G, and if any x, as defined in theorem 2, is prime to $2m$, then G involves a corresponding invariant subgroup generated by $2m$ conjugate substitutions of degree less than $2m$.*

ON GRAEFFE'S METHOD FOR THE NUMERICAL SOLUTION OF ALGEBRAIC EQUATIONS†

By C. A. HUTCHINSON, University of Colorado

In recent times, the problem of finding the numerical solutions of algebraic equations has become of great importance. To mention but two instances, in aerodynamics and in electric circuit analysis, it becomes necessary to solve equations of fairly high degree, and to find all of the roots. Particularly in the electrical case, the complex roots are of equal importance with the real ones. The methods ordinarily considered in courses in College Algebra, such as Horner's method, are inadequate for this purpose, being for practical purposes incapable of determining the complex roots at all, except in the case where only one pair of complex roots is present.

In the present paper, an exposition will be given of an old method, not usually mentioned in the ordinary textbooks, but very useful in practice. The method is due to Germinal Dandelin‡ (1794–1847), although the fundamental idea goes back to Edward Waring§ (1734–1798). Dandelin's paper was not widely circulated, and the process goes under the name of Carl Heinrich Graeffe (1799–

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* A. Sinkov, Transactions Amer. Math. Soc., vol. 35 (1933), p. 372.

† To have been presented, by invitation, at the Los Angeles meeting of the Mathematical Association of America, August 29–30, 1932; read by title when flood conditions prevented the author from reaching the meeting. Read at the Fort Collins, Colorado, meeting of the Mathematical Association of America, April 14–15, 1933.

‡ Mém. de l'Acad. Royale de Bruxelles, vol. 3 (1826), p. 48.

§ *Miscellanea analytica*, 1762; *Meditationes analyticae*, 1776, p. 311.

1873), who published it as a prize paper.* The method was also suggested independently by Nicholaus Ivanovich Lobachevski.† Later contributions were made by Johann Franz Encke,‡ the astronomer (1791–1865). The following discussion is almost entirely expository, but contains a few results on the presence of roots of equal modulus which may be new. I am particularly indebted to the books of Whittaker and Robinson,§ Runge and König,|| Scarborough¶ and Willers.** Other references, some of which discuss applications and extensions of Graeffe's method, are indicated in the footnote.††

The basis of Graeffe's method is the "root-squaring" process. The equation to be solved (assuming for the moment that no two roots have the same absolute value) is subjected to a transformation which replaces it by a new equation, in which the roots are more widely separated in absolute value. By repeated application of the transformation, an equation is obtained in which one root is very large in comparison with the next largest. Let the equation proposed for solution be

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{1}$$

Transposing all odd powers of x to one side of the equation, squaring, and replacing x^2 by $-y$, we get

$$y^n + b_1y^{n-1} + b_2y^{n-2} + \cdots + b_{n-1}y + b_n = 0, \tag{2}$$

where

$$\begin{aligned}
b_1 &= a_1^2 - 2a_2,\\
b_2 &= a_2^2 - 2a_1a_3 + 2a_4,\\
b_3 &= a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6,\\
&\cdots\cdots\cdots\cdots\\
b_k &= a_k^2 - 2a_{k-1}\cdot a_{k+1} + 2a_{k-2}\cdot a_{k+2} + \cdots + (-1)^m 2a_{k-m}\cdot a_{k+m} \pm \cdots + (-1)^k 2a_{2k},\\
&\cdots\cdots\cdots\cdots\\
b_n &= a_n^2.
\end{aligned}$$

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* *Die Auflösung der höheren numerischen Gleichungen.* Zürich, 1837.

† *Algebra or Calculus of Finites*, Kasan, 1834, 157.

‡ Berl. Astr. Jahrbuch, 1841; Ges. Abh. 1, Berlin, 1888, pp. 125–187; Jour. für. Math., Bd. 22, 1841, p. 193.

§ *The Calculus of Observations*, London, 1924, pp. 106–120.

|| *Vorlesungen über numerisches Rechnen*, Berlin, 1924, pp. 164–173.

¶ *Numerical Mathematical Analysis*, Baltimore, 1930, pp. 198–217.

** *Methoden der praktischen Analysis*, Berlin, 1928, pp. 205–212.

†† Carvallo, *Méthode pratique pour la Résolution numérique complète des Équations.* Paris, 1920.
Bauer, *Vorlesungen über Algebra*, 1903, p. 244.
Bairstow, *Applied Aerodynamics*, p. 558.
Berg, *Heaviside's Operational Calculus*, New York, 1929, pp. 140–163.
Brodetsky and Smead, Proc. Camb. Phil. Soc., 22 (1924), p. 83.

Let the roots of the original equation, with their signs reversed, (these have been called the "Encke roots") be $a, b, c, \cdots$. Then the Encke roots of the new equation are $a^2, b^2, c^2, \cdots$. The process may be repeated m times, giving an equation whose Encke roots are the 2^mth powers of the Encke roots of the original equation. This is the root-squaring process referred to above. It may be reduced to a working rule for the first stage of the process as follows:

Write in line (1) the coefficients of the given equation, missing powers of x being supplied with zero coefficients, and below them put their squares [line (2)]. For the third line, write in column k minus twice the product of the numbers in columns $k-1$ and $k+1$ of line (1). Column k of line (4) contains twice the product of the coefficients of columns $k-2$ and $k+2$ of line (1). Continuing in this way, alternately subtracting and adding the doubled products, we obtain in all

$$[2n + 7 + (-1)^n]/4$$

lines, where n is the degree of the equation to be solved. The columns are now summed, omitting line (1), and the sums are the coefficients of the first transformed equation.

Example:

$$x^3 - 2x^2 - x + 2 = 0, \text{ Encke roots } -1, 1, -2.$$

(1)	1	− 2	− 1	2
(2)	1	4	1	4
(3)		2	8	
	1	6	9	4

The transformed equation, $y^3+6y^2+9y+4=0$, has Encke roots 1, 1, 4.

Now let us form an equation whose Encke roots are $a^m, b^m, c^m, \cdots$. It is

$$(x + a^m)(x + b^m)(x + c^m) \cdots = 0,$$

or

$$x^n + [a^m]x^{n-1} + [a^m b^m]x^{n-2} + [a^m b^m c^m]x^{n-3} + \cdots + [a^m b^m c^m \cdots] = 0,$$

where

$$[a^m] = a^m + b^m + \cdots,$$
$$[a^m b^m] = a^m b^m + a^m c^m + \cdots + b^m c^m + \cdots,$$

etc.

Let us assume at first that the roots are all real and numerically unequal, with $|a| > |b| > |c| > \cdots$. Then, if m is sufficiently large, we may write, approximately, in the sense that the ratio of the term retained to the sum of the terms rejected is large,

$$[a^m] = a^m,$$
$$[a^m b^m] = a^m b^m,$$

etc., and our equation becomes

$$x^n + a^m x^{n-1} + a^m b^m x^{n-2} + \cdots + (a^m b^m \cdots) = 0.$$

This represents the equation obtained above by one or more applications of the root-squaring process. We can now determine $|a|$ from the second coefficient, $|b|$ from the next coefficient, and so on.

One little difficulty appears: the signs of the actual roots are not determined. This is not serious, as a rough check in the original equation will usually settle the question of sign. The ordinary rules, such as Descartes' Rule of Signs, are often useful.

A numerical example will bring out the technique. For the sake of simplicity, a synthetic example is chosen:

$$x^3 - 2x^2 - 5x + 6 = 0.$$

The work is arranged as shown below. The numbers in parentheses in the first column indicate the powers to which the Encke roots are raised. The notation 9.604^3 is an abbreviation for

$$9.604 \times 10^3, \text{ etc.}$$

(1)	1	−2	−5	6
	1	4 10	25 24	36
(2)	1	14	49	36
	1	196 −98	2401 −1008	1296
(4)	1	98	1393	1296
	1	9.604^3 -2.786^3	1.940^6 -0.254^6	1.680^6
(8)	1	6.818^3	1.686^6	1.680^6
	1	4.649^7 -0.337^7	2.843^{12} -0.023^{12}	2.822^{12}
(16)	1	4.312^7	2.820^{12}	2.822^{12}
	1	1.859^{15} -0.006^{15}	7.952^{24} -0.000	7.964^{24}
(32)	1	1.853^{15}	7.952^{24}	7.964^{24}

The procedure is that indicated in the rule above: the first line contains the coefficients of the original equation; the second line has the squares of these numbers. Next are placed the doubled products, with changed signs, of the

coefficients in the two adjacent columns. Adding the last two lines, we have equation (2), which is now treated as equation (1) was. The root-squaring process is stopped at the stage where the doubled products, $-2a_2$, $-2a_1, a_3$, etc., no longer affect the figures retained. It will be noted that it is not necessary to carry a large number of significant figures. The index indicating a power of ten is of great importance.

We now determine the absolute values of the roots, using four-place logarithms.

$$a^{32} = 1.853\cdot 10^{15},\ \log a^{32} = 15.2679,\ \log |a| = 0.4771,\ |a| = 3.000;$$

$$a^{32}b^{32} = 7.952\cdot 10^{24},\ \log b^{32} = 24.9005 - 15.2679 = 9.6326,$$

$$\log |b| = 0.3010,\ |b| = 2.000;$$

$$a^{32}b^{32}c^{32} = 7.964\cdot 10^{24},\ \log c^{32} = 24.9011 - 24.9005 = 0.0006,$$

$$\log |c| = 0.0000,\ |c| = 1.000.$$

The original equation must have one negative root, and the sum of the three roots is $+2$; therefore, the roots are 3, -2, 1.

The same process can be used to exhibit the presence of complex roots, and to determine their values. It is our purpose to explain how Graeffe's method shows the existence of complex roots, or of roots of equal moduli, and how it calculates the numerical values of these roots in the same schedule of operations used for simple real roots.

To illustrate the argument, let us take the simplest possible case, that of a cubic with one real root and a pair of conjugate complex roots. Let the Encke roots be

$$a,\ re^{\pm i\phi},\ r > 0.$$

Then the "mth power" equation is

$$(x + a^m)(x^2 + 2r^m \cos m\phi\cdot x + r^{2m}) = 0,$$

or

$$x^3 + (a^m + 2r^m \cos m\phi)x^2 + (r^{2m} + 2a^m r^m \cos m\phi)x + a^m r^{2m} = 0.$$

Let us suppose that m is taken so large that we can make the necessary approximations, as was done in the first case. If $|a| > r$, the approximate equation (that in which only the dominant terms are retained) is

$$x^3 + a^m x^2 + 2a^m r^m \cos m\phi\cdot x + a^m r^{2m} = 0,$$

while if $|a| < r$, it is

$$x^3 + 2r^m \cos m\phi\cdot x^2 + r^{2m}x + a^m r^{2m} = 0.$$

In this case all that is meant by "dominant terms" is: for *some* sufficiently large values of m, $|2r^m \cos m\phi|$ will be large in comparison with a^m, even if for some other large values of m the relations are reversed.

In the previous case (that of real roots, with different moduli), the coefficients, for large m, become at each stage the squares of the coefficients in the preceding step. This is still true here, except in one column of coefficients, that containing $\cos m\phi$. Furthermore, in the real case, all coefficients of the transformed equations after the given equation, must be positive. Here the coefficients in the exceptional column will be irregular in sign, that is, minus signs will appear at intervals. This irregularity shows that a pair of complex roots is present. Also, the position of the column containing the irregularities of sign (second or third column) shows the relative magnitude of the modulus of the complex roots and that of the real root. We may summarize thus:

If, in applying the root-squaring process to a cubic equation, one column of coefficients shows minus signs in the transformed equations, after the first row, there is a pair of complex roots present. If this occurs in the second column, the modulus of the complex roots is greater than that of the real root; if it is the third column which is affected, the modulus of the complex roots is less than that of the real root.

Again an example, also synthetic, will show how the roots are actually determined:

$$x^3 - 12x^2 + 61x - 150 = 0.$$

Only the transformed equations are shown in the schedule below, as the intermediate steps offer nothing of interest.

(1)	1	-1.200^1	6.100^1	-1.500^2
(2)	1	2.200^1	1.210^2	2.250^4
(4)	1	2.420^2	-9.754^5	5.0625^8
(8)	1	2.010^6	7.064^{11}	2.563^{17}
(16)	1	2.627^{12}	-5.313^{23}	6.569^{34}
(32)	1	7.964^{24}	-6.286^{46}	4.315^{69}
(64)	1	6.356^{49}	-6.478^{94}	1.862^{139}

The minus signs in the third column show that the equation has a real root, a, and two complex roots, $-re^{\pm i\phi}$, and that $|a| > r$. In determining the proper stopping place for the root-squaring, no attention need be paid to the column in which the negative signs occur. A comparison with the coefficients of the literal equation worked out on page 157 shows that

$$a^{64} = 6.356 \times 10^{49},$$

and hence

$$|a| = 6, \quad a = +6,$$

since the given equation has no negative roots. Also

$$a^{64}r^{128} = 1.862 \times 10^{139},$$
$$r^{128} = (1.862 \times 10^{139}) \div (6.356 \times 10^{49}) = 2.930 \times 10^{89},$$
$$r^2 = 25.$$

Let the actual roots be 6, $u \pm iv$. Then, since the sum of the roots is $+12$,

$$6 + 2u = 12, \quad u = 3,$$
$$v = \sqrt{r^2 - u^2} = 4.$$

The roots are 6, $3 \pm 4i$.

It will be noticed that the column of irregular coefficients has served its purpose in giving warning of the presence of complex roots, and is not used in the final calculation. If we did try to use the last figure in this column, we should have

$$2a^{64}r^{64} \cos 64\phi = -6.478 \times 10^{94},$$

$$\cos 64\phi = \frac{-6.478 \times 10^{94}}{2\sqrt{6.356 \times 10^{49}}\sqrt{1.862 \times 10^{139}}} = -0.9415.$$

But we have no means of determining the quadrant of 64ϕ, or of the multiple of 360° to be added to $\cos^{-1}(-0.9415)$ before dividing by 64.

We need not consider the case of multiple roots, as these can be removed. If $f(x)=0$ represents the given equation, the highest common factor of $f(x)$ and $f'(x)$ can be found by the Euclidean Algorithm. The zeroes of the H. C. F. are multiple roots of $f(x)=0$, and the corresponding factors can be removed from $f(x)=0$. However, the presence of unequal roots, of equal absolute value, is of interest. As before, let us start with the simplest possible case, that of a cubic. Let the Encke roots be

$$a, \ -a, \ b, \ |a| \neq b.$$

The mth-power equation is

$$x^3 + (2a^m + b^m)x^2 + (a^{2m} + 2a^mb^m)x + a^{2m}b^m = 0,$$

which is approximated for large m by

$$x^3 + 2a^mx^2 + a^{2m}x + a^{2m}b^m = 0,$$

if $|a| > |b|$, and by

$$x^3 + b^mx^2 + 2a^mb^mx + a^{2m}b^m = 0,$$

if $|a| < |b|$. If m is so large that the doubled products of the coefficients are negligible, one of the columns exhibits a new kind of peculiarity. Since

$$2a^{2m} = \tfrac{1}{2}(2a^m)^2, \text{ and } 2a^{2m}b^{2m} = \tfrac{1}{2}(2a^mb^m)^2,$$

the coefficient in this column is not squared by another root-squaring transformation, but becomes one-half of the square of its former value. This behavior in the magnitudes of the coefficients of one column, unaccompanied by any irregularity in sign, shows that a "doublet" is present; by this we mean a pair of roots of equal magnitude, but opposite sign. As in the case of complex roots, the column of irregular coefficients serves its purpose in indicating the character of the roots; no use is made of it either in the actual determination of the roots, or in determining the number of root-squarings to be performed. Example:

(1)	1	-3	-4	12
(2)	1	17	88	144
(4)	1	1.130^2	2.848^3	2.074^4
(8)	1	7.073^3	3.424^6	4.301^8
(16)	1	4.318^7	5.640^{12}	1.850^{17}
(32)	1	1.854^{15}	1.583^{25}	3.422^{34}

$$b^{32} = 1.854 \times 10^{15}, \quad |b| = 3.000,$$
$$a^{64}b^{32} = 3.422 \times 10^{34}, \quad |a| = 2.000.$$

Two roots are therefore 2 and -2. Since our equation can have only one negative root, the third root is $+3$.

An alternative procedure is available here. As soon as the presence of a doublet is disclosed, the root-squaring is stopped, and equation (2) is treated as an equation with equal roots. The H.C.F. of

$$x^3 + 17x^2 + 88x + 144$$

and

$$3x^2 + 34x + 88$$

is $x+4$. Hence (2) has the Encke roots 4, 4, 9, and (1) the roots ± 2, ∓ 2, and 3 or -3.

There is some danger of confusing a doublet with a pair of pure imaginary roots. Consider, for example, a cubic with the Encke roots

$$a, \; re^{\pm i\phi}, \; \phi = \frac{\pi}{2}.$$

The mth-power equation is

$$x^3 + \left(a^m + 2r^m \cos \frac{m\pi}{2}\right)x^2 + \left(r^{2m} + 2a^m r^m \cos \frac{m\pi}{2}\right)x + a^m r^{2m} = 0.$$

If $m=2$, this becomes

$$x^3 + (a^2 - r^2)x^2 + (r^4 - 2a^2r^2)x + a^2r^4 = 0.$$

Either one or two of these coefficients is negative. For $m=2, 4, \cdots$, we have

$$x^3 + (a^m + 2r^m)x^2 + (r^{2m} + 2a^m r^m)x + a^m r^{2m} = 0.$$

For large m this is approximated by

$$x^3 + a^m x^2 + 2a^m r^m x + a^m r^{2m} = 0, \text{ if } |a| > r,$$

and

$$x^3 + 2r^m x^2 + r^{2m} x + a^m r^{2m} = 0, \text{ if } r > |a|.$$

Hence, after the first transformation, the coefficients show the behavior characteristic of a doublet. If $|a| = r$, the mth-power equation is, for $m > 2$

$$x^3 + 3a^m x^2 + 3a^{2m} x + a^{3m} = 0,$$

which shows the characteristics of a triplet.

The presence of pure imaginary roots is therefore detected by means of the minus signs in the first transformed equation, followed by the behavior appropriate for multiple roots.

We have now exhausted the possibilities of the cubic equation. The cubic has been used for illustrations, because of its simplicity, in spite of the fact that we have other perfectly good methods for the solution of this equation. We now proceed to equations of higher degree.

Let us first dispose of the case of multiple roots. In view of what has been said about algebraically equal roots, nothing worse than a doublet really needs to be treated. Nevertheless, the behavior in the presence of multiple roots is so interesting that a brief discussion will be given. The term "triplet" will be used to indicate three roots of the same absolute value, whether algebraically equal or not. Similar definitions are to be understood for quadruplet, etc. In order to have a convenient terminology, we shall speak of the behavior of the coefficients of the transformed equations in the first case (all roots real, no two of the same absolute value) by saying that they increase at "normal rate." In the case of a doublet, we shall say that the coefficients in one column increase ultimately at one-half of normal rate.

If a cubic has the Encke roots a, a, $-a$, the mth-power equation is

$$x^3 + 3a^m x^2 + 3a^{2m} x + a^{3m} = 0.$$

Here two adjacent columns each increase ultimately at one-third of normal rate, since

$$3a^{2m} = \tfrac{1}{3}(3a^m)^2.$$

With a quadruplet, the corresponding equation is

$$x^4 + 4a^m x^3 + 6a^{2m} x^2 + 4a^{3m} x + a^{4m} = 0;$$

three adjacent columns increase ultimately at one-fourth, one-sixth, one-fourth normal rate, respectively. The fractions appearing here are obviously the reciprocals of binomial coefficients. It is easy to see that this behavior extends to multiplicities of any order.

It remains to consider the question of the simultaneous presence of two or more peculiarities, say, a pair of complex roots and a doublet, or two pairs of complex roots, etc. Let us treat first this latter case. Suppose the Encke roots of a quartic are

$$re^{\pm i\phi},\ se^{\pm i\theta},\ r > 0,\ s > 0.$$

The mth-power equation is

$$x^4 + 2(r^m \cos m\phi + s^m \cos m\theta)x^3 + (r^{2m} + 4r^m s^m \cos m\phi \cos m\theta + s^{2m})x^2$$
$$+ 2r^m s^m(r^m \cos m\theta + s^m \cos m\phi)x + r^{2m}s^{2m} = 0.$$

The approximate equations are

$$\text{if } r > s,\ x^4 + 2r^m \cos m\phi \cdot x^3 + r^{2m}x^2 + 2r^{2m}s^m \cos m\theta \cdot x + r^{2m}s^{2m} = 0;$$
$$\text{if } r < s,\ x^4 + 2s^m \cos m\theta \cdot x^3 + s^{2m}x^2 + 2r^m s^{2m} \cos m\phi \cdot x + r^{2m}s^{2m} = 0.$$

In either case, we have two columns behaving irregularly with respect to sign; these two columns are separated by one regular column. The details of determining the roots are exhibited in the following example. For the first transformation, all steps are shown; thereafter, only the transformed equations are given. Example:

$$x^4 + 6x^3 + 26x^2 + 46x + 65 = 0.$$

(1)	1	6	26	46	65
	1	36	676	2116	4225
		−52	−552	−3380	
			+130		
(2)	1	−16	254	−1264	4225
(4)	1	−252	3.252^4	-5.486^5	1.785^7
(8)	1	-1.536^3	8.168^8	-8.600^{11}	3.186^{14}
(16)	1	-1.632^9	6.652^{17}	2.191^{23}	1.015^{29}
(32)	1	1.333^{18}	4.432^{35}	-8.704^{46}	1.030^{58}

The minus signs in columns two and four show the presence of two pairs of complex roots. Comparison with the approximate equations above shows that

$$r^{64} = 4.432 \times 10^{35},\ r^2 = 13;$$
$$r^{64}s^{64} = 1.030 \times 10^{58},\ s^2 = 5.$$

Let the actual roots be $u_1 \pm iv_1$, $u_2 \pm iv_2$, so that

$$u_1^2 + v_1^2 = r^2,\ u_2^2 + v_2^2 = s^2.$$

The equation with these roots is

$$x^4 - 2(u_1 + u_2)x^3 + (r^2 + 4u_1u_2 + s^2)x^2 - 2(u_2r^2 + u_1s^2)x + r^2s^2 = 0.$$

Therefore,

$$- 2(u_1 + u_2) = 6,$$
$$- 2(s^2u_1 + r^2u_2) = - 10u_1 - 26u_2 = 46.$$

Hence,

$$u_1 = - 2,\ v_1 = \sqrt{13 - 4} = 3;$$
$$u_2 = - 1,\ v_2 = \sqrt{5 - 1} = 2.$$

The roots are $-2 \pm 3i,\ -1 \pm 2i$.

If we have a pair of complex roots and a doublet, we naturally wonder if the two kinds of peculiarities will show up separately, or whether there will be confusion, or overlapping, of the indications. Let us build up, step by step, an equation whose Encke rocts are the mth-powers of the Encke roots

$$a,\ b,\ - b,\ re^{\pm i\phi}.$$

First, the mth-power equation for the roots $re^{\pm i\phi}$ is

$$x^2 + 2r^m \cos m\phi \cdot x + r^{2m} = 0.$$

Multiplying by

$$x^2 + 2b^m x + b^{2m},$$

we get

$$x^4 + 2(r^m \cos m\phi + b^m)x^3 + (r^{2m} + b^{2m} + 4b^m r^m \cos m\phi)x^2$$
$$+ 2b^m r^m(r^m + b^m \cos m\phi)x + b^{2m}r^{2m} = 0.$$

The approximate equations are

$$\text{if } |b| > r: \quad x^4 + 2b^m x^3 + b^{2m}x^2 + 2b^{2m}r^m \cos m\phi \cdot x + b^{2m}r^{2m} = 0;$$
$$\text{if } |b| < r: \quad x^4 + 2r^m \cos m\phi x^3 + r^{2m}x^2 + 2b^m r^{2m}x + b^{2m}r^{2m} = 0.$$

It appears that the characteristics of a doublet, and those of a pair of complex roots, are both present, and that the relative magnitudes of $|b|$ and r are indicated by the position in which the irregular columns appear. Now let us multiply by $x+a^m$:

$$x^5 + (a^m + 2b^m + 2r^m \cos m\phi)x^4$$
$$+ (2a^m r^m \cos m\phi + 2a^m b^m + r^{2m} + b^{2m} + 4b^m r^m \cos m\phi)x^3$$
$$+ (2b^m r^{2m} + 2b^{2m} r^m \cos m\phi + a^m r^{2m} + a^m b^{2m} + 4a^m b^m r^m \cos m\phi)x^2$$
$$+ (b^{2m}r^{2m} + 2a^m b^m r^{2m} + 2a^m b^{2m} r^m \cos m\phi)x + a^m b^{2m} r^{2m} = 0.$$

The approximate equations are

$$|a| > |b| > r: \quad x^5 + a^m x^4 + 2a^m b^m x^3 + a^m b^{2m} x^2 + 2a^m b^{2m} r^m \cos m\phi \cdot x + a^m b^{2m} r^{2m} = 0;$$
$$|a| > r > |b|: \quad x^5 + a^m x^4 + 2a^m r^m \cos m\phi \cdot x^3 + a^m r^{2m} x^2 + 2a^m b^m r^{2m} x + a^m b^{2m} r^{2m} = 0;$$

$$|b|>|a|>r:\quad x^5+2b^mx^4+b^{2m}x^3+a^mb^{2m}x^2+2a^mb^{2m}r^m\cos m\phi\cdot x+a^mb^{2m}r^{2m}=0;$$
$$|b|>r>|a|:\quad x^5+2b^mx^4+b^{2m}x^3+2b^{2m}r^m\cos m\phi\cdot x^2+b^{2m}r^{2m}x+a^mb^{2m}r^{2m}=0;$$
$$r>|a|>|b|:\quad x^5+2r^m\cos m\phi\cdot x^4+a^mr^{2m}x^2+2a^mb^mr^{2m}x+a^mb^{2m}r^{2m}=0;$$
$$r>|b|>|a|:\quad x^5+2r^m\cos m\phi\cdot x^4+r^{2m}x^3+2b^mr^{2m}x^2+r^{2m}b^{2m}x+a^mb^{2m}r^{2m}=0.$$

We find that in each case there is one column of coefficients irregular in sign, and one column in which the coefficients increase at one-half of normal rate. Furthermore, unless $|a|$ lies between $|b|$ and r, these two columns are separated by one column of regular coefficients; if $|a|$ is between $|b|$ and r, there are two such intervening columns.

The chief conclusion to be drawn from this example is, that each characteristic behavior remains unaltered when the equation is multiplied by other factors. Enough has been done to illustrate the method to be used in investigating more complicated cases.

In conclusion, we give as a summary a list of several of the simpler behaviors of the coefficients of the transformed equations in the root-squaring process. The terminology used has already been explained. Not all of the cases in the list have been discussed in this paper, but all are routine results of the method of investigation used.

Summary

1. All signs plus after the given equation, and all columns increase at normal rate: all roots real, and of unequal absolute values.

2. A single column irregular in sign: one pair of complex roots.

3. Two adjacent columns irregular in sign: one pair of complex roots, with modulus equal to the modulus of a real root.

4. One column increases eventually at one-half of normal rate: doublet.

5. Two adjacent columns each increase eventually at one-third of normal rate: triplet.

6. Two non-adjacent columns each increase at one-half of normal rate: two doublets (not a quadruplet).

7. Three adjacent columns increase at one-fourth, one-sixth, and one-fourth of normal rate, respectively: quadruplet.

8. One column increases at one-half of normal rate, and a non-adjacent column is irregular in sign: doublet and pair of complex roots.

9. Two non-adjacent columns irregular in sign: two pairs of complex roots, with unequal moduli.

10. Three adjacent columns irregular in sign: two pairs of complex roots, with equal moduli.

11. One column irregular in sign, and one column adjacent on each side regular in sign, but irregular in magnitude: doublet and a pair of complex roots with the same modulus as the doublet.

After the character of the roots is thus determined, the relative magnitude of the absolute values of the various roots is read off from the location of the columns in which the irregularities occur, as already illustrated. Then a literal equation of the desired type is set up, and approximated for large m to accord with the known relative magnitudes of the roots. This approximate equation then indicates at once how the roots (or their moduli, in the case of complex roots) are to be calculated from the results of the root-squaring schedule.

The method of determining the actual values of complex roots, after their moduli have been found has been shown in the case of one and two pairs of such roots. If a greater number of complex roots is present, the same method can be followed, making use of the relations between the symmetric functions of the roots and the coefficients of the given equation.

The method of Graeffe has these advantages: (1) it determines all roots, both real and complex, in one schedule of operations; (2) the nature of the roots shows up in this same schedule, without preliminary examination; (3) it is adapted to either logarithmic or machine calculation; and (4) it does not require the carrying of a large number of significant figures.

NOTE ON A POINT IN THE THEORY OF SAMPLING

By E. S. KEEPING, University of Alberta

It is commonly assumed* that the best value to adopt for the standard deviation σ of an infinite population of variates, as estimated from observations on a finite sample of n, is

$$\sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

where $\bar{x}$ is the arithmetic mean of the measurements x. This, however, is doubtful.

It is true that, as shown by "Student,"† the best estimate of σ^2 is $\sum(x-\bar{x})^2/(n-1)$, but it does not follow that the best estimate of σ is the square root of this, since the distribution of $\sum(x-\bar{x})^2$ for various possible samples is skew.

By the "best" estimate is meant, as usual, that which gives a minimum standard deviation from the true value.

Let the standard deviation of the sample be given by

$$s^2 = \frac{1}{n}\Sigma(x - \bar{x})^2.$$

Then the distribution of s is given‡ by

* For example, see Fisher, *Statistical Methods for Research Workers*, 4th ed., p. 46.

† Student, Biometrika, vol. 6, p. 1, 1908.

‡ Student, loc. cit.

$$p(s) = As^{n-2}e^{-ns^2/(2\sigma^2)}$$

where $p(s)ds$ is the probability of a value between s and $s+ds$, and A is a constant. The mean value of s from an unlimited number of samples is therefore

$$\bar{s} = \frac{\int_0^\infty s^{n-1}e^{-ns^2/(2\sigma^2)}ds}{\int_0^\infty s^{n-2}e^{-ns^2/(2\sigma^2)}ds} = k\sigma$$

where

$$k = \sqrt{\frac{2}{n}}\cdot\left(\frac{n-2}{2}\right)! \bigg/ \left(\frac{n-3}{2}\right)!,$$

and the symbol $x!$, when x is not integral, means

$$\int_0^\infty u^x e^{-u}du.$$

On the other hand,

$$\overline{s^2} = \frac{n-1}{n}\sigma^2,$$

whence

$$\overline{(s-\sigma)^2} = \overline{s^2} - 2\sigma\bar{s} + \sigma^2 = \sigma^2\left[2(1-k) - \frac{1}{n}\right].$$

Also

$$\overline{(s-k\sigma)^2} = \sigma^2\left[(1+k)(1-k) - \frac{1}{n}\right]$$

and since $k<1$ for all values of n, $1+k<2$, and

$$\overline{(s-k\sigma)^2} < \overline{(s-\sigma)^2}.$$

This means that s is a better estimate of $k\sigma$ than of σ, but if what we want is an estimate of σ itself, it is not true that s/k is better than s. For

$$\overline{\left(\frac{s}{k} - \sigma\right)^2} = \frac{\sigma^2}{k^2}\left[1 - k^2 - \frac{1}{n}\right]$$

and therefore

$$\overline{\left(\frac{s}{k} - \sigma\right)^2} - \overline{(s-\sigma)^2} = \frac{\sigma^2}{k^2}\left[1 - k^2 - \frac{1}{n} - 2k^2 + 2k^3 + \frac{2k^2}{n}\right] = \frac{\sigma^2}{k^2}\left[(k-1)^2(2k+1) + \frac{1}{n}(2k^2-1)\right]$$

and for all values of n greater than 2, $2k^2-1>0$ and the expression above is positive. Hence, as estimated by the sum of squares of the residuals, the error is less if we take s as an estimate than if we take s/k.

A table of values of k for a few values of n is appended, and corresponding values of $\sqrt{(n-1)/n}$ are tabulated for comparison.

n	k	$\sqrt{(n-1)/n}$
1	0	0
2	0.5642	0.7071
3	0.7236	0.8165
4	0.7979	0.8660
5	0.8408	0.8944
6	0.8686	0.9129
7	0.8882	0.9258
8	0.9027	0.9354
9	0.9139	0.9428
10	0.9227	0.9487
20	0.9619	0.9747
50	0.9849	0.9899
100	0.9925	0.9950

If n is even $(=2m)$,

$$k = \frac{2m-1}{2m}\sqrt{\frac{1}{\pi m}}\,\frac{2^{2m}(m!)^2}{(2m)!}$$

and if n is odd $(=2m+1)$,

$$k = m\sqrt{\frac{2\pi}{2m+1}}\,\frac{(2m)!}{2^{2m}(m!)^2}.$$

Since, by Wallis's formula,

$$\lim_{m\to\infty}\frac{2^{2m}(m!)^2}{(2m)!\sqrt{(2m)}} = \sqrt{\frac{\pi}{2}}$$

it is clear that $k\to 1$ as n increases.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

TWO MORE PERFECT NUMBERS

By HANSRAJ GUPTA, Government College, Hoshiarpur, India

Numbers which are equal to the sum of all their possible factors (including unity but excluding the number itself), are called Perfect Numbers. Euclid has given the following formula for such numbers (even):

"$2^{p-1}(2^p-1)$ is a perfect number when 2^p-1 is a prime."

Marin Mersenne in 1644 gave the first eight perfect numbers. A ninth was found by P. Seelhoff in 1885; and a tenth by R. E. Powers in 1912. Manguram Gupta under my guidance has recently found two more which are communicated herewith.

For the first of these $p=107$, and for the second $p=127$.

XI. 131640364585696483372397534604587229102234723183869431177837728128.
XII. 14474011154664524427946373126085988481573677491474835889066354349131199152128.

Editor's Note. The fact that 2^p-1 is a prime for $p=107$ and $p=127$ was proved in 1914 by Powers and Fauquembergue. See Dickson, *History of the Theory of Numbers*, vol. I, p. 32. I cannot find where the decimal representation is given elsewhere, so they are set forth here for purpose of record. I have checked the smaller one and the larger has been checked by Mrs. E. S. Quade, formerly of Brown University.

THE DEGREE OF THE HIGHEST COMMON FACTORS OF TWO POLYNOMIALS*

By W. V. PARKER, Georgia School of Technology

1. *Introduction.* In a recent paper, T. A. Pierce† gives a method for computing the resultant of two polynomials which is especially practical in case one of the polynomials is of low degree. His method is based on a theorem of Frobenius‡ that if $f(x)=0$ is the characteristic equation of a square matrix A, and $g(x)$ is any polynomial, the resultant of $f(x)$ and $g(x)$ is $R(f, g)=|g(A)|$. The purpose of this note is to give a method for determining the degree of the highest common factor of $f(x)$ and $g(x)$ from the resultant in this form, which is analogous to the method given when the resultant is in the usual form.§

2. *The rank of a function of a matrix.* The equation

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

is the characteristic equation of the square matrix

$$A = \begin{pmatrix} -a_1 & -a_2 & -a_3 \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 & 0 \end{pmatrix}.$$

We assume the matrix A to be in this form throughout this discussion.

* An abstract of this paper appeared in the October, 1934, number of this MONTHLY.

† *The Practical Evaluation of Resultants*, this MONTHLY, vol. 39 (1932), p. 161.

‡ Frobenius, Journal für Mathematik, vol. 84, p. 11.

§ M. Bôcher, *Introduction to Higher Algebra*, p. 197, theorem 1.

As a preliminary to our theorem we will establish the Lemma: *If $\phi(x)$ is an s degree factor of $f(x)$, the matrix $\phi(A)$ is of rank $n-s$.*

Denote the zeros of $f(x)$ by $\alpha_1, \alpha_2, \cdots, \alpha_n$, and suppose that they are so ordered that the zeros of $\phi(x)$ are $\alpha_1, \alpha_2, \cdots, \alpha_s$. We may then write

$$\phi(A) = (A - \alpha_1 I)(A - \alpha_2 I) \cdots (A - \alpha_s I) = N_1 N_2 \cdots N_s,$$

where $N_i = A - \alpha_i I$ $(i = 1, 2, \cdots, s)$. Let $\Sigma_{i,j}$ denote the sum of the products of the zeros $\alpha_{i+1}, \alpha_{i+2}, \cdots, \alpha_n$ taken j at a time, and let $S_{i,j}$ denote the sum of the products of the zeros $\alpha_1, \alpha_2, \cdots, \alpha_i$ taken j at a time. Let M_k denote the matrix, of order n, obtained from the identity matrix by replacing the elements to the right of the principal diagonal in the kth row by $-\Sigma_{k,1}, \Sigma_{k,2}, -\Sigma_{k,3} \cdots, (-1)^{n-k}\Sigma_{k,n-k}$. It is readily verified that

$$P_1 = M_1 N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\ 1 & -S_{1,1} & 0 & 0 & \cdots 0 & 0 & 0 \\ 0 & 1 & -S_{1,1} & 0 & \cdots 0 & 0 & 0 \\ 0 & 0 & 1 & -S_{1,1} & \cdots 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots 0 & 1 & -S_{1,1} \end{pmatrix},$$

and that if P_2 denote the product $M_2 M_1 N_1 N_2$,

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\ 1 & -S_{2,1} & S_{2,2} & 0 & \cdots 0 & 0 & 0 \\ 0 & 1 & -S_{2,1} & S_{2,2} & \cdots 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots 1 & -S_{2,1} & S_{2,2} \end{pmatrix}.$$

Let us denote by P_{k-1} the product $M_{k-1}M_{k-2} \cdots M_1 N_1 \cdots N_{k-2}N_{k-1}$ and assume that P_{k-1} has the following form: all the elements in the first $k-1$ rows are zero, in the $(k+j)$th row, the first j elements are zero, the next k elements are $1, -S_{k-1,1}, S_{k-1,2} \cdots (-1)^{k-1}S_{k-1,k-1}$ and the last $(n-j-k)$ elements are zero $(j=0, 1, 2, \cdots, n-k)$. If P_k denotes the product $M_k P_{k-1} N_k$, it is readily seen that P_k has the same form as P_{k-1} with $k-1$ replaced by k, but P_1 and P_2 are of this form and hence it follows by induction that all the P's are of this form. P_s is, therefore, of rank $n-s$. Since each of the M's is non-singular, the rank of $\phi(A)$ is the same as the rank of P_s, and our lemma is established.

3. *The degree of the highest common factor of two polynomials.* Let $g(x) \equiv \phi(x)\psi(x)$ be a polynomial such that $\phi(x)$ is the highest common factor of $f(x)$ and $g(x)$. We may then write $g(A) = \phi(A)\psi(A)$. If $\phi(x)$ is of degree s, $\phi(A)$ is of rank $n-s$ by our lemma, and $\psi(A)$ is non-singular. The rank of $g(A)$ is, there-

fore, $n-s$. If r is the rank of $g(A)$, n the degree of $f(x)$, and s the degree of the highest common factor of $f(x)$ and $g(x)$, they are connected by the relation $r=n-s$, or $s=n-r$. We have, therefore, the

THEOREM. *If $f(x)$ is the characteristic function of the square matrix A, of order n, and $g(x)$ is any polynomial, and r is the rank of the matrix $g(A)$, then the highest common factor of $f(x)$ and $g(x)$ is of degree $n-r$.*

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

A History of Mathematics in America before 1900. By David Eugene Smith and Jekuthiel Ginsburg. Carus Mathematical Monographs, No. 5. Chicago, The Open Court Publishing Company, 1934. viii+210 pages. $2.00; to members of the Association, $1.25, if purchased through the Secretary's office.

The authors of this slim volume have contributed an admirable, though necessarily brief sketch of mathematical developments in the United States down to the close of the 19th century. The presentation is well organized, systematic, and furnishes, with considerable clarity, an excellent overview of the subject in proper historical perspective. The style, unfortunately, is somewhat matter-of-fact and impersonal, in contrast to the more intimate and at times dramatic account given by the late Prof. Cajori in *The Teaching and History of Mathematics in the United States* (Washington, D. C., 1890). While the treatment of Cajori is more random, it is replete with details which one wishes were not lacking in the present volume. The comparison, however, is perhaps unfair, since the purpose of Professors Smith and Ginsburg is slightly different, although this is not apparent from the title. Incidentally, it seems rather strange that the 400-page pioneer treatise of Cajori's has apparently not been drawn upon by these authors; while Professor Lao Simons' modest monograph on the *Introduction of Algebra into American Schools in the Eighteenth Century* is freely alluded to, there appear no footnote references to Cajori's work, and only a brief paragraph (pp. 160–161) acknowledges his historical and expository contributions.

The first chapter is devoted to a very brief discussion of conditions in the 16th and 17th centuries, the needs of the early colonists, the causes of the low degree of intellectual efforts in America in general, and the roles played by early astronomy and astrology. The second chapter is concerned with the mathematical work done during the 18th century at Harvard, William and Mary, Yale, Princeton, Pennsylvania, Columbia, Brown, Rutgers, and Dartmouth.

The influence of interest in astronomy, navigation and geodesy is pointed out, as well as the pioneer movements in the establishment of learned societies and scientific periodicals. Among the prominent personalities mentioned are the astronomers John Winthrop and David Rittenhouse, and those public leaders, Benjamin Franklin and Thomas Jefferson, the latter of whom has not been as fully appreciated, as a scientist, as he deserves to be. In summing up the developments of this period the authors suggest that "modern mathematics was substantially unknown in America in the 18th century, but that questionable manners then and later existed both in the Mother Country and in her offspring."

Chapter III gives a general survey of developments during the 19th century. It is pointed out that while the first three centuries of our history were barren of mathematical achievement, the first half of the nineteenth century was a time of "preparation for action," the third quarter gave evidence of an impending awakening, and the last quarter saw a genuine blossoming of activity. The earlier part of that century was definitely affected by French influence, as shown in the translations of texts on elementary geometry and algebra, mathematical astronomy, and descriptive geometry. Among the important personages appearing at this time were Robert Adrain, whose interests lay chiefly in pure mathematics; Nathaniel Bowditch, known for his work in astronomy, mathematical physics, geodesy, navigation, theory of curves, and his translation of the *Mécanique Céleste* of Laplace. Then too, a few lesser known figures: Charles Gill, the first American actuary, interested also in the theory of numbers; Alexander Bache, contributor to the fields of surveying and astronomy; and Ferdinand Hassler, known for his pioneer work in connection with the U. S. Coast Survey and what eventually was to become the U. S. Bureau of Standards.

We now come to Chapter IV, which describes the developments from 1875 to 1900, bringing the account more up to date than Cajori could have given it. This is by all odds the most significant portion of the book, and fortunately embraces some hundred pages, or half the entire volume. Here we find various influences affecting the revolutionary improvement in the study of mathematics in this country, chief among them the inspirational vision of President Gilman of Johns Hopkins and of President Eliot of Harvard; the founding of the American Mathematical Society in 1894, and its subsequent rapid growth and expansion; the Chicago Congress of 1893; and, by no means the least significant, the effect of our contacts with European scholars. By this last we refer not only to the influence of Sylvester and Cayley, but also the contributions of those more serious graduates of Harvard, Yale and other American universities who completed their mathematical education in Berlin, Leipzig, Göttingen, or Jena, and returned home animated with new zeal for research in undreamed of fields of mathematics.

The epoch-making achievements of Benjamin Peirce and the brilliant discoveries of J. J. Sylvester and Simon Newcomb are appropriately presented next, as are also the contributions of Emory McClintock, J. H. Van Amringe,

G. B. Halsted, F. N. Cole, E. H. Moore, Maxime Bôcher, and many others. Also adequately treated is the work of J. Willard Gibbs in creating the field of vector analysis in 1881, despite the criticisms of the staunch defenders of quaternion notation in a controversy not altogether unlike that between Newton and Leibniz more than two hundred years earlier. Finally, and perhaps most pertinent and valuable of all, there follows a very satisfactory treatment, in some detail, of the trends and specific accomplishments in the following fields, each separately discussed: algebra, function theory, quantics, equations, transformations, calculus, differential equations, theory of numbers, theory of groups, determinants, quaternions, and vector analysis, probability and approximation methods, geometry.

On the whole, it would seem only honest to state that while the book leaves a little to be desired in the matter of style and details, nevertheless, considering its brevity (necessitated by conforming with companion volumes of the series) it unquestionably represents a welcome addition to the literature of historical and expository mathematics.

W. L. SCHAAF

The Theory of Functions. By E. C. Titchmarsh. Oxford University Press, 1932. x+454 pages. $7.50.

This thick volume treats both the complex variable and the real variable, with somewhat greater emphasis on the former. It takes the reader with a knowledge of elementary analysis (specifically, the material found in Hardy's *Pure Mathematics*) and prepares him for reading and understanding the literature of a variety of fields in which recent work has been done.

Following an initial chapter on infinite processes, the theory of functions of a complex variable is built up in a series of eight chapters (Chapters II–IX). The first three of these lay the usual foundations—contour integration, residues, isolated singularities, analytic continuation, and the like—in an acceptable manner.

In Chapter V, entitled *The Maximum Modulus Theorem*, the treatment breaks away from the usual text book matter. Here are brought together many results on the absolute value of a function: Schwarz's lemma, Hadamard's three-circles theorem, convex functions, the work of Vitali and Montel on sequences, inequalities associated with the names of Carathéodory, Phragmén, Lindelöf—these give the flavor of the chapter. Here is specific equipment for work in modern fields.

Chapter VI, on *Conformal Representation*, treats linear transformations, *schlicht* functions, and the mapping of regions on a circle. The author gives a compact and readable proof of the fundamental mapping theorem but exhibits a curious reluctance to give an adequate statement of the theorem. *Any region with a suitable boundary can be represented on a circle by a simple analytic function* is unduly vague, since *suitable* is not given a meaning. This is at variance with the author's usual carefulness of statement. The words "simply connected" seem nowhere to appear.

Chapter VII on *Power Series* . . . introduces the reader to a field much cultivated since Hadamard's thesis in 1892,—singularities on the circle of convergence, gap theorems, over-convergence, and the like. Two substantial chapters on *Integral Functions* and *Dirichlet Series* are modern introductions to these two subjects. Many results of fairly recent date will be found.

Elliptic functions, which bulk so large in the older texts, find no place here. They can well be spared, since their elimination makes room for many other subjects not so readily available.

The last four chapters of the book (Chapters X–XIII) deal with the real variable. Two of these are concerned chiefly with the Lebesgue integral. The discussions of measure and of the definition of the integral are clear and readable, and the author pauses now and then (as on p. 334) to orient the reader with a few well-chosen remarks. Chapter X might well serve as a first text on the Lebesgue integral. In the final chapter of the book some forty or fifty pages are devoted to *Fourier Series*. These are sufficient to establish some of the major theorems of this field and to give the reader an acquaintance with its growing literature.

The preceding sketch gives some idea of the richness of content of this volume. The author has a very readable style. He is a skillful expositor; and he has the knack of illuminating an abstruse matter with a well turned phrase or a happy illustration. The presentation might have been further enhanced by the use of well-chosen figures, of which there are none.

The book seems to be singularly free from errors. The theorem of Weierstrass (p. 93) is unfortunately stated; there is an erroneous statement about the convergence of a series at the middle of page 141; there is a suggestion (p. 142) that an analytic function is, in general, many-valued. The reviewer wonders in what sense, if any, this last is true. The printing of the many formulae has been very carefully done.

Various readers will regret the omission of divers topics. The reviewer would have wished to find a fundamental discussion of curves—Jordan curves and rectifiable curves. This would have given precision to the theorems on contour integration. It must be recognized, of course, that not everything can be included.

It is easy to become enthusiastic in praise of this book. It is modern in its point of view. It admirably achieves its purpose of initiating the reader into various fields of mathematical enquiry. In particular it is valuable as an introduction to certain activities of the British school. It should be in the hands of every worker in the theory of functions and of every student of the subject.

L. R. FORD

Elementary Statistics. An Introduction to the Principles of Scientific Method. By J. G. Smith. New York, Henry Holt, 1934. x+517 pages. $3.50.

The theory of statistics includes the theory of probability in a generalized sense which is virtually synonymous with the theory of probable inference. Be-

cause many people are interested in uncertain inference but lack mathematics, there has been a rapid succession of books on statistics, scientific method, and induction, mostly designed as textbooks for elementary students of economics and other subjects conceived of as non-mathematical, and definitely attempting to avoid mathematics. Professor Smith's is one of the most impressive of these books. To the usual material of books on economic statistics is added something like 150 pages on the foundations and evolution of scientific method. The classical logical classification of arguments and of fallacies is presented. Mixed with the discussion are many allusions to current economic problems and recent history, which should add to the interest.

The attempt to avoid mathematics is never entirely successful in books on statistics. There are many formulae. These indeed constitute the very heart of what the student learns to do. What is avoided is proving the formulae and justifying the methods taught. The evaluation to be placed on such books and the system of education they represent will therefore involve one's opinion of *ex cathedra* promulgation of mathematical ideas and methods without proof or derivation, and without accurate statement of premises from which they may be derived. If it be granted that promulgation without proof is desirable, it would seem that special care should be taken to see that the promulgators are of the highest mathematical competence in the field involved. Now the authors of this category of textbooks would almost unanimously admit their own lack of competence in higher mathematics of any variety. They have however acquired a body of mathematical ideas, concerning which they have the impression that sound proofs have somewhere, somehow, been given by mathematicians specializing in that sort of thing, and these ideas are turned into arithmetical procedures in which the younger generation is being drilled all over the land. These procedures are in turn made the basis of statistical investigations on which depend important questions of science and of public policy.

In recent years a revolution has taken place in the theory of statistics. New methods have been discovered which are better than the old in every respect—in making more accurate estimates, utilizing more of the information in data, providing exact tests of significance instead of crude approximations to probabilities, making possible inferences where no inferences were possible by the old crude methods, requiring less numerical work, necessitating a fundamental reorganization of the whole theory of probable inference. Some of the old formulae have been retained with modified meanings, others have been found false; others are true, but inefficient, and should be discarded as obsolete. To understand the new efficient methods it is no more necessary to understand the old inefficient ones than it is necessary to drive horses before one can drive an automobile. Yet the teaching of statistics is so organized that the new methods have made little impression on the classroom. Obsolete methods continue to be taught and to appear in the new textbooks.

In addition to perpetuating inefficient methods, the non-mathematical school of writers on the theory of statistics naturally makes mistakes in details.

Of such mistakes the volume under review probably has less than the normal quota. The original source is of course seldom or never given, though quotations of secondary or tertiary sources appear; in this, less complaint than usual is to be made. It is however most unfortunate that the great errors involved in many inferences from the correlation coefficient receive so little attention. The probable error formula for r is given, and its numerical calculation illustrated, with only a brief footnote reference to indicate the profound modifications needed in using this crudely approximate formula. An even more flagrant case is the formula for the standard error of a standard deviation on p. 312. In addition to having shortcomings similar to those of the formula for the probable error of the correlation coefficient, such as being only the first term of an asymptotic expansion for which a limit of error is lacking, assuming normality where normality cannot exist, and so forth, this formula is presented in contradistinction to rather than as a generalization of the formula on the preceding page, which applies for a normal or other mesokurtic distribution. The condition for the validity of the special formula is given as symmetry; it should be mesokurtosis, an altogether different property. Many pages are devoted to calculation of seasonal variation by the link relative method, which in the reviewer's opinion is inefficient, inaccurate, incapable of interpretation in terms of probability, excessively laborious in computation, and altogether objectionable. Obsolete methods are set forth for testing linearity of regression, estimating moments, and fitting trend lines by least squares, among other things. It is stated that "The mathematical proof of the law of error is based on the hypothesis that any 'accidental error consists of the algebraic sum of a very large number of infinitesimal errors, all of equal magnitude, and as likely to be positive as negative'." This statement of the hypotheses is far too restrictive, though failing to mention the requirements relating to finite moments and independence.

Judged by the standards of its class, the book is a good one.

HAROLD HOTELLING

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB TOPICS

1935 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By W. C. EELLS, Stanford University

In continuation of previously published lists (See this MONTHLY, Vol. 41 (1934), pp. 260–61 for a list of 1934 centennial events, and for references to previous volumes for corresponding lists from 1925 to 1933) of centennial dates

in the history of mathematics, the following group of significant 1935 centennial dates is presented.

A.D. 235. Censorinus flourished, author of *De Die Natali* which deals with mathematical and chronological questions and which has been useful in determining the principal epochs of ancient history.

A.D. 535. Chön Luan, Chinese mathematician, wrote his *Suan-king*.

A.D. 735. Birth of Alcuin, who wrote on arithmetic, geometry, and astronomy and whose name is connected with a famous collection of mathematical recreations which has influenced the writers of textbooks for over a thousand years.

A.D. 735. Death of Baeda, commonly known as the Venerable Bede, to whom "we are indebted for the best work on the calendar written during the Dark Ages, and for the best work up to his time on digital notation," (Smith, D. E., *History of Mathematics*, I, 185).

A.D. 1435. Ulugh Beg, Persian royal astronomer, flourished, author of astronomical tables and founder of observatory at Samarkand.

A.D. 1535. Tartaglia, Italian mathematician, discovered method of solving any equation of the type $x^3+ax^2=c$. Defeated Florido in a contest involving the solution of cubics.

A.D. 1635. Birth of Hooke, English mathematician, discoverer of "Hooke's Law" concerning the proportionality of stress to strain in a stretched string, and inventor of the conical pendulum.

A.D. 1635. Publication of Cavalieri's *Geometria indivisibilibus continuorum nova qoadam ratione promota*, Bologna, containing his enunciation of the principle of indivisibles, one of the forerunners of the calculus.

A.D. 1635. Richelieu founds the French Academy. Death of Metius, who published his father's computation of pi.

A.D. 1735. Birth of Vandermonde, French mathematician, who (according to Cajori) was the first to give a connected and logical exposition of the theory of determinants.

A.D. 1735. Euler, famous Swiss mathematician, solved in three days an astronomical problem proposed by the Academy, a problem for which several eminent mathematicians had demanded several months time, but at the expense of a fever which cost him the sight of his right eye—a warning to modern students against too intensive mathematical efforts!

A.D. 1835. Birth of Beltrami, Italian mathematician, who extended and simplified the theory of non-Euclidean geometry.

A.D. 1835. Birth of Simon Newcomb, American astronomer and mathematician, director of the *Nautical Almanac*, professor of mathematics at the Johns Hopkins University, and editor of the *American Journal of Mathematics*.

A.D. 1835. Publication of Plücker's *System der Analytischen Geometrie*, which contained a complete classification of plane curves of the third order based on the nature of the points at infinity.

A.D. 1835. Sir William Hamilton's paper on *General Method in Dynamics* and statement of the "*Hamiltonian* Equations."

A.D. 1835. Organization of the London Statistical Society which is now the Royal Statistical Society.

A.D. 1835. Division of the chair of mathematics and natural philosophy at Yale University into two chairs, choice of Anthony D. Stanley as professor of mathematics.

A.D. 1835. Requirements for admission at the University of North Carolina raised to include all of arithmetic and "Young's Algebra to simple equations." After three years this excessive requirement in algebra, however, was withdrawn not again to be required until 1855.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON and W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 142. *Proposed by W. B. Clarke, San Jose, California.*

From the midpoint of a side of a triangle a line is drawn through the incenter of the triangle, intersecting the altitude to that side at the point P. Prove that the distance from P to the vertex opposite the side already referred to, is equal to the radius of the inscribed circle.

E 143. *Proposed by H. T. R. Aude, Colgate University.*

If x, y and z are restricted to positive integers, find how many solutions exist for the equation $6x+3y+2z=49$. Also find the maximum and the minimum values of $(x+y+z)$ and of (xyz).

E 144. *Proposed by W. P. Udinski, University of Texas.*

Show that a triangle must be equilateral if any pair of the following centers coincide: Incenter, Circumcenter, and Centroid.

E 145. "*Proposed humbly and anonymously by one, presumably able, but actually unable, to do it himself,*" *West Lafayette, Indiana.*

A cube is circumscribed about a sphere of radius unity. At the eight points where the sphere is pierced by the radial line from the center to the vertices,

tangent planes are drawn cutting off the corners, thus forming a polyhedron with fourteen faces and twenty-four vertices. The process is then repeated; that is, at the twenty-four points where the sphere is pierced by the radial lines from the center to these new vertices, tangent planes are drawn cutting off the corners and forming a polyhedron with thirty-eight faces. Find its volume in simplest form in terms of radicals.

E 146. *Proposed by C. W. Foard, Youngstown College, Ohio.*

A cow is tethered by a seventy foot rope which passes over a long straight fence seven feet high, to a stake in the ground, twenty-four feet back from the fence. The ground is level. It is required to find the area over which the cow can graze.

E 147. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a certain college of under five thousand total enrollment, a third of the students were freshmen, two-sevenths of the students were sophomores, a fifth of them were juniors and the rest seniors. The history department offered a popular course in which there were registered a fortieth of all the freshmen in college, a sixteenth of all the sophomores and a ninth of all the juniors, while the remaining third of this history class were all seniors. How many students were there in the history class?

SOLUTIONS

E 113 [1934, 517]. *Proposed by E. T. Krach, Hasbrouck Heights, N. J.*

Prove that if three circles are so arranged that their six external tangents are real (each tangent touching two circles), then the three points of intersection of the three pairs of corresponding tangents are collinear.

Solution by M. J. Turner, Ball State Teachers College.

Denote the centers of the three circles by A, B and C, and let their radii be P, Q and R respectively. Let L, M and N be the respective intersections of pairs of external tangents to circles centered at B and C, C and A, and A and B. Then from similar triangles, $AN/BN=P/Q$, $CM/AM=R/P$, and $BL/CL=Q/R$. Consequently $(AN/BN)(CM/AM)(BL/CL)=1$, so that the theorem of Menelaus requires that L, M and N must be collinear.

Editorial Note. This problem is not new in projective geometry. C. A. Richmond points out that it appears solved on page 56 et seq. of G. Monge's Géométrie Descriptive, fifth edition, 1827, Paris, Bachelier. It may also be found in H. Dörrie's Triumph der Mathematik, page 155 et seq., 1933, Breslau, Hirt, where it is credited to D'Alembert. While the plane geometry solution above is simple, it seems new.

Also solved by S. J. Bernat, W. E. Buker, L. M. Kelly, Roy MacKay, C. W. Trigg, Simon Vatriquant, Maud Willey and E. B. Worthington.

E 114 [1934, 517]. *Proposed by Maud Willey, Long Beach, Mississippi.*

What is the locus of the parametric equations,

$$x = k + h \sin A,$$
$$y = k + h \sin (A + 2\pi/3),$$
$$z = k + h \sin (A + 4\pi/3),$$

where h and k are constants and A is the parameter?

Solution by E. P. Starke, Rutgers University.

If the equations are added, we have at once

$$x + y + z = 3k. \tag{1}$$

If each k is transposed, the members squared and the equations then added, the result is

$$(x - k)^2 + (y - k)^2 + (z - k)^2 = 3h^2/2. \tag{2}$$

Consequently, the required locus is the great circle of the sphere, (2), lying in the plane (1). Its center is (k, k, k) and radius $(h/2)\sqrt{6}$.

Also solved by C. W. Trigg, Simon Vatriquant, H. H. Walker and the proposer. Many incorrect solutions were received to this problem, "proving" the locus to be one or the other of the surfaces (1) and (2), instead of their curve of intersection.

E 115 [1934, 517]. *Proposed by J. E. Trevor, Cornell University.*

Four rectangular buildings have lengths x_i, widths y_i, and heights z_i, $i=1, 2, 3, 4$. These dimensions are positive integers in a unit equivalent to ten feet, and $z_i<y_i<x_i$. The owner proposes to add to each building a top story of height y_i-z_i. The values of x_i, y_i, and z_i are such that, for each new story, the numbers expressing its volume, the combined areas of its east and south faces, and its height, add up to 165. What are the dimensions in feet of each building before this construction?

Solution by C. W. Trigg, Cumnock College, Los Angeles.

$$xy(y - z) + (x + y)(y - z) + (y - z) = (x + 1)(y + 1)(y - z) = 165.$$

Here $(y-z)<(y+1)<(x+1)$, and each quantity is a positive integer. The corresponding sets of factors of 165 which meet these conditions are $3\cdot 5\cdot 11$, $1\cdot 3\cdot 55$, $1\cdot 5\cdot 33$ and $1\cdot 11\cdot 15$. Hence the four different possible sets of values of x, y and z are (10, 4, 1), (54, 2, 1), (32, 4, 3) and (14, 10, 9), and the dimensions of the four different buildings before construction were $100\times 40\times 10$ feet, $540\times 20\times 10$ feet, $320\times 40\times 30$ feet and $140\times 100\times 90$ feet.

Also solved by W. E. Buker, A. J. Lewis, Roy MacKay, F. L. Manning, E. P. Starke, Simon Vatriquant, Maud Willey and the proposer.

E 116 [1934, 517]. *Proposed by V. Thébault, Le Mans, France.*

Find the number whose square and cube together require the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, once each, to write them, and show that the solution is unique.

Solution by W. E. Buker, Leetsdale, Pa.

$46 < N < 100$, since only numbers within these limits have ten digits in N^2+N^3.

The right hand digit of N cannot be 0, 1, 5 or 6, for then the right hand digits of N^2 and N^3 would be the same. Consequently, if N exists, it must be one of the thirty-three integers between 46 and 100 ending in 2, 3, 4, 7, 8 or 9.

Since N^2+N^3 is divisible by 9, it follows that N is either a multiple of 3 or else one less than a multiple of 9. This reduces our list of suspects from thirty-three to the following fifteen: 48, 53, 54, 57, 62, 63, 69, 72, 78, 84, 87, 89, 93, 98 and 99.

An examination of the cubes of these fifteen numbers in a table shows that all but 69, 84 and 93 are eliminated because they show a repeated digit in their cubes. Then an examination of the squares of these remaining three suspects rules out the last two, since they each show a digit in common between their squares and their cubes, so that the only number which satisfies all the conditions of the problem is 69, since $69^2=4761$ and $69^3=328{,}509$.

Also solved by Hansraj Gupta, Beatrice Rosenberg, E. P. Starke, C. W. Trigg, W. J. Thome, J. E. Trevor, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 117 [1934, 517]. *Proposed by Otto Dunkel, Washington University.*

The shorter parallel side of a trapezoid is produced each way a distance equal to the length of the longer parallel side, and the longer parallel side is produced each way a distance equal to the shorter parallel side. Prove that the diagonals of the new trapezoid built on these new parallel sides, intersect at the centroid of the original trapezoid.

Solution by Simon Vatriquant, Brussels, Belgium.

Let $ABCD$ be the given trapezoid, AB being the longer parallel side, M and N the respective midpoints of AB and CD, E the centroid of ABC, F the centroid of ACD, G the centroid of the trapezoid $ABCD$, h the common height of the three figures, and x and y the respective distances from G to AB and CD.

G lies at the intersection of EF and MN. Taking moments with respect to the lines AB and CD respectively, we may write

$$ACD(2h/3) + ABC(h/3) = ABCD(x), \text{ and } ACD(h/3) + ABC(2h/3) = ABCD(y).$$

If we now set $AB=m$ and $CD=n$, and cancel the common factor $(h/6)$, these two equations become $h(m+2n)=x(m+n)$ and $h(2m+n)=y(m+n)$. Consequently $x/y=(m+2n)/(2m+n)$, and since $GM/GN=x/y$, we have $GM/GN=(m+2n)/(2m+n)$.

Now if we produce DC a distance $CE=m$ and BA a distance $AF=n$, EF cuts MN in the ratio $NE/MF=(m/2+n)/(m+n/2)=(m+2n)/(2m+n)$, and hence at the point G. Similarly, the second diagonal of the enlarged trapezoid passes through G, and the statement is proved.

Also solved by L. J. Adams, W. B. Clarke, L. M. Kelly, A. J. Lewis, C. W. Trigg, and E. P. Starke.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3726. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

The vertices of a triangle inscribed in a given circle are the points of tangency of a triangle circumscribed about the circle. Prove that the product of the perpendiculars from any point on the circle to the sides of the inscribed triangle is equal to the product of the perpendiculars from the same point to the sides of the circumscribed triangle.

3727. *Proposed by J. R. Musselman, Western Reserve University.*

It is well known that, if P is a point on the circle O circumscribing a quadrilateral $ABCD$, where AB is parallel to CD, the feet of the perpendiculars from P to AC, AD, BC and BD are concyclic. Show that the locus of the center Q of this circle is that diameter of O which is perpendicular to AB and CD; in fact for a given point P to construct the point Q associated with it, one merely drops the perpendicular from P upon the aforementioned diameter of O.

3728. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the principal axes of an ellipse and the angle between two conjugate diameters; by a ruler-compass construction locate the diameters.

3729. *Proposed by J. Rosenbaum, Hartford Federal College, Hartford, Conn.*

In an orthocentric n-dimensional simplex the centroid G divides the line segment joining the orthocenter H with the circumcenter O so that $HG/GO = 2/(n-1)$.

This is a generalization of problem 3697 [1934, 453].

3730. *Proposed by William Chisholm, Quincy, Mass.*

There is said to be a short and simple method of finding one solution x of each of the equations

$$x^5 - (m+n)(5x^2+5x+1) = (m+n)^2, \tag{1}$$

$$x^2 + m^2 + n^2 - 2(mn + mx + nx) = 8\sqrt{2mnx(x+m+n)}. \tag{2}$$

What is the method?

SOLUTIONS

3576 [1932, 549]. *Proposed by J. J. L. Hinrichsen, Iowa State College.*

Given any triangle with three line-segments concurrent in a point interior to the triangle and joining each vertex to its opposite side; prove that the length of the longest of these three lines cannot be less than $\sqrt{3}/2$ times the length of the opposite side of the triangle.

3637 [1933, 496]. *Proposed by Otto Dunkel, Washington University.*

Triangle ABC is isosceles with the vertical angle C greater than 60° and less than arc cos 1/3. Through a point P in its interior the segments of straight lines $AL=p$, $BM=q$, $CN=r$ are drawn to the opposite sides. Show that there are two points on its interior such that $p=q=r$, and that each of these segments has the length $3^{1/2}c/2$, where $c=AB$.

Show also that, adjacent to each of the equal sides $AC=BC$, there is in the interior of the triangle a region bounded by a part of that side and by parts of two curves which meet in one of the above two points, such that for each interior point r is the longest of the three segments. See problem 3576 [1932, 549].

Solution of 3576 *by E. H. Cutler, Lehigh University*

We show that if a segment is less than $\sqrt{3}/2$ times opposite side it is not the longest of three concurrent segments. Take length of the side as 2, then that of segment is $u<\sqrt{3}$. Place the segment OA along the y axis so that the opposite side A_1A_2 passes through the origin. The coordinates of the three vertices of the triangle are $A:(0,\ u)$, $A_1:(a\cos\theta,\ a\sin\theta)$ and $A_2:\{(a-2)\cos\theta,\ (a-2)\sin\theta\}$, where $0<a<2$. Let the point of concurrence of the three segments be $P:(0,\ t)$, where $0<t<u$. Then the feet of the two other segments are:

$$Q_1:\left(\frac{-a(2-a)(u-t)\cos\theta}{2t+a(u-t)},\ \frac{2ut-a(2-a)(u-t)\sin\theta}{2t+a(u-t)}\right)$$

$$Q_2:\left(\frac{a(2-a)(u-t)\cos\theta}{2u-a(u-t)},\ \frac{2ut+a(2-a)(u-t)\sin\theta}{2u-a(u-t)}\right).$$

We have for the lengths of the two segments:

$$(A_1Q_1)^2=\frac{4u^2[a^2-2at\sin\theta+t^2]}{[t(2-a)+au]^2}$$

$$(A_2Q_2)^2=\frac{4u^2[(2-a)^2+2(2-a)t\sin\theta+t^2]}{[at+(2-a)u]^2}.$$

The requirements $(A_1Q_1)^2<u^2$ and $(A_2Q_2)^2<u^2$ yield:

$$(4-a)t^2-2[u(2-a)+4\sin\theta]t+a(4-u^2)<0,$$

$$(2+a)t^2-2[ua-4\sin\theta]t+(2-a)(4-u^2)<0.$$

Adding, we have

$$6t^2 - 4ut + 2(4 - u^2) < 0.$$

In order that this inequality admit a solution t we must have $16(u^2-3)>0$, but we assumed $u<\sqrt{3}$. Hence there is no point of concurrence for which OA is the longest segment. The theorem is proved.

If we set $u=\sqrt{3}$ and admit equality signs we obtain conditions satisfied by a family of isosceles triangles, each triangle having 3 concurrent line segments of length $\sqrt{3}$, the base being 2 and the equal legs between $\sqrt{3}$ and 2. Here the ratio to opposite side is for one of the equal segments exactly $\sqrt{3}/2$; for the other two it is greater.

Solved also by J. K. Peterson.

Editorial Note. In the solution above, if $AA_1=AA_2$ and $A_1Q_1=A_2Q_2=OA=u$, it is easily found in one case that $a=1$, which means that OA is an altitude of the isosceles triangle. Setting aside this case, the other result is given by $u=\sqrt{3}$ and $t=1/\sqrt{3}$; and the other two equal sides have lengths between $\sqrt{3}$ and 2 as found above. These results together with those given in the Note by the Editors [1934, 194] furnish a solution of problem 3637.

The solution by Peterson is of considerable merit and shows great care in the tedious reductions which are necessary in his form of proof. A brief account of his solution will be given.

Let α, β, γ be the lengths of the concurrent segments to the sides of lengths a, b, c, which are divided by the segments in the ratios $x:1-x$; $y:1-y$; $z:1-z$. Each of these last six quantities is greater than zero and less than unity, and they satisfy the relation

$$(1) \qquad xyz = (1 - x)(1 - y)(1 - z).$$

There are three equations such as

$$(2) \qquad -x(1 - x)a^2 + xb^2 + (1 - x)c^2 = \alpha^2,$$

the other two being obtained by cyclic interchange of the letters. The determinant D of the system (2) is easily shown to be positive by the use of (1), and

$$(3) \qquad a^2D = \alpha^2 l + \beta^2 m + \gamma^2 n.$$

The cofactors m and n are seen to be positive; and, if we now assume that $\alpha \geqq \beta$, $\alpha \geqq \gamma$,

$$a^2D \leqq \alpha^2 d, \qquad d = l + m + n.$$

In the determinant $4D-3d$, z is replaced by its expression in x and y; and after reduction there is obtained

$$(4) \qquad (1 - x - y + 2xy)^2(4D - 3d) = y(1 - x)(1 - y + xy)(1 - y - 2xy)^2.$$

On the right the first three factors are each greater than zero, and so is also the first parenthesis on the left. Hence the second parenthesis on the left is not negative, and we have

$$\frac{\alpha^2}{a^2} \geqq \frac{D}{d} \geqq \frac{3}{4}, \qquad \frac{\alpha}{a} \geqq \frac{\sqrt{3}}{2}, \tag{5}$$

and this completes the proof of 3576.

If $2\alpha=\sqrt{3}a$, then from (5) it follows that $4D-3d=0$, $a^2D=\alpha^2d$, $\alpha=\beta=\gamma$, and from (4), $1-y-2xy=0$. The equations (1) and (2) then show that $b=c$, and we now have

$$\frac{b^2}{a^2} = \frac{3}{4} - x^2 + x, \qquad \frac{\sqrt{3}}{2} < \frac{b}{a} < 1; \tag{6}$$

and there actually exist such isosceles triangles.

We may also extend Peterson's solution to cover 3637 by considering the converse $b=c$ and $\alpha=\beta=\gamma$. In this case the determinant of the equations in x, y, z obtained from (2) must be zero. After eliminating z, removing the denominator $(1-x-y+2xy)^2$, and considerable reduction, there follows

$$y(1 - y)(1 - x)(xy + 1 - y)(2x - 1)(2xy + y - 1) = 0.$$

From the restriction already imposed the only solutions are those given by setting each of the last two factors equal to zero. The first solution gives α as an altitude, and the second gives $\alpha=\sqrt{3}a/2$ and the results in (6).

3653 [1933, 611]. *Proposed by L. S. Johnston, University of Detroit.*

Given the triangle ABC and the points A', B', and C' on BC, CA, and AB respectively such that $AC'/C'B=BA'/A'C=CB'/B'A=m/n$ (m less than n for convenience). Let AA' and BB' intersect at C'', BB' and CC' intersect at A'', and CC' and AA' intersect at B''. Prove that

$$AB'' : B''C'' : C''A' = BC'' : C''A'' : A''B' = CA'' : A''B'' : B''C',$$

and calculate the division ratios in terms of m and n.

I. *Solution by Maud Willey, Long Beach, Miss.*

The theorem of Menelaus applied to triangle ABA' and the transversal $C'B''C$ gives $AC'\cdot BC\cdot B''A'=C'B\cdot A'C\cdot AB''$. In the same manner triangle $AA'C$ and transversal $BC''B'$ gives $AC''\cdot BA'\cdot B'C=C''A'\cdot BC\cdot AB'$. Hence we have

$$\frac{AB''}{m(m+n)} = \frac{B''A'}{n^2} = \frac{AA'}{m^2+mn+n^2}, \quad \frac{AC''}{n(m+n)} = \frac{C''A'}{m^2} = \frac{AA'}{m^2+mn+n^2},$$

and from these equations we have

$$\frac{B''A'}{n^2} = \frac{C''A'}{m^2} = \frac{B''C''}{n^2 - m^2}.$$

Therefore

$$\frac{AB''}{m(m+n)} = \frac{B''C''}{n^2 - m^2} = \frac{C''A'}{m^2}.$$

Similar results follow for the other two segments by cyclic interchange of the letters in the numerators above. Hence the common division ratio required in the problem is

$$m(m+n):n^2 - m^2:m^2.$$

II. *Solution by Otto J. Ramler, The Catholic University of America, Washington, D. C.*

Using areal coordinates with ABC as reference triangle, the coordinates of A'', B'' and C'' are found to be respectively,

$$(mn:m^2:n^2), \quad (n^2:mn:m^2), \quad \text{and} \quad (m^2:n^2:mn).$$

Whence

$$A'C'':A'A = m^2:m^2 + mn + n^2; \; A'B'':A'A = n^2:m^2 + mn + n^2;$$

$$B''C'':AA' = (n^2 - m^2):(m^2 + mn + n^2); \; AB'':AA' = (m^2 + mn):(m^2 + mn + n^2);$$

and therefore

$$AB'':B''C'':C''A' = (m^2 + mn):(n^2 - m^2):m^2.$$

Similarly

$$BC'':C''A'':A''B' = CA'':A''B'':B''C' = (m^2 + mn):(n^2 - m^2):m^2.$$

The required result is thus established.

It may be remarked that the triangle $A''B''C''$ is similar to a triangle having the transversals AA', BB', CC' as sides. $A''B''C''$ and ABC have the same centroid. Allowing the ratio $m:n$ to assume all values from $-\infty$ to $+\infty$ the point A'' moves on an ellipse tangent to AB at B and to AC at C and passing through the centroid of ABC. Similarly B'' and C'' each move on an ellipse. Employing Stewart's theorem we readily calculate the length of AA' to be

$$[m^2b^2 + n^2c^2 + mn(c^2 + b^2 - a^2)]^{1/2}/(m+n),$$

which when $m = n$ becomes the well known formula for the length of a median.

Solved also by J. W. Clawson, T. C. Esty, A. S. Householder, Olga Larson, J. Rosenbaum, F. Underwood, and the proposer.

Editorial Note. The solutions by Clawson and Larson used Menelaus' theorem, while Rosenbaum's solution used barycentric coordinates as in II. Those by Householder, Esty and the proposer used vectors. Underwood's solution used areas so that it also is somewhat similar to II.

This problem is related to 3392 solved [1930, 264] by the use of barycentric coordinates. That solution shows that as m/n varies the envelope of $B'C'$ is a parabola tangent to AB at B, to AC at C, and to $B'C'$ at A_1, where $B'A_1/A_1C' = m/n$. The straight line AA'' may be shown to pass through A_1.

3654 [1934, 49]. *Proposed by N. A. Court, University of Oklahoma.*

The tangent and the normal at a variable point of a certain curve determine a pair of conjugate points of a given involution on a given straight line. Find the curve.

Solution by L. Green, University of Chicago.

Choose the given line as the x-axis and the center of involution as the origin. If (x_1, y_1) is a point on the curve, the equations of the tangent and the normal are

$$y - y_1 = y_1'(x - x_1) \text{ and } y_1'(y - y_1) = x_1 - x$$

respectively. Their intercepts on the x-axis are

$$x = \frac{x_1 y_1' - y_1}{y_1'} \text{ and } x = x_1 + y_1 y_1'.$$

Hence

$$\left(\frac{x_1 y_1' - y_1}{y_1'}\right)(x_1 + y_1 y_1') = k^2,$$

where k^2 is the power of involution. This may be written

$$\text{(1)} \qquad xyp^2 + (x^2 - y^2 - k^2)p - xy = 0, \qquad p = \frac{dy}{dx},$$

with the subscripts removed. This becomes, on differentiating,

$$\text{(2)} \qquad 2xyp\frac{dp}{dx} + (x^2 - y^2 - k^2)\frac{dp}{dx} + xp^3 - yp^2 + xp - y = 0.$$

If we replace $(x^2 - y^2 - k^2)$ by its value as given in (1), this equation simplifies into

$$\text{(3)} \qquad xy\frac{dp}{dx} + xp^2 - yp = 0.$$

Since this may be written

$$\text{(4)} \qquad x^2\frac{d}{dx}\left(\frac{yp}{x}\right) = 0,$$

then $yp/x=c$ and $p=cx/y$. If we substitute this value of p in (1), the resulting equation becomes

$$y^2 = cx^2 - \frac{k^2c}{c+1}$$

which represents a family of central conics.

Solved also by A. Gelbart, O. J. Ramler, W. P. Udinski, and F. Underwood.

Editorial Note. The solutions differ mainly in the manner of solving the differential equation. Gelbart, Ramler and Udinski set $u=x^2$, $v=y^2$; while Underwood puts $x^2+y^2=2v$, $x^2-y^2=2u$. The resulting Clairaut equation in dv/du is then easily solved. Underwood notes that the singular solution is a pair of point circles. Ramler adds that the solution is a system of confocal conics whose foci are the double points of the involution; and that this result proves that the tangent and normal at any point of a central conic are the bisectors of the angles formed by the focal radii to that point. Udinski considered the generalization in which the normal is replaced by a line through the point of the curve with the slope $1/[kf'(x)]$, where $f'(x)$ is the slope of the curve at the point and k is a constant. The curves are central conics.

This problem is the same as 3078 by the same proposer, for which a solution by C. K. Robbins was printed [1925, 141]. It was also solved by Theodore Bennett, through whose courtesy this repetition was discovered.

3656 [1934, 49]. *Proposed by Raphael Robinson, University of California at Berkeley.*

If the vertices of a simplex S in n dimensions are numbered from 0 to n, the lengths of the edges are $e_{01}, e_{02}, \cdots, e_{n-1,n}$, and the lengths of the medians $m_0, m_1, \cdots, m_n$, then

$$m_0 = \frac{1}{n}\sqrt{n(e_{01}^2 + e_{02}^2 + \cdots + e_{0n}^2) - (e_{12}^2 + e_{13}^2 + \cdots + e_{n-1,n}^2)},$$

and $m_1, \cdots, m_n$ are given by similar formulas.

Solution by J. K. Peterson, Nashville, Tenn.

Let $\mathbf{a}_i$ be the vector from the centroid of the simplex to the vertex numbered i. Since the sum of all the vectors $\mathbf{a}_i$ is null, the vector from the centroid to the other end of the median from the vertex i is

$$\frac{1}{n}(\mathbf{a}_0 + \cdots + \mathbf{a}_{i-1} + \mathbf{a}_{i+1} + \cdots + \mathbf{a}_n) = -\frac{1}{n}\mathbf{a}_i.$$

The length, m_i, of the median from the vertex marked i is therefore that of the vector $(n+1)\mathbf{a}_i/n$, and

$$(1) \qquad n^2m_i^2 = (n+1)^2\mathbf{a}_i\cdot\mathbf{a}_i.$$

The square of the length of the edge joining the vertices i and j is

$$e_{ij}^2 = (\mathbf{a}_j - \mathbf{a}_i)\cdot(\mathbf{a}_j - \mathbf{a}_i) = \mathbf{a}_i\cdot\mathbf{a}_i - 2\mathbf{a}_i\cdot\mathbf{a}_j + \mathbf{a}_j\cdot\mathbf{a}_j.$$

The sum of the squares of the lengths of all the edges from the vertex marked i is

$$(2)\qquad \sum_{j=0}^{n} e_{ij}^2 = (n+1)\mathbf{a}_i\cdot\mathbf{a}_i + \sum_{j=0}^{n} \mathbf{a}_j\cdot\mathbf{a}_j.$$

The sum of the squares of the lengths of all the edges is therefore

$$(3)\qquad \frac{1}{2}\sum_{i=0}^{n}\left(\sum_{j=0}^{n} e_{ij}^2\right) = (n+1)\sum_{j=0}^{n}\mathbf{a}_j\cdot\mathbf{a}_j = \frac{n^2}{n+1}\sum_{j=0}^{n} m_j^2,$$

which gives the immediate generalization of problem E66, solved in this MONTHLY [1934, 329.]

By subtraction of (2) from (3), the sum of the squares of the lengths of the edges opposite the vertex i is found to be

$$(4)\qquad O_i^2 = \sum_{i\neq j<k\neq i} e_{jk}^2 = n\sum_{j=0}^{n}\mathbf{a}_j\cdot\mathbf{a}_j - (n+1)\mathbf{a}_i\cdot\mathbf{a}_i.$$

From (2), (4), and (1),

$$n\sum_{j=0}^{n} e_{ij}^2 - O_i^2 = (n+1)^2\mathbf{a}_i\cdot\mathbf{a}_i = n^2 m_i^2,$$

whence

$$(5)\qquad m_i = \frac{1}{n}\left(n\sum_{j=0}^{n} e_{ij}^2 - O_i^2\right)^{1/2}.$$

Solved also by Erna Jonas, J. Rosenbaum, Maud Willey, and the proposer.

Editorial Note. In the above solution the proof is valid even if the $n+1$ points lie on a straight line, and we then have a proof of the identity:

$$\left[nx_0 - \sum_{i=1}^{n} x_i\right]^2 = n\sum_{i=1}^{n}(x_0 - x_i)^2 - \sum\sum(x_i - x_j)^2 \qquad (1 \leqq i < j = 2, 3, \cdots, n),$$

where the x's are independent variables.

The solution by the proposer starts with this identity, which is easily verified directly; and, if we now suppose that each x is provided with a superscript k so that $x_i^{(k)}$, $k=1, 2, \cdots, n$, gives the coordinates of the ith vertex, then there follows at once

$$n^2 m_0^2 = n\sum_{i=1}^{n} e_{0i}^2 - \sum\sum e_{ij}^2 \qquad (1 \leqq i < j = 2, 3, \cdots, n).$$

If each side is summed we obtain the result (3) in the solution above.

In Rosenbaum's solution G denotes the centroid of all the vertices omitting A_0, H denotes the centroid of all omitting A_0 and A_1. Then $m_0 = A_0G$, $a_1 = A_1H$, $b_0 = A_0H$, and G lies on a_1 so that $nA_1G = (n-1)a_1$. From the two triangles A_1GA_0 and A_1HA_0 we have after applying the law of cosines to each

$$m_0^2 = \frac{e_{01}^2}{n} - \frac{n-1}{n^2} a_1^2 + \frac{n-1}{n} b_0^2,$$

where a_1 and b_0 are medians for simplexes in $n-1$ dimensions. By use of this result mathematical induction shows that, if the required result of the problem is true for $n-1$ dimensions, it is true for n; and, since it is true for $n=2$, it is true for any n.

The solution by Jonas used mathematical induction in a somewhat similar manner, while that of Willey expressed the lengths in terms of coordinates of the points.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Flavelle medal of the Royal Society of Canada has been awarded to Professor L. V. King, of McGill University.

Professor Cassius Jackson Keyser, of Columbia University, spoke on "Mind, the Maker, the World Theory of the late William Benjamin Smith," at Teachers College, Columbia University, on December 10, 1934, under the auspices of the Forum of the Friends of Scripta Mathematica.

The second All-Soviet Mathematical Congress was held at Leningrad, June 24–30, 1934. The only non-Soviet participant was Professor S. Lefschetz, of Princeton University, who delivered an invited address on *Algebraic geometry, its methods, problems, and tendencies.*

The Academy of Science of Virginia, under the leadership of Dr. J. Shelton Horsley, a few years ago raised a trust fund for the purpose of encouraging and developing research in Virginia, and appointed a committee to administer the fund. Dr. T. McN. Simpson, of Randolph-Macon College, is the representative of mathematics on that committee.

At Wesleyan University, Professor B. H. Camp has been given funds from the university's endowment for research, to carry on work on problems which have been submitted by the research committee of the American Statistical Association. These are chiefly economic problems for which a statistical analysis is desired.

Professor G. A. Baker of Shurtleff College has been appointed head of the department of mathematics at Mississippi Woman's College.

Dr. Saloman Bochner has been appointed assistant professor of mathematics at Princeton University.

Dr. Richard Brauer, formerly of the University of Königsberg, has been appointed an assistant at the Institute for Advanced Study, Princeton.

Dr. G. A. Hedlund has been appointed associate professor of mathematics at Bryn Mawr College.

The following have recently entered the field of secondary teaching: Dr. T. S. Peterson at the Shipley School, Bryn Mawr; Dr. M. F. Rosskopf, at the John Burroughs School, Clayton, Missouri; Dr. Edna Kramer, Dr. Mabel F. Schmeiser, and Dr. James Singer, in the New York City high school system.

Professor Louis Ingold, professor of mathematics at the University of Missouri, a charter member of the Mathematical Association, died January 25, 1935.

The death of Sir Horace Lamb occurred on December 3, 1934. He was well known to American mathematicians because of the wide use of his texts on mechanics and hydrodynamics.

Professor E. A. Lyman, for many years professor of mathematics at the Michigan State Normal College, Ypsilanti, died October 9, 1934. He was a charter member of the Mathematical Association.

In the death of Daniel Alexander Murray, author of the well known *Introductory Course of Differential Equations* and of several other textbooks which have had a wide use, this Association loses one of its oldest members. Professor Murray, who was born in Nova Scotia in 1862, was a graduate of Dalhousie University and a Ph.D. of Johns Hopkins University. Professor Murray taught at Cornell (1894–1901), at Dalhousie (1901–1907), and at McGill (1907–1930). At the time of his death on October 19, 1934, he was professor emeritus at McGill University.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.
Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., March.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE NINTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The ninth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania on Saturday, December 1, 1934, Professors Smith and Morris presiding.

The attendance was fifty-seven, including the following thirty-eight members of the Association: J. A. Benner, A. A. Bennett, Wm. Beverley, H. W. Brinkmann, L. H. Bunyan, P. A. Caris, G. G. Chambers, J. W. Clawson, J. E. Davis, L. J. Deck, Tomlinson Fort, J. A. Gardiner, J. R. Kline, P. A. Knedler, V. V. Latshaw, D. H. Lehmer, A. E. Meder, Jr., H. H. Mitchell, C. N. Moore, Richard Morris, C. A. Nelson, C. O. Oakley, F. W. Owens, Helen B. Owens, T. S. Peterson, G. E. Raynor, C. J. Rees, George Rosengarten, J. A. Roulton, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, Vivian Spencer, E. P. Starke, R. M. Walter, A. H. Wilson, C. R. Wilson.

At the business meeting the following officers were elected for next year: Chairman, Richard Morris, Rutgers University; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Fort, Shohat, Smith.

The next meeting will be held on Saturday, November 30, 1935, at Easton, Pa.

The following papers were presented:

1. "On some applications of Taylor's Formula" by Professor J. A. Shohat, University of Pennsylvania.

2. "On successive approximations in differential equations" by Professor C. O. Oakley, Haverford College.

3. "Some geometry associated with $\lim_{N\to\infty}(1+1/N)^N$" by Professor J. A. Benner, Lafayette College.

4. "Mathematics and poetry" by Professor C. N. Moore, Institute for Advanced Study.

Abstracts of the papers follow:

1. Professor Shohat showed that the well-known trapezoidal formula and Simpson's Rule for approximate evaluation of a definite integral, say, (1) $\int_a^{a+h}f(x)dx$, can be derived by a simple application of Taylor's Formula. We construct a function $\psi(h)=$ the product of h into a linear combination of $f(a)$, $f(a+h)$, $f(a+\theta_1 h)$, where $0<\theta_1<1$, with coefficients independent of h. The function $\psi(h)$ is so constructed that a certain number of terms, from the beginning, in its development by Taylor's Formula coincide with those in the Taylor development of (1).

2. One way of demonstrating the existence of a continuous, unique function $y(x)$ satisfying the differential system (1)

$$y^{[n]} + p_1 y^{[n-1]} + \cdots + p_n y = q; \tag{1}$$

$$y(c) = \alpha,\ y'(c) = \alpha',\ \cdots,\ y^{[n-1]}(c) = \alpha^{[n-1]}, \tag{2}$$

is by the so-called method of successive approximations—a method which, dating back to Cauchy and Liouville, was developed in its most general form by Picard and Bôcher. In so treating such a system as (1) (2), it has been customary to impose the (same) boundary conditions (2) on each approximating function $y_n(x)$ and its derivatives. In the present paper Professor Oakley removes this restriction and proves the existence theorem for system (1) (2) where the estimate-functions are required to satisfy a sequence of boundary conditions, i.e.,

$$y_n^{[i]}(c_n) = \alpha_n^{[i]}, \text{ with } c_n \to c,\ \alpha_n^{[i]} \to \alpha^{[i]}.$$

3. The polar equation of the curve whose radius vector is the magnitude of the limit approached by $(1+1/N)^N$ as N becomes infinite and the equation of the path traversed by the element $1/N$ as this limiting value is approached, were obtained by Professor Benner. Certain properties of these curves were examined and their relation to the geometry of growth discussed.

4. Professor Moore makes a comparison of mathematics and poetry on the basis of certain common aesthetic elements. The most important of these is considered to be the combination of breadth of thought with compact statement. Others utilized are elegance of form and wealth of creative imagination.

P. A. CARIS, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Maryland, on Saturday, December 8, 1934. The Chairman, Professor F. M. Weida of the George Washington University, presided over both sessions, morning and afternoon. President Ames of the Johns Hopkins University officially welcomed the members and their guests at the morning session. Five papers were read at the morning session while in the afternoon, at the invitation of the association, Professor G. T. Whyburn of the University of Virginia delivered a lecture on "Analytic topology."

The attendance was sixty-two, including the following thirty-six members of the Association: O. S. Adams, Beatrice Aitchison, G. A. Bingley, C. C. Bramble, Abraham Cohen, Tobias Dantzig, Alexander Dillingham, J. L. Dorroh, J. A. Duerksen, J. H. Edmonston, Mary Ewin, Michael Goldberg, T. N. E. Greville, Harry Gwinner, Isabel Harris, E. K. Haviland, L. M. Kells, W. D. Lambert, A. E. Landry, Florence Lewis, J. J. Luck, Florence M. Mears, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, A. W. Richeson, R. E. Root, J. M. Stetson, J. H. Taylor, John Tyler, J. F. Wardwell, F. M. Weida, C. H. Wheeler, G. T. Whyburn, John Williamson, E. W. Woolard.

The spring meeting will be held on May 11, 1935 at the George Washington University, Washington, D.C.

The following six papers were read:

1. "Properties of geodesics on polyhedrons" by Richard Kershner, The Johns Hopkins University, introduced by Professor Murnaghan.

2. "The Kakeya minimal problem" by Michael Goldberg, The Navy Department.

3. "The nth derivative of a function evaluated by means of the indeterminate equation $k_1+2k_2+\cdots+pk_p=0$" by Professor John Tyler, U. S. Naval Academy.

4. "Some irrational covariants" by Professor Frank Morley, The Johns Hopkins University, introduced by the Secretary.

5. "Remarks concerning the Liouville theorem on transcendentals" by C. W. Williams, University of Maryland, introduced by Professor Dantzig.

6. "Analytic topology" by Professor G. T. Whyburn, University of Virginia.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The paper discussed the geodesics on any polyhedron, which has the property that the sum of all angles at any corner is a rational multiple of π. Mr. Kershner showed that if a geodesic on such a surface be extended indefinitely it will be closed or lie everywhere dense on a subregion Γ of the original surface. This subregion must be the sum of a finite number of strips of parallel geodesics and be bounded by geodesics terminated by two corners. Further, any parallel geodesic in Γ is also everywhere dense in Γ. This subregion Γ need not be the whole polyhedron, but on any given polyhedron the exceptions to this case are countable. Any geodesic on one of the regular polyhedrons whose sides are triangles or squares is either closed or everywhere dense on the whole polyhedron. The case of the dodecahedron remains undecided.

2. The Kakeya minimal problem is the determination of the least area within which a straight line of given length can be completely rotated. Although the solution of the discontinuous case was published in 1920, and the slight modification to make it continuous was published in 1927, the authors of several recent publications seem to be unaware of the solution. Mr. Goldberg gave an exposition of the solution of Besicovitch (Math. Zeitschrift, 1927) in which it is shown that, by proper overlapping of the elementary triangles of a rotation, the area may be made as small as we please.

3. The object of the paper is to show that when we know the integer solutions of the indeterminate equation

$$k_1 + 2k_2 + 3k_3 + 4k_4 + \cdots + pk_p = n$$

we can write the nth derivative of a function. Professor Tyler divided his paper into two parts: (1) The evaluation of the nth derivative of a function when the independent variable is zero. (2) The evaluation of the nth derivative of a function for any value of the independent variable.

4. Consider four points on a circle, say α, β, γ, δ. They form four triangles, and each triangle has four contact circles. It is known that the sixteen centers of these circles lie by fours on eight lines, which form a rectangular system. Thus if we mean by $\bar{\alpha}\beta\gamma$ the escribed circle opposite to α, the scheme $\alpha\bar{\beta}\gamma\delta\bar{\alpha}\beta$, that is $\alpha\bar{\beta}\gamma$, $\bar{\beta}\gamma\delta$, $\gamma\delta\bar{\alpha}$, $\delta\bar{\alpha}\beta$, gives four circles whose centers lie on a line. Professor Morley proved that these four circles, and the original circle, are touched by a circle.

5. Liouville in 1851 established the existence of transcendentals, departing from a certain inequality which bears his name. It is the object of Mr. Williams to apply the method of Liouville to generating transcendentals by means of infinite series on the one hand, and infinite continued fractions on the other. In the first case, it is found that given any power series with positive terms, convergent for $|x|<1$, it is possible, by a process of elimination of whole groups of terms, to generate a transcendental number, even if the original series converges toward an algebraic limit. Similar results are obtained for the infinite continued fractions.

6. Analytic topology might be described as the group of results of topology which are conveniently expressible in the language of continuous transformations. Professor Whyburn covered certain aspects of the subject which will be indicated in what follows. Let A and B be bounded continua in a euclidean space and let $T(A)=B$ be a continuous transformation of A into B. The first problem considered is that of determining the possible images B of A when A is a linear interval and the converse problem for A when B is a linear interval. This brings in the continuous image theorem of Hahn-Mazurkiewicz, and for the converse problem results due to Čech and others are cited. The next general problem may be stated as follows: Given $T(A)=B$, what conditions on the transformation and its inverse will insure that B will be homeomorphic with A? The known solutions for this problem for the cases where A is a circle or a sphere due to R. L. Moore are given. They lead naturally to a consideration of monotone and of non-alternating transformations and of the relationships between A and its image B under such transformations when A is a circle, sphere, cactoid or boundary curve. It is shown that no satisfactory solution to the general problem may be expected so long as we limit ourselves to conditions on the sets $T^{-1}(b)$ alone. The condition that for each point b of B and each point x of A the sets $A-x$ and $A-T^{-1}(b)$ be homeomorphic is proposed for further study. This reduces, in every case where the problem has been solved, to the condition suited to that particular case; and it is hoped that it or some of its many possible modifications may yield solutions in other important cases, if not of the general problem.

JOHN WILLIAMSON, *Secretary*

FIRST MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The first regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on Saturday, February 10, 1934. Sessions were held at 10:30 and at 1:30, with a luncheon at 12:30. Professor C. S. Atchison, chairman of the section, presided at both sessions.

Seventy-one representatives of twenty-three educational institutions and research laboratories of Western Pennsylvania, West Virginia, and Eastern Ohio attended the meeting, including the following thirty-four members of the Association: C. S. Atchison, O. F. H. Bert, Helen Calkins, W. E. Cleland, Elizabeth B. Cowley, L. L. Dines, J. A. Eiesland, C. W. Foard, F. A. Foraker, N. C. Grimes, E. E. Hess, H. C. Hicks, R. C. Hildner, B. P. Hoover, R. P. Johnson, A. V. Karpov, C. W. MacGregor, W. W. McCormick, W. I. Miller, David Moskovitz, J. H. Neelley, E. G. Olds, N. C. Riggs, J. B. Rosenbach, E. A. Saibel, H. C. Shaub, C. S. Shively, J. C. Stayer, R. G. Sturm, J. S. Taylor, R. W. Thomas, C. H. Vehse, E. D. Wells, E. A. Whitman; and W. H. Cramblet, institutional member representative.

The following papers were presented:

1. Welcoming address by President Thomas S. Baker, Carnegie Institute of Technology.

2. "Numerical solution of partial differential equations" by Professor H. C. Hicks, Carnegie Institute of Technology.

3. "A note on parabolic anti-collineations" by Professor J. C. Stayer, Juniata College.

4. "A note on anti-involutions" by E. E. Hess, Huntingdon High School.

5, 6. Round Table Discussion. "What can the college do for students without adequate training in high school mathematics?" Discussion led by Professor W. H. Cramblet, Bethany College, and Professor F. A. Foraker, University of Pittsburgh.

7. "Flat-sphere geometry in non-Euclidean N-space" by R. H. Downing, West Virginia University, introduced by Professor Eiesland.

8. "An alignment chart for homogeneous functions" by Professor David Moskovitz, Carnegie Institute of Technology.

9. "A new method for evaluating double integrals" by W. L. Morris, Gulf Research Laboratory, introduced by Professor Taylor.

10. "The Ruled V_{n-1}^{n-1} in S_n with an $(n-2)$-fold S_{n-2}" by J. K. Stewart, West Virginia University, introduced by Professor Eiesland.

Abstracts of the papers follow:

1. Mathematics is playing a role of increasing importance in higher education. In addition to the increasing mathematical needs of engineering and physical science students, students of biology and of the social sciences are making a much wider use of mathematics. In addition, many students pursuing

a cultural course desire to become informed concerning the field of science in general, and thus find a need for the study of mathematics to a larger extent than formerly was the case.

2. The elements of the method of approximation to the solution of boundary value problems by means of finite differences were developed by Professor Hicks. The characteristic multipliers introduced by L. F. Richardson were determined for the solution of Laplace's equation in a rectangular region. The several types of approximation involved and their relation to applications of the method in physical problems were discussed. Conclusions were drawn as to the computational merits of the method in comparison with graphical and harmonic analysis methods.

3. With each parabolic anti-collineation there is associated, Professor Stayer pointed out, a pair of perpendicular lines through the fixed point having the property that the family of circles passing through the fixed point and with centers on one of these lines is left invariant as a family. Any point of the Argand plane will be the intersection (other than the fixed point) of two circles, one from each of the invariant families. The transform of the point will be the intersection of the transforms of these two circles, these transforms being readily obtained in terms of two constants associated with the anti-collineation.

4. The transform of a point under an anti-involution is very simply exhibited in terms of the circles through and the circles "about" a pair of interchanging points or a pair of fixed points (if the latter exists). In addition a very convenient method was deduced by Mr. Hess for distinguishing between the type of anti-involution having a circle of fixed points and the type having no fixed points.

5. Some colleges admit students of high scholastic standing who have taken little or no work in high school mathematics. Experiments at Bethany College in permitting some students of this type to undertake work in first year college mathematics have been more than fifty per cent successful.

6. An experiment with a small group of students who had failed in college algebra one or more times met with a certain amount of success. Emphasis was placed on the concept that there are only some twenty principles involved in the study of algebra. After some six weeks devoted to the study of these principles as applied to arithmetic operations it proved possible to complete those parts of college algebra needed in the succeeding course in mathematics, the mathematics of investment.

7. Taking a certain quadric as the absolute in a space S_{n-1} of odd dimensions (n even) the representation was found of the configurations of that space in an image space S_{n-1}. Following Study's method as applied in the case $n=4$ this orientation was found analytically by considering certain generators. Correspondences between surface elements in the two spaces were developed by Mr. Downing, certain contact transformations investigated, and a correspondence deduced between oriented "spheres" and $(n-2)/2$ flats in image space.

8. This paper is a summary and generalization of "An Alignment Chart for

Various Means" published in the American Mathematical Monthly for December 1933, pp. 592–596.

9. Among the various means employed for the evaluation of double integrals which are not directly integrable the method of transforming the region in question has been used primarily with a view to simplifying the nature of the region. If, instead, a transformation is sought whose jacobian is the reciprocal of the integrand, the transformed integral becomes a simple area integral and can be evaluated approximately by the use of a planimeter. Suitable transformations of this kind were exhibited by Mr. Morris for the types of integrands met most frequently in engineering practice.

10. The locus of the ∞^{n-2} lines incident to n given generic $(n-2)$-flats in S is a ruled V_{n-1}^{n-1}. The case was studied by Mr. Stewart of the V_{n-1}^{n-1} with an $(n-2)$-fold $(n-2)$-flat. For $n=3$ this is the quadric surface and for $n=4$ it is the Segre Variety in S_4 with a double plane. The equation of the surface was derived for $n=5$ and the geometry of the variety studied by the analytic method. The equation of the variety was then found in S_n and studied in a similar fashion.

J. S. TAYLOR, *Secretary*

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The second meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Washington and Jefferson College, Washington, Pennsylvania, on Saturday, May 5, 1934. Sessions were held at 10:30 and at 2:00, with a luncheon at 12:45. The morning session was opened with a welcoming address by President Ralph C. Hutchison of Washington and Jefferson College. Professor C. S. Atchison, chairman of the Section, presided at both sessions. The section was particularly honored by the presence of Professor Arnold Dresden, President of the Association, who delivered the principal address of the meeting.

Ninety-one persons were in attendance, including the following thirty-one members of the Association: C. S. Atchison, L. C. Bagby, O. F. H. Bert, Helen Calkins, Elizabeth B. Cowley, L. L. Dines, Arnold Dresden, J. A. Foberg, F. A. Foraker, W. O. Gordon, N. C. Grimes, E. E. Hess, B. P. Hoover, R. P. Johnson, W. I. Miller, T. W. Moore, David Moskovitz, L. T. Moston, E. G. Olds, D. M. Rasel, J. B. Rosenbach, E. A. Saibel, H. C. Shaub, C. S. Shively, J. C. Stayer, R. G. Sturm, J. S. Taylor, R. W. Thomas, C. C. Wagner, E. D. Wells, E. A. Whitman, and H. L. Black, institutional member representative.

The following seven papers were presented:

1. "A note on the four-space representation of chain congruences" (preliminary report) by Professor R. W. Thomas, Washington and Jefferson College.
2. "A problem in differential geometry" by Professor L. T. Moston, Waynesburg College.
3. "Generalizations of the calculus of variations" by Professor Arnold

Dresden, Swarthmore College, President of the Mathematical Association of America.

4. "A probability experiment for classes in college algebra" by Professor J. S. Taylor, University of Pittsburgh.

5. "A note on mixed boundary value problems in logarithmic potential theory" by Dr. Morris Muskat, Gulf Research Laboratory, introduced by the Secretary.

6. "Mathematics in research and research in mathematics" by R. G. Sturm, Aluminum Company of America.

7. "Should more algebra be taught in college?" by Professor H. L. Black, Westminster College.

The abstracts of these papers follow:

1. A chain congruence of points in the plane of two complex variables is a set of points projectively equivalent to the set of real points of the plane. Such a set of points may be represented in an ordinary real four-space by a two dimensional surface of the fourth order. The nature of this type of surface is illustrated by a numerical example, the method involving a "line-wise" factoring of the four-space interpretation of the projective transformation generating the chain congruence into the product of a set of transformations, each of which leaves the plane of reals invariant as a whole and transforms its associated line of points into a circle, and a set of "analytic rotations" of the four-space, each of which rotates its associated circle into its proper position on the surface in question.

2. A brief development of differential geometry is given based on an invariant method due to Professor Graustein. Two families of curves on the spherical representation of a surface are chosen as the curves of reference for the surface and a trihedral of reference at a point of the sphere selected. Modified directional derivatives are introduced to simplify the relation, in terms of ordinary directional derivatives, corresponding to the equality of second partial derivatives, of mixed variety, in reverse orders. Fundamental equations and the corresponding Gauss and Codazzi Equations are stated. In conclusion, an application of the general theory is made which results in an analogue of Dini's Theorem in terms of geometric quantities intrinsic to the spherical representation.

3. This paper discusses the work of Volterra, Hadamard, and Tonelli with reference to the maxima and minima of functionals, the generalized Lebesgue Problem in both the explicit and implicit form, the network method of Caratheodory applied to the problem of a k-fold integral as interpreted in $(n+k)$-space, and the Tonelli generalizations to functionals of the concepts of the theory of functions of a real variable. Throughout the discussion emphasis is placed on questions awaiting investigation.

4. Most students who study the elementary theory of probability do not learn the significance of the probability values which they compute. A new type

of class experiment in coin tossing is devised which not only leads to better understanding of fundamental concepts but also involves all the factors to be met in any scientific experiment. An analysis of these factors is made and the results of such an experiment presented.

5. A method is given for solving logarithmic potential problems in which the potential is preassigned over part of the boundary and the normal derivative over the remainder. A "mixed" Green's function is defined such that it vanishes over part of the boundary of interest and its normal derivative vanishes over the remainder, and the solution for the logarithmic potential is expressed in terms of this function and the boundary conditions by means of Green's theorem. Taking the case of the infinite quadrant as the "prototype" region the solutions for other regions such as the infinite half plane, infinite and semi-infinite strips, and the circle are obtained by mapping them on the infinite quadrant; these are illustrated by several specific examples.

6. Two different attitudes toward mathematics are indicated, one leading to research in mathematics, the other leading to the use of established mathematical laws in the research problems of industry. Closer cooperation between the workers in these two fields will result in the enrichment of both.

7. There is a place in undergraduate college mathematics for many portions of the subject of algebra which are not included in present curricula. A survey of such topics is given and an analysis of their values presented.

J. S. TAYLOR, *Secretary*

THE FIRST FALL MEETING OF THE MICHIGAN SECTION

The first fall meeting of the Michigan Section was held at the Michigan State College, East Lansing, Michigan, on Saturday December 1, 1934, Professor L. S. Johnston presiding.

The attendance was about seventy including the following thirty-two members of the Association: W. L. Ayres, W. D. Baten, J. B. Brandeberry, R. V. Churchill, R. W. Clack, A. H. Copeland, C. C. Craig, S. E. Crowe, J. D. Elder, Peter Field, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, H. S. Kaltenborn, C. E. Love, Paul Muehlman, A. L. Nelson, H. L. Olson, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, R. C. Shellenbarger, E. R. Sleight, G. G. Speeker, H. E. Vaughan, Jr., T. O. Walton, J. B. Winslow.

The morning session was devoted to the reading of papers and the afternoon session to an invited address by Julius D. Madaras and to a general session in which different members present spoke briefly on topics which they thought might interest the group present. For the Madaras lecture the Section had as its guests those attending the meetings of the Society for the Promotion of Engineering Education.

The following program was presented:

1. "Configurations connected with the symmetric group G_{24}" by C. A. Jacokes, Michigan State College, introduced by Professor V. G. Grove.

2. "Equations characterizing the 2-dimensional involution" by J. B. Thomson, Wayne University, introduced by Professor A. L. Nelson.

3. "Expansion of certain logical functions" by Professor A. H. Copeland, University of Michigan.

4. "A problem in series summation" by Leonide Vinokooroff, Albion College, introduced by Professor E. R. Sleight.

5. "The differential" by Professor V. C. Poor, University of Michigan, introduced by the Secretary.

6. "Some mathematical implications of the Chinese musical scale of 3000 B.C. as compared with our modern equally tempered scale" by Professor R. W. Clack, Alma College.

7. "The imaginary exponential function" by Professor A. F. Frumveller, University of Detroit, introduced by Professor L. S. Johnston.

8. "What University of Toledo students think about mathematics" by Professor J. B. Brandeberry, University of Toledo.

9. "Recent interests in mathematical statistics" by Professor W. D. Baten, University of Michigan.

10. "Application of fluid dynamics to obtain electric power from the air" by Julius D. Madaras, President, Madaras Rotor Power Corporation, by invitation of the Program Committee.

11. General session.

In the absence of the author, Professor A. L. Nelson presented the paper of Mr. Thomson. Abstracts of the papers follow, the numbers corresponding to those above:

1. The 24 points determined by applying the symmetric G_{24} to the homogeneous coordinates of a point in three dimensions were studied by Mr. C. A. Jacokes. The subgroups of order two associate the points by pairs on lines whose configurations were determined. One subgroup of order four associates the points by fours into tetrahedra having unique perspective properties. The subgroups of order eight link the various lines set up by determining those lines which intersect, and those which lie in planes.

2. The one-dimensional involution, defined as a self-inverse projective transformation of a line into itself, is characterized by the fact that the non-homogeneous coordinates of corresponding points satisfy the symmetric bilinear equation $Axy+B(x+y)+C=0$. Mr. Thomson derives the analogous symmetric bilinear system of equations for the two-dimensional involution.

3. Certain propositional functions admit of expansions closely analogous to power series developments. The logic used is that of Whitehead and Russell in *Principia Mathematica*. Professor Copeland defines two additive operators which can be applied to these expansions. One of these operators can be interpreted as the probability of an event (or propositional function) and the other

as the number of ways in which an event can occur. The application of these operators to the logical expansions produces a generalization of King's formula, a generalization of a formula developed by Whitworth, and a generalization of a formula of Whitney.

4. Mr. Vinokooroff showed that the sum of $n+1$ terms of the type $f(n+1)\cdot x^n$ may be expressed as

$$y_{n+1} = f(1) + xf(2 + xD)\frac{x^n - 1}{x - 1},$$

where $D=d/dx$; the convergence of this expression depends upon x and upon $f(n)$ itself. The summation process is further analyzed for $x=1$, and also for infinite series. Relations are developed between this and Euler's summation formula, which is treated as a solution of the type differential equation,

$$(I - e^{-D})y_n = f(n)\cdot x^{n-1},$$

where $D=d/dn$. This process is quite useful when $f(n)$ is algebraic, or expressible in a rapidly convergent power series.

5. In this paper Professor Poor showed how the differential may be introduced into the calculus earlier than is usually done, and how the confusion arising from current procedure may largely be obviated. It also attempted to show that the differential is more fundamental than the derivative.

6. Keyboard instruments introduced into music the problem of tuning so as to make possible modulation into any key. If the vibration rates of the tones in the true Major and Natural Minor Scales are written out for each of the 12 semitones of the octave as keynote, using standard International Pitch, to play them exactly would require 42 keys to the octave. The dilemma has been satisfactorily met since the time of Bach by using the Equally Tempered Scale, which divides the octave into 12 equal intervals, the successive ratios being as 1 to $2^{1/12}$. According to the Book of Rites (c. 8th cent. B.C.) Ling Lun, a Minister of State in about 2700 B.C. systematized the Chinese scale by cutting 12 pipes, each 2/3 the length of the one next longer. These gave tones differing by the ratio 3/2 (a perfect fifth), and when all are brought down into the same octave they give the chromatic scale, the upper notes being slightly flat. This scale differs from the equally tempered scale considerably less than the equally tempered differs from the diatonic.

7. Professor Frumveller believes that De Moivre's Theorem and Euler's formulas in terms of exp (ix) should be included in every course in Trigonometry. As the only proofs offered for Euler's equations are based on series expansions for $\sin x$, $\cos x$, and exp (x), which must be assumed, the paper presents a simple deduction of Euler's formula from De Moivre's Theorem without using infinite series. A rigorous deduction would still demand infinite series, but it is interesting to observe that the general form of Euler's equation can be deduced without their aid.

8. Professor Brandeberry reported that, in a survey of students at the Uni-

versity of Toledo made at the end of the second semester of 1932–33, out of 39 departments the one in Mathematics ranked first on the question "how satisfactory were the type of quizzes" and fifth on the question "to what extent do the grades given represent the proper standing of the students."

9. Professor Baten illustrated how interest in probability and in mathematical statistics had grown in the past two decades by the following facts; (a) most universities and colleges in the United States have introduced these subjects into their curricula during this period; (b) six new journals devote their pages entirely to these subjects; (c) more articles on these topics appeared in scientific journals in this interval than formerly; (d) a large majority of books in these subjects in the library of the University of Michigan have been written in this period; (e) one of our largest universities permits students in certain fields to replace one foreign language requirement by one year of mathematical statistics; (f) a large number of other sciences use these subjects as tools; and (g) research in these fields is being done the world over.

11. Professor Norman Anning mentioned the report of Lietzmann on the teaching of secondary mathematics in the United States. Professor H. L. Olson spoke on the condition that a tetrahedron be orthocentric. Professor E. R. Sleight called attention to the importance of college geometry in the preparation of high school teachers. Mr. D. Kazarinoff spoke of some relations between maxima and minima problems and mechanics. Professor S. E. Crowe told of an experiment being tried at Michigan State College of having the papers graded by others instead of the instructor of the course. Professor L. S. Johnston mentioned several topics, particularly a simple device for finding the angle of rotation to simplify the equation of a conic. Professor C. C. Richtmeyer told of the success of an experiment in teaching extension classes at Central State Teachers College. A small group wanted several different courses so the class was put on an individual basis and the length of the courses was adapted to individual needs.

W. L. AYRES, *Secretary*

A PROGRAM FOR MATHEMATICS*

By ARNOLD DRESDEN, Swarthmore College

As the end of the term drew near during which I have had the honor to serve as president of the Association and as it thus became imperative to reflect upon a subject for the address by means of which the retirement could be expected to be carried through safely, I was cheered by the thought that the occasion represents an opportunity and thus a real obligation as well as a purely formal duty. For it furnishes a larger and presumably more attentive audience than can usually be expected for the presentation of personal views. It is not often

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that I have been able to indulge my thoughts publicly at so little risk. In order to avoid undue strain on the indulgence of the audience, I have tried to select a topic which is of concern to a large part of my listeners as well as to myself. I hope that I have not been too far wrong in my estimate that the problems which I am about to discuss with you will commend themselves to you as demanding serious consideration.

The unrest and the dissatisfaction with the teaching of mathematics in the schools and the colleges, which are making themselves felt with increasing strength throughout the land, confront us with a serious and difficult problem. When schools in many parts of the country remove mathematics from the list of required subjects and replace the requirement by a feeble recommendation, when colleges and universities open their gates to students who have not studied this subject, we must at least take notice. Having taken notice of the phenomenon, one may take the view that there is no reason why mathematics should occupy a more sheltered position in the curriculum than physics or chemistry, latin or history, sewing and typewriting; that our subject should stand on its own merits and that in so doing it would attract students who bring to it a strong interest and special ability, and that there is not much gain for any one in bothering with the others. This laissez-faire point of view can be supported by strong arguments. I am not without appreciation of it. There may be a good deal of wisdom in the belief that the best way to treat the opposition to mathematics is to ignore it, that by giving it free scope it will most rapidly meet the fate it is thought to deserve. But let us not make the mistake of thinking that all this opposition arises from ignorance of the subject and from a lack of understanding of its significance. The laissez-faire attitude would have a stronger claim on my support, and mathematics could more easily be expected to stand on its merits, if the subject had a better chance to present its merits than it actually has.

There is no denying the fact that the sheltered position which mathematics has occupied for many generations in our schools and colleges has brought about a good deal of decay—not a rare occurrence in human organisms and among groups of men, who live on special privilege. Too much reliance has been placed on accepted standards of excellence both in teaching procedure and in choice of subject matter, too little contact has been maintained with the developments and the changes in the science and in related fields, too little notice has been taken of the profound modifications in attitude and aspirations of the population with which we have to deal.

I am not forgetful of the important work done by the committee of which one of my regretted predecessors was chairman. But the report of this committee was published in 1923 and was concerned exclusively with secondary education. Moreover, even for that restricted field it must be looked upon as a first step, not as the only one necessary for the rehabilitation of our subject. And it is precisely because I want to see it rehabilitated that my sympathy with the laissez-faire school is considerably tempered.

Young people in school and also in college, certainly if they do not go beyond the work of the freshman year, rarely have the opportunity to learn what mathematics is really about, to become imbued with its spirit. Because it is moreover my firm conviction that both the content and the spirit of mathematics have a great deal to contribute to the education of the individual, and that given a proper share in the education and training of our people, mathematics can wield a powerful influence in that reshaping of our world of which the present witnesses perhaps the first significant stirrings, that is why I must look upon the problem which the happenings of the day put before our profession as one that calls for an attitude more active than that of laissez-faire.

For those of you who share these convictions, no further argument is necessary. Others should be reminded of the fact that, quite apart from the doubtless sordid but none the less real bread-and-butter considerations, the mathematical profession cannot expect to prosper in times when its aims are misunderstood, its practices maligned and its development hampered in powerful places, nor in an environment in which it lacks the esteem that should be accorded an occupation which is devoted in a very real way to the highest aspirations of mankind.

These convictions suggest at once a very definite and three-partite program for mathematicians: 1. To make clear the specific aspects through which mathematics contributes to these high purposes; 2. to determine the relation of these aspects to the rest of the subject matter, and 3. to devise ways in which they can be made effective in the teaching of the subject at its various levels. It is in the nature of the present occasion that I should not attempt more than to sketch some of the ideas that may serve as guides in the elaboration of such a program. This I shall do by calling attention to a few important characteristics of mathematical procedure.

In insisting upon an abstract formulation of the problems with which it deals, mathematics gives to such problems a quality of universality. The particular problem is thought of as a special case in a set of problems—the set is the abstraction derived from the particular cases. In the same sense the abstract concepts of mathematics and the "universals" of the logician are to be understood. A striking illustration of this character of abstract concepts is found in the definition of a natural number. Whereas in earlier definitions, "three" was the "quality common to all triples," the form most commonly accepted now is that of the "class of all triples" (see, e.g., Russell, *Introduction to mathematical philosophy*, p. 18).

But we do not have to go to the philosophy of mathematics for illustrations of our position. It is a commonplace to say that we have not fully understood the problem involved in the solution of linear equations until we have formulated and solved it for any number of equations, any number of variables and coefficients arbitrary in a field, i.e., until we have dealt with it abstractly. In elementary geometry, the determination of a triangle from three given sides has not been completely understood by a pupil who has been limited to the con-

struction of triangles of 4, 5 and 7 inches, or 12, 15 and 20 centimeters, or any number of special cases, but who has not considered the "general" problem of constructing a triangle whose sides are a, b and c. I trust that it would be superfluous to enlarge still further upon the simple point that throughout mathematics, from the most elementary to the most advanced, the abstract formulation of concepts and problems carries with it extension of their scope of application. This breadth of scope goes hand in hand quite naturally with the use of a symbolism which is independent of the national limitations of language.

There seems to be little doubt that this preoccupation with concepts of ever widening scope, this concern with matters which transcend so largely restricted group interests, gives to mathematics a very individual quality, not easy to describe or to specify but manifest at least occasionally to discerning observers outside the field of mathematics. In a fine essay on "cosmopolitanism," the contemporary Spanish philosopher Ortega y Gasset observes that in the second half of the nineteenth century, when there was an unprecedented separatism in the intellectual life in Europe, the "mathematicians—a handful of people, who were scattered over the entire surface of the earth—formed a most remarkable and spontaneous community so small and so intimate in character as to have the nature of a family, with the mutual confidence and the home atmosphere which characterize such a group," (see, Ortega y Gasset, *Das Buch des Betrachters*, page 11). It is perhaps this same quality of mathematics which is at least partly responsible for the attitude which the Hitler regime in Germany has taken towards the subject—mathematics does not contribute sufficiently to the national purposes to find favor with the advocates of unrestricted nationalism.

But, in spite of the apparent success of excessive nationalistic aspirations in many parts of the world at the present time, I venture the opinion that if there is to be true progress, any fundamental betterment in human existence, the advance must be in the direction of enlarged application of general principles, of the discovery of new principles of wider scope, and of a world wide envisagement of the important problems. Thus in no small measure will the development of man's existence on this earth depend upon his ability to grasp and appreciate more inclusive generalities, to deal intelligently with vaster ranges of human and natural phenomena. In the cultivation of such ability the abstract formulation of problems should play an increasingly important role. The clarification of the bearing of this aspect of mathematics on those general problems constitutes indeed an important part of point 1 in my program, viz., the specification of the contribution of mathematics to the development of human life. It is hardly necessary to say anything concerning its significance for the second part, since the formulation of abstract concepts is the very essence of the subject of mathematics, its warp and woof, the source of its power and of its appeal for many of us.

But it may not be superfluous to consider this aspect of mathematics with reference to the third part of the program; that is, to give some indication of the

way in which it can be made to influence effectively even the elementary parts of the subject. It is a common but in my judgment erroneous belief that abstract concepts cannot be grasped by young people. The error arises perhaps from the mistaken absolute notions as to concrete and abstract. What forms a familiar part of our experience, that is concrete. The abstractions based on such experience and which are extrapolations from it begin by being unfamiliar; gradually as experience grows up around them, they lose their strangeness and they are not distinguishable except by mental effort from the concrete environment out of which they arose. Thus the abstractions of last year become concretions for next year. Without involving ourselves in the metaphysical question as to whether there is anything absolutely concrete, we recognize at least that for our purpose there is no sharp contrast between abstract and concrete. As soon at least as a child remembers experiences and confronts memories with actualities, he has started his abstracting. What is important is that there be a fairly accurate estimate of what is familiar in his experience and what is still strange; to this estimate the abstractions are to be adjusted. What is needed furthermore is that the abstract concepts of which certainly even young children have a good many, be recognized as such and that the inclusiveness of the concepts with which the teaching is concerned, receives sufficient emphasis. The natural numbers, with which such a large part of the school years is occupied, provide an excellent opportunity for exemplification of this inclusiveness. If the exercises in arithmetic were imbued with the spirit of mathematics, they would not be merely tests of skill, but also challenges to the imagination. Once a child has seen that the sums in arithmetic with which he is made to wrestle have significance in a great diversity of conditions, in other times and places besides those in which he happens to find himself, they are illumined by a new light. Am I wrong in thinking that much of the teaching of arithmetic in our schools fails to extract this significance? In similar manner the first contacts with algebra would give a thrill to the young student if they could make him realize that the formulas with which he is asked to operate are not tricks to be memorized, but universal statements whose universe of significance spreads over a vast field in human experience. As our mathematical concepts are extended, the abstractions of the earlier stage become concrete instances of those in the next stage. Ever wider becomes the scope of the concepts, ever more general the methods. But that is a familiar story. What is perhaps not familiar is that we do not have to wait until the years of graduate study to realize that mathematics deals with abstract concepts and that the significance of this fact can be made to penetrate the earlier years of mathematical experience. No longer need instruction in arithmetic merely serve the purpose of drilling children in the skills necessary in the shop and the counting-house; indeed this purpose itself is usually defeated by the excess of emphasis on purely formal aspects. No longer should the instruction in geometry be dominated by drill in memorizing "proofs" of propositions; nor that in algebra by the "doing" of long rows of dull exercises. Most teachers know that the skills obtained in this way rarely survive

a summer vacation, and that formal manipulative ability not based on insight into the abstract character of the concepts, furnishes no foundations for later work; neither are they in themselves valuable accomplishments. Keeping actively before the mind the connection between the abstract concepts and the concrete instances from which they are derived, both the skill which the mathematical technician needs and the insight which gives mathematics its non-technical values, can be secured at every level of mathematical training. The highly trained specialist should have acquired enough experience in mathematics to have a concrete substratum for the abstract concepts with which he deals, or to obtain such background as he may need. A person who has had only limited experience in the field of analysis, for whom the central facts of real variable theory have not become concrete, lacks a basis for the understanding of any form of "general analysis." He may be able to follow the developments of the theory and the arguments in a proof, step by step; but he will be bewildered, he will not become enlightened. The knowledge he may acquire will not stay with him for very long; it is bound to die because it has not struck root in his mind. This example is introduced not because it in itself constitutes a serious problem, but rather because the situation it represents is perhaps more familiar to many of my listeners than are the conditions in the earlier stages of mathematical education. A person such as we have pictured will recognize his defects and will know how to remedy them.

At the lower levels great care must be taken that the experience of the student will contain the elements from which mathematical concepts may be abstracted. It may indeed be necessary to set up types of experience artificially so that mathematical content may be drawn from them. If this is to be carried out effectively, it must be done in a different spirit from that which animates much of the "practical" work that has been introduced in some schools, such as keeping accounts for a newspaper or the school theatricals. Not to provide applications, which are to take the place of mathematics, but to supply foundations on which a mathematical structure can be built, should be their aim. A similar remark could be made concerning the "applied problems" in many parts of school and college mathematics, which frequently use the language of the real world only to introduce situations which are very far removed from any actual experience. This then must be an essential number on this part of our program: so to select and to arrange experiences from what is available at each stage that the abstract concepts of mathematics can be extracted from them. If this is done systematically from the very beginning of mathematical instruction, then no matter at what period the instruction ceases, something uniquely valuable has been secured. Moreover, there has been acquired a basis on which a sound and reliable technique can be built.

I must now pass on to the consideration of another part of the program, viz., that arising from the unique way in which mathematics reaches its conclusions. It is usually said that mathematics is a deductive science. As far as can be judged by actual practice in the schools and colleges, geometry is treated

to some extent at least in accordance with this characterization, algebra in a manner in flagrant contradiction to it and the other fields of undergraduate instruction seem to occupy a very uncertain position with respect to it. The fact that it is a deductive science is of tremendous significance for the role of mathematics in the educational scheme. For it cannot be doubted that to be able to see or not to be able to see the logical, i.e., tautological and therefore incontrovertible, deductions to be obtained from a given proposition makes a very great difference in man's capacity to deal with his problems. It is not a question here whether or not logic is to reign supreme, nor whether we shall always want to follow the logical deductions of a principle we have adopted. Neither does it matter which particular type of logic we adopt, whether Aristotelian or not, whether based on a two-valued or a multiple-valued truth-function. Rather are we concerned with the ability to derive and to envisage logical consequences. Deficiency in that ability is sadly evident in many of the utterances of public men the world over; to remove it is a sore need if the new order towards which we are tending is to be nearer to perfection that the present. To this task mathematics has an important contribution to make. Consequently, the subject should be so presented as to bring out prominently its deductive character, so that no matter to what stage mathematical education is carried, it will have exerted an influence towards that end. Can this be done? The study of this question is a significant part of the program with which mathematics is confronted at present; I think it can.

Let us not be frightened by the old bogey of non-transfer of training. If anything can have a positive effect on young people, the force of example and the formation of habits certainly can. Having had to occupy themselves over a period of many years with a subject permeated by the deductive method cannot remain entirely without effect. Insistence upon deductive reasoning in their own thinking can have the same result as the use of habit-forming drugs. It may become a pernicious habit; and it may very well turn out to be a serious problem to limit its effects. But perhaps we need not yet be alarmed on that score; we can safely urge the deductive procedure as the dominant method in mathematical teaching. Moreover there seems to be an interesting and significant link between the deductive method in mathematics and the abstract character of its content. For one of the characteristics of a deductive science is the fact that it is based on a set of undefined ideas and unproved propositions. Within limits with which we are all familiar and which have caused and are still causing untold distress, these ideas and propositions may be chosen quite arbitrarily. But, as a matter of fact, the arbitrariness of the choice is always a very mild one. The particular concepts and propositions which are chosen have some particular reason for being chosen; and the reason is that they are abstractions from experience in a restricted portion of mathematics, or in the wider fields of man's struggle with his environment. Thus the two characters of mathematics which have thus far entered into our program are not without connection.

The absorption into the teaching of mathematics of the second certainly

would carry the first one along with it. I will not stress the point that the converse of this assertion is also defensible. It would be a worthy task for any mathematician so to arrange instruction in the subject from the earliest stages to the more advanced ones as to put both the abstract character of its content and the deductive nature of its methods in the strongest possible light. It goes without saying that this requires that we be not afraid to put something hard before our pupils—the experienced and interested teacher will always know how far to go in straining the capacities of his audience. It would certainly lift algebra out of the sad state of being a collection of tricks; it would remove the fundamental concept of limit out of the anaemic condition in which it now exists for most college students into a more healthy life, and the simple theorems concerning limits would come out of the libido of unfulfilled desires.

It is an ineffective remedy for the unpopularity of mathematics to remove from it the difficult portions. It was not with this in mind, I am sure, that the National Committee recommended some years ago the elimination of a good many "hard" portions of the traditional curriculum; what determined this suggestion was their unrelatedness to the principal aim the Committee laid down for the teaching of mathematics in the secondary schools. On page 10/11 of the report of the Committee we read: "The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual." Again, "Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take." There is little that I should want to change in these statements, but perhaps the emphasis is too exclusively on the physical aspects of our environment and this is thought of too much as in a fixed state—conditions were that way half a generation ago. It is not only relations of quantity and space that we want to understand, but also relations among human beings and among groups of them. Moreover it must not be forgotten that the experience which young people have had with the physical environment by the time they study mathematics in school and college is of a rather restricted kind. We must expect the teaching of mathematics to contribute to the insight into the forces which influence other phases of human life besides those which involve quantitative and spatial relations. Among the most important of these contributions is an understanding of the significance of the abstract formulation of problems and of the character of a deductive science.

The third element of mathematics upon which I want to dwell is the existence theorem. It has taken mathematicians a good many years to learn to distinguish between a purely formal constructive process and a constructive process which rests upon an existence theorem. Examples of this sort are well

known in the theory of differential equations and in the operations with infinite series. It has less frequently been recognized that the formal processes used in the solution of linear equations have value only because there is a theorem which assures the existence of solutions under certain conditions. There is a temptation to carry these ideas further, and also to discuss the relation between existence theorems and constructive processes in general. The latter temptation I must resist, the former I shall postpone for a few moments. What is of consequence for our immediate purpose is to recognize that an existence theorem, particularly if it asserts unique existence, gives significance to procedures which otherwise would be purely formal. Before we set out on an elaborate quest for something or other we may well inquire as to the conditions under which the thing we are looking for is known to exist. The application of such a point of view to the study of social questions would unquestionably have a profound effect. We have only to suggest that those who are directing the affairs of nations should consider it as an essential part of the efforts to establish world peace to inquire whether this ideal state can exist no matter what may be the economic structure of society; or that those who are responsible for the educational policies of a district should inquire as to the conditions necessary for the existence of the educational ideal which they pursue. Insistence upon at least a consideration, if not a solution of the existence problem in connection with social and economic policies would exercise a very significant, and I think wholesome influence upon the development of human life. Mathematical education can implant an understanding of the significance of this problem; we should not be satisfied with teaching that does not make this contribution to the education of its pupils.

It is part of our program to determine how this can be done on the various levels of mathematical instruction. As a possible contribution to this part I shall submit a few remarks. One reason why there is sense in learning the multiplication table is that, given any two integers, there always exists one and only one integer equal to the product of these two. In other words, in the set of natural numbers multiplication is possible uniquely and without restrictions, i.e., we have an existence theorem for a unique product of two natural numbers in the set of natural numbers. Incidentally, a teacher aware of this uniqueness could make good use of it to lend an adventurous interest to multiplication exercises; children are readily thrilled by the hunt for something unique. On the high school level, the solution of equations which occupies so large a part of time spent on algebra, very definitely calls for an existence theorem. For an equation (which, by the way, should always be followed by an interrogation mark because it is not a statement but a question) may have no answer, one or several. It is rather important, before we spend much time on developing skill in "solving equations" to know whether any answers exist. The experienced teacher will easily recognize that such a point of view can moreover be of great help in the clarification of several points in connection with the solution of different types of equations. There is no reason then why the point of view which

is generally accepted in more advanced parts of mathematics, e.g., in the theory of differential equations, should need to be abandoned in the more elementary parts of the subject. If it is abandoned, we sacrifice to a great extent the influence which mathematics should exert and the place it should occupy in the general educational scheme.

These are some of the points of the program for mathematics of which the times appear to force a consideration. I hope that I have gone far enough to indicate a general outline, although incomplete and perhaps vague at several points. The task of specification and of detailed working out must be left to the mathematicians of the country. It is not an easy one; if it were there would be little point in suggesting it. It calls for hard work over a long period of time; indeed it does not appear to have a natural termination. Once we adopt the point of view which it represents, we shall find more and more ways in which mathematics can and should contribute to the clarification and perhaps the solution of the problems with which mankind has to struggle. The refashioning of the world in which we live is not a task which we can think of as ever completed; it will probably continue indefinitely. As the process develops, it will call for increasing understanding of the forces through which it operates. If my view is not altogether wrong, mathematics can do a great deal to bring into active participation those qualities of the human mind which will have to come into play.

Of the mathematician this program demands a great deal of work, a technical preparation far beyond the immediate needs of any teaching work, constant occupation with new developments in his own field and in related domains, an attitude of hospitality towards new ideas and new presentations of old ideas, in short an active scientific life. Quite apart from its significance for the questions we have been discussing, this sort of activity brings its own rewards.

The need for a shift in emphasis in the teaching of mathematics of which I have spoken is one phase of a more general change in orientation which is demanded of all science teaching. From a recent speech of John Dewey, I should like to quote the following (see John Dewey, The supreme intellectual obligation, Science, March 16, 1934, vol. 79, page 40): "The obligations incumbent upon science cannot be met until its representatives cease to be contented with having a multiplicity of courses in various sciences represented in the schools, and devote even more energy than was spent in getting a place for science in the curriculum, to seeing to it that the sciences which are taught are themselves more concerned about creating a certain mental attitude than they are about purveying a fixed body of information or about preparing a small number of persons for the further specialized pursuit of some particular science. I do not mean of course that every opportunity should not be afforded the comparatively small number of selected minds that have both taste and capacity for advanced work in a chosen field of science. But I do mean that the responsibility of science cannot be fulfilled by educational methods that are chiefly concerned with

the self-perpetuation of specialized science to the neglect of influencing the much larger number to adopt into the very make-up of their minds those attitudes of open-mindedness, intellectual integrity, observation and interest in testing their opinions and beliefs that are characteristic of the scientific attitude." There are many more utterances of this and of other thinkers which definitely foreshadow a widespread need and demand for an attitude quite different from the traditional one towards the role of education in our social organization. It must have become clear that in my opinion mathematics has a very significant part to play in such a reorientation of the educational enterprise, and I have tried to indicate some of the ways in which the part may be envisaged. It is a challenge to the intelligence, the insight and the devotion of mathematicians. To the end that you may accept this challenge, to the end that mathematics may bring to bear upon these important matters the wisdom it has garnered and abstracted from the age-long experience of the race, I have submitted this program to you.

THE NEW MATHEMATICS "REQUIREMENT" AT THE UNIVERSITY OF WISCONSIN

By R. E. LANGER, University of Wisconsin

The movement to abolish the requirement of algebra and geometry for entrance to the College or the University is widespread. It has therefore become a matter of serious concern to those who are firm in their conviction that in a study of these subjects there is value, as well in the facts which they present as in the mode of thought which is characteristic of them. The question matured at Wisconsin during the past academic year to the point where new legislation by the University regarding entrance requirements became necessary. Whether or not the solution reached is a fair and sane one may perhaps be left to individual opinion; to some, at least, it seems that it may be the basis for a real strengthening rather than a weakening of the position of mathematics in the secondary school curriculum. The following discussion falls essentially into two parts and consists, in the first place, of a brief review of an analysis of the problem. This is followed, in the second place, by a statement of the conclusions which were recently adopted. The discussion is submitted here in the belief that it may be of interest, and that some measure of guidance to others faced with the prospect of having to grapple with similar situations may be contained in it.

The facts, symbolisms, and operational processes of mathematics are universal, peculiarly permanent, and of unquestioned importance. They are indeed the very web upon which the fabric of our modern scientific civilization is woven. Modest though it may be, an acquaintance with the language and mode of reasoning of mathematics, or a manipulative facility with its processes should be unchallengeable as a worthy element of the equipment of any person of general culture. Traditionally the study of mathematics has almost universally

been regarded as necessary, and in this respect as secondary only to the study of the mother tongue. It has accordingly been a "requirement" for admission to College,—one of which the justice and desirability are now widely under attack.

Mathematics plays continually in the dual roles of *the cultural subject* and of *the tool.* Not that these aspects are unrelated or divorceable,—certainly not that they are antagonistic. On the contrary, much of the richness of mathematical study is ascribable to the interplay of the aesthetically pleasing with the logically satisfying and the concretely useful. As a *cultural subject* mathematics must share the field with many others, and for some of these similar and perhaps even equally significant claims may be put forth. Mathematical thought is a distinct type of thought, a prominent, serious, and important type of thought. Yet the fact must be faced that the range of worthy human interests is large, and that no student, however earnest, can sample all subjects. A selection is imperative, mathematics in this role is one of many,—certainly a commendable one, but perhaps hardly a necessity.

In the capacity of a *tool* mathematics is the servant of every individual, and is an essential in the large majority of fields of practical pursuits and intellectual interests. There is no need to recapitulate here the manifold instances and situations in daily life in which a knowledge of and facility in mathematical facts and processes is a desideratum. If applications do not always or even frequently confront us in the trite form and garb of the examples of the geometry or algebra textbook, they are nevertheless there. Our reactions to the geometrical matters of shape, proportion, orientation, etc., are in large measure as subconscious as they are to matters of syntax, color and melody. Moreover, an important, even if incidental, function of the study of algebra is that of establishing and fixing an intelligent understanding and control over the subject matter of arithmetic. Unless the mathematics courses of the high school are taken, a student comes to graduation from four to six years removed from his experience with arithmetic in the grade school. He lacks the most elementary conceptions and techniques of numerical relationships and operations, of proportionalities, of graphical representations or of tables,—indeed, of any kind of quantitative thinking. These subjects are eminently suited to the high school. With the trend of all sciences, social as well as natural, toward concepts and methods of greater precision, it seems inevitable that educated thought of all kinds will follow much the same trend. Can it be that a judgment ascribing to algebra and geometry little or no value for students not planning to go beyond the high school, is a superficial one? It may perhaps be a defensible opinion that in so far as the college entrance requirement has indirectly induced many students not actually bound for college to study the mathematical subjects, it has been a salutary influence in many instances.

With regard to the college "requirement" as it applies immediately to students destined for college, one must admit that there are such things as high and useful careers in which mathematical thought and activity play but a minor role. To the extent that it bears upon students unalterably dedicated to

such careers it is difficult to justify the "requirement" and the desires of secondary school administrators for greater flexibility in the preparation of such students would seem to deserve sympathetic consideration. Such students, however, are rare. It is not generally the case that correct and abiding judgments as to one's ultimate interests can be made at the high school stage. Where an error in the matter of preparation precludes further work in a single subject or in a few related subjects, the matter is perhaps of lesser consequence, inasmuch as the subjects concerned probably will and can be avoided. Of much greater moment is such an error when as a result of it the student closes upon himself the door to the great majority of fields of serious university activity. A failure to study the mathematics of the high school has precisely this effect, as will be evident from the statement which is given below. Most college subjects can, to be sure, be studied in their elementary aspects without mathematics, but the situation is vastly different when advanced study is concerned. Unrestricted use of the University's faculties and facilities is possible only for those who have made the mathematical background their own. Without algebra and geometry the student is restricted essentially to the fields of letters, languages, history, music, and journalism.

It seems important under the circumstances, firstly, that the facilities for the study of the mathematical subjects in the high school be guarded against any impairment, and that under no circumstances they be allowed to deteriorate from their present status. Secondly, it seems important that students be brought at an early stage to a realization of the role played by mathematics both as a subject in itself, and as a buttress to innumerable others many of which are not obviously related to it. If these matters are properly safeguarded, and it seems certain that under any real educational statesmanship they will be, the removal of the "requirement" of mathematics may not be of extreme moment nor entirely an unmixed evil. Compulsion carries with it all too frequently an odium potent in engendering antipathy and ill will. With the advent of difficulties the "requirement" becomes irksome and resentment may be born where a challenge to better effort might otherwise have sprung forth. Learning in such cases can hardly be other than perfunctory and valueless, and reflection upon a train of such experiences may often lie at the root of a lifelong disdain and antagonism for the subject.

The manner in which the situation is to be met at Wisconsin is briefly summarized in the statement which follows. The new and salient feature will be found in the differentiation between "unrestricted" and "restricted" admission or standing in the University determined on the basis of whether mathematics was or was not included in the secondary preparation. It should be stated that inclusion in the list of subjects for which "unrestricted" standing, i.e., *a background of mathematics*, is to be a requisite was in each case made at the instance of the department concerned. In no case was any solicitation or influence of "professional mathematicians" involved. In conclusion, be it mentioned that the Letters and Science faculty at Wisconsin—a body of linguists,

historians, men of letters, philosophers, economists, scientists, etc.—expressed itself almost unanimously as favoring and recommending for prospective college students the study of algebra and plane geometry in the high school.

METHODS OF ADMISSION TO THE UNIVERSITY OF WISCONSIN

Statement

Admission to and standing in the University may be on the "restricted" or the "unrestricted" basis, the designations having reference to the freedom with which the various courses and fields of work offered by the University may be chosen. "Unrestricted" standing is available only to students who have included mathematics (as specified below) in their preparatory training. "Restricted" standing may at any time at the option of the student be changed to "unrestricted" standing by the mastery, subsequent to high school graduation, of the content of high school algebra and plane geometry by private study, tutoring, or correspondence study. The University will not provide resident instruction in this preparatory work.

Definitions

"Unrestricted" admission to the University is admission which opens to the student all Colleges, Courses, and fields of study to which freshmen are eligible and insures full freedom of choice among all the college majors and fields of specialization.

"Restricted" admission opens to the student such Colleges, Courses, and fields of specialization *as do not require high school mathematics as background.* It does *not* give admission to the College of Agriculture or the College of Engineering or the Course in Chemistry, and does *not* permit the student to major or specialize in chemistry, commerce, economics, mathematics, pharmacy, political science, pre-medicine, philosophy, psychology or sociology, or in any of the other natural sciences including physical geography and geology, or to graduate from the School of Education with a major or minor in any of these fields.

A "major" consists of three or more units in one field of study.

A "minor" consists of two units in one field of study.

Fields of study:

"Group A"

(1) English and Speech	(3) History and the Social Sciences	(5) Natural Sciences
(2) Mathematics	(4) Foreign Languages	(6) Advanced Applied Music or Art

"Group B"

(1) Agriculture	(3) Home Economics	(5) Mechanical Drawing
(2) Commercial Subjects	(4) Industrial Arts	(6) Optional (2 units)

Methods

I. By presenting a certificate of graduation from an accredited high-school

bearing the principal's recommendation of the candidate's fitness for college, and showing satisfaction of the following requirements:

(i) For graduates of regular four-year high schools. Sixteen units, including not more than six from fields in group B, and including two majors and two minors in fields of group A. One major or minor shall be in English and Speech; and

for "unrestricted" admission one major or minor shall be mathematics, i.e., one unit of algebra and one unit of plane geometry, with an additional half-unit of algebra in the case of those who seek unqualified admission to the College of Engineering.

(ii) For graduates of high schools which maintain a senior high school division. Twelve units from this division including one major and two minors or four minors in fields of group A. One major or minor shall be in English and Speech; and

for "unrestricted" admission one major or minor shall be mathematics (as described above) unless, before entering the senior high school, the entrant has completed one of the two units in mathematics specified in the preceding sub-section, in which case the completion of the second unit will suffice.

II. High school graduates need not meet the above requirements if, on the combined basis of rank in graduating class and aptitude and achievement tests satisfactory to the University, they stand in the upper twenty-five per cent of the average freshman class entering the University and are recommended for college by the high school principal; *but their admission will necessarily be "restricted" unless their preparation includes the two units in mathematics.*

III. On the basis of entrance examinations, as in the past.

CONFORMAL AND EQUIAREAL WORLD MAPS*

By B. H. BROWN, Dartmouth College

1. *Introduction.* The surface of the earth is not developable; that is, it can not be spread out in a plane without stretching. If we consider even an infinitesimal region of the earth, the map of this region must, in general, be distorted either in area or in shape; it may be distorted in both. But if for every infinitesimal region there is no distortion in shape the map is said to be *conformal*: if there is no distortion in area, *equiareal.* An equiareal map is equiareal over all. Conformality, however, is a property of an infinitesimal region only; it is wrong to say that a conformal plane map of Greenland has the shape of Greenland, for Greenland does not possess a plane shape. A conformal map is desirable whenever the accurate representation of directions is required as in navigation, meteorology, and in artillery fire. An equiareal map is desirable for political, agricultural, and general statistical purposes.

On our earth the system of lines of latitude and longitude has a significance

* An address presented by invitation at the meeting of the Association at Williamstown, Mass., Sept. 4, 1934.

which makes this system more important, geographically, than any other possible co-ordinate system. A map of the earth is effected as soon as the system or network of curves in the plane corresponding to the lines of latitude and longitude has been formulated and drawn. Other things being equal, it is desirable that this plane system consist of curves which are easily constructed. Such curves are the line, the circle, the conic sections. It is then very natural to propose the *Problem*: *to find all the conformal, and all the equiareal maps which carry the lines of latitude and longitude into plane systems composed of lines, circles, or conics.*

As will appear in paragraphs 2 and 3, two basic mappings, the conformal map of Mercator, and the cylindrical equiareal map of Lambert, reduce the problem proposed to the study of transformations in the plane. It is then sufficient to give the results of four investigations each concerned with the problem of finding all the plane transformations which carry the rectangular network:

I conformally into a system of lines and circles, Lagrange,*
II conformally into a system of lines and conics, Von der Mühll,†
III equiareally into a system of lines and circles, Gravé,‡
IV equiareally into a system of lines and conics, the author.§

A detailed bibliography of the projections mentioned in this paper would run to excessive length. The reader interested in knowing more of the subject is advised to begin with the article "Map" in the Encyclopaedia Britannica, then turn to the section on Kartographie by Bourgeois and Furtwanger in volume 6 of the Encyklopädie der Mathematischen Wissenschaften.

Finally we include illustrations of all the maps which can be constructed under I, II, III, and a few under IV. Mention is made of all known cases where cartographers have constructed such maps. The remainder of the maps (as constructions) are new. The formulation of IV is new. In all of the maps we have purposely included as much of the earth's surface as was conveniently possible, so that the amount of distortion is greater than the average user of maps is accustomed to see. This should be kept in mind when considering the usefulness of a map for limited portions of the earth's surface. The maps bear numbers corresponding to the equations which define them.

2. *The Basic Conformal Map.* The well-known Mercator projection (Fig. 1)

$$\text{(C)} \qquad \begin{cases} u = \lambda \\ v = \log \tan (\pi/4 + \phi/2), \end{cases}$$

where ϕ and λ are latitude and longitude respectively, maps the sphere on the plane conformally, and carries the lines of latitude and longitude into a rec-

* Lagrange, *Sur la construction des Cartes géographiques*, Oeuvres, t. 4, pp. 635–692.

† Von der Mühll, *Ueber die Abbildung von Ebenen auf Ebenen*, Crelle, vol. 69, pp. 264–285.

‡ Gravé, *Sur la construction des Cartes géographiques*, Liouville, 5 ser., t. 2, pp. 317–361.

§ Brown, *Equiareal maps with conic meridians and parallels*, presented to Am. Math. Society. 1933.

tangular co-ordinate system (u, v). The extension here, and in paragraph 3, to an ellipsoid of revolution is immediate, and can be effected in terms of elementary functions.

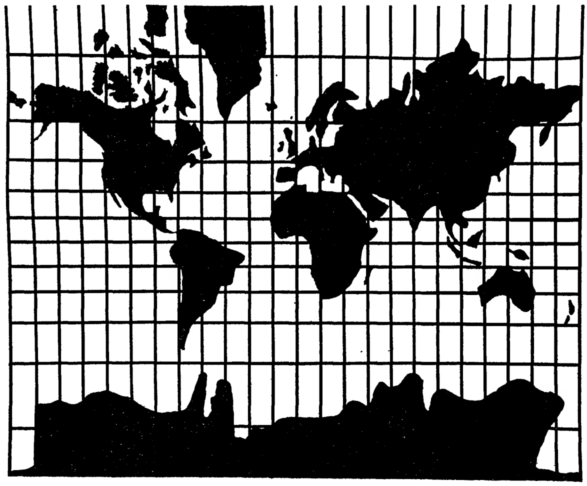

FIG. 1

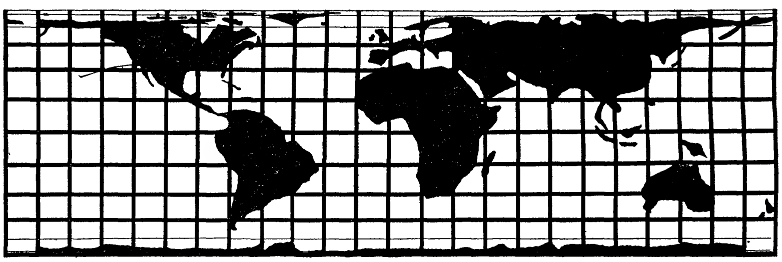

FIG 8

3. *The Basic Equiareal Map.* The Lambert cylindrical equiareal projection (Fig. 8)

$$\text{(E)} \qquad \begin{cases} u = \lambda \\ v = \sin \phi, \end{cases}$$

carries the lines of latitude and longitude of a unit sphere into a rectangular co-ordinate system and is equiareal. The projection may be effected by wrapping a tangent cylinder about the equator, and projecting the sphere onto this cylinder by rays through the axis which are parallel to the equatorial plane. This mapping was first formulated by Lambert, but the principle which allows us to conclude that the resulting map is equiareal is due to Archimedes.

4. *The Problem of Lagrange.* Lagrange showed that all the conformal plane

transformations which carry the rectangular network (u, v) into a system of lines or circles fall into 4 types. We may with no essential loss in generality factor out translations, rotations, expansions, and reflections, and enumerate these types as follows:

$$\begin{cases} x = u \\ y = v, \end{cases} \tag{1}$$

$$\begin{cases} x = e^v \cos u \\ y = -e^v \sin u, \end{cases} \tag{2}$$

$$\begin{cases} x^2 + y^2 - x/u = 0 \\ x^2 + y^2 + y/v = 0, \end{cases} \tag{3}$$

$$\begin{cases} x^2 + y^2 - 2x \tan u = 1 \\ x^2 + y^2 + 2y \coth v = -1. \end{cases} \tag{4}$$

Geometrically the networks thus obtained are:

(1) the identical rectangular network,
(2) a polar co-ordinate network,
(3) a family of circles tangent to the x-axis at the origin, and a family of circles tangent to the y-axis at the origin,
(4) a family of circles through the two real points $(0, \pm 1)$, and a family of circles through the two imaginary points $(\pm i, 0)$.

The identity (1) combined with (C) gives of course the Mercator projection. The transformation (2) similarly gives a stereographic projection from the North Pole on a plane tangent at the South Pole. Substitution of $-u$ for u and $-v$ for v, reverses the poles. Substitution of cu for u and cv for v gives the general case, the Lambert conformal conical projection (Fig. 2). The sector in which the map is confined is arbitrary and depends only on c. The so-called Lagrange projection is a special case of (4), see Fig. 4.

If in the transformation (2) we replace u by v, and v by $-u$, we obtain an entirely different appearing map (Fig. 2′), based on the same network, but with the rôles of the lines of latitude and longitude reversed. And in general, whenever the system of curves in the plane consists of two essentially different types of families, two distinct types of maps are possible. This is true for (2) and (4), but not for (1) and (3). We thus obtain six fundamental maps.

5. *Von der Mühll's Extension of Lagrange's Problem.* Generalizing Lagrange's problem to include conics, Von der Mühll showed that in addition to transformations (1) to (4), the only conformal transformations involving systems of conics are:

$$\begin{cases} x^2/\cos^2 u - y^2/\sin^2 u = 1 \\ x^2/\cosh^2 v + y^2/\sinh^2 v = 1, \end{cases} \tag{5}$$

$$(6)\qquad \begin{cases} y^2 + 4u^2x = 4u^4 \\ y^2 - 4v^2y = 4v^4, \end{cases}$$

$$(7)\qquad \begin{cases} x^2 - y^2 = u \\ 2xy = v. \end{cases}$$

Geometrically the networks thus obtained are:

(5) confocal ellipses and hyperbolas,
(6) confocal parabolas,
(7) equilateral hyperbolas, the axes of either family being asymptotes of the other family.

Transformation (5) thus yields two new maps, (6) and (7) yield each a single map.

From the standpoint of functions of a complex variable it is interesting to note that the transformations (1) to (7) are, without loss in generality, given by

(1)	$Z = z,$	(5)	$Z = \cos z,$
(2)	$Z = e^z,$	(6)	$Z = z^2,$
(3)	$Z = 1/z,$	(7)	$Z = \sqrt{z},$
(4)	$Z = \tan z,$		

where $Z = x + iy$, and $z = u + iv$.

6. *The Problem of Gravé.* Gravé showed that all the equiareal plane transformations which carry the rectangular network into a system of lines and circles, are of 6 types, as follows:

$$(8)\qquad \begin{cases} x = c_1u + c_2v \\ y = c_3u + c_4v, \text{ where } c_1c_4 - c_2c_3 = 1, \end{cases}$$

$$(9)\qquad \begin{cases} y/x = (c_3v + c_4)/(c_1v + c_2) \\ c_3x - c_1y = \sqrt{2u}, \text{ where } c_2c_3 - c_1c_4 = 1, \end{cases}$$

$$(10)\qquad \begin{cases} x = c_1u \\ x^2 + (y - v/c_1)^2 = c_2^2, \end{cases}$$

$$(11)\qquad \begin{cases} x^2/(2u) + y^2/(2u) + 2c_1x/\sqrt{2u} + 2c_2y/\sqrt{2u} + c_3 = 0 \\ y = V(v)x, \text{ where } (1 + V^2)/V' + 2(c_1 + c_2V)/\sqrt{V'} + c_3 = 0, \end{cases}$$

$$(12)\qquad \begin{cases} x \cos u/c_1 + y \sin u/c_1 = c_2 \\ x^2 + y^2 = c_2^2 + 2c_1v, \end{cases}$$

$$(13)\qquad \begin{cases} x^2 + y^2 = 2c_1v + c_2 \\ \{x - c_3 \cos (u/c_1)\}^2 + \{y - c_3 \sin (u/c_1)\}^2 = c_4^2. \end{cases}$$

Geometrically, the networks thus obtained are:

	u-curves	*v-curves*
(8)	Parallel lines,	Parallel lines,
(9)	Parallel lines,	Concurrent lines,
(10)	Parallel lines,	Displaced congruent circles tangent to 2 u-lines,
(11)	Circles tangent to 2 v-lines,	Concurrent lines,
(12)	Lines tangent to a v-circle,	Concentric circles,
(13)	Congruent circles with centers on a v-circle,	Concentric circles.

The Lambert projection is of course the identity under (8) combined with (E), see Fig. 8. The Collignon projection is a special case of (9). Certain very special forms of (11) were given by Lambert, and maps based on these bear his name, as the Lambert zenithal equiareal map, and the Lambert conical equiareal map. The Albers projection is also a special case of (11). The general transformation (8) applied to the Lambert projection is shown in Fig. 8A. Each of the transformations (9) to (13) yields two distinct types of maps as illustrated.

7. *The Generalization of Gravé's Problem.* The author has recently determined all the equiareal transformations in the plane which carry the co-ordinate network into a system of lines and conics. Since the transformation (8) will carry a system of conics into a system of conics with preservation of areas, it is possible to simplify materially the formulation *if we assume that any transformation is subject further to* (8). It should be pointed out that with proper use of (imaginary) constants in (8) it may be possible to carry a family of ellipses into a family of hyperbolas. For convenience we define:

$$f_1(x) = c_1x^2 + c_2x + c_3, \qquad g_1(x) = c_4x^2 + c_5x + c_6,$$

$$f_2(x) = c_1x + c_2 + c_3/(x + c_4), \qquad g_2(x) = c_5x + c_6 + c_7/(x + c_4),$$

$$f_3(x) = c_1x + c_2 + c_3\sqrt{c_4x^2 + c_5x + c_6}, \qquad g_3(x) = c_7x + c_8 + c_9\sqrt{c_4x^2 + c_5x + c_6}.$$

The symbol $P_2(S, T)$ shall stand for the most general quadratic polynomial in S and T with constant coefficients. With these conventions, the complete formulation is given by means of the following fourteen types:

$$(14)\quad \begin{cases} x = c_1u + c_2v \\ y = c_3u + c_4v, \text{ where } c_1c_4 - c_2c_3 = 1, \end{cases}$$

$$(15)\quad \begin{cases} x = (c_1v + c_2)\sqrt{2u} \\ y = (c_3v + c_4)\sqrt{2u}, \text{ where } c_2c_3 - c_1c_4 = 1, \end{cases}$$

$$(16)\quad \begin{cases} \int f_i(x)dx = u \\ y = vf_i(x) + g_i(x), \end{cases}$$

$$(17)\quad \begin{cases} y = Vx;\ P_2(1/\sqrt{V'},\ V/\sqrt{V'}) = 0 \\ P_2(x/\sqrt{u},\ y/\sqrt{u}) = 0, \end{cases}$$

$$(18)\quad \begin{cases} x \sin (v/c_1) + y \cos (v/c_1) = c_2 \\ x^2 + y^2 = c_2^2 + 2c_1 u, \end{cases}$$

$$(19)\quad \begin{cases} y = (c_3 + c_1 v^{1/3})x + c_2 v^{2/3}/2, \text{ where } u = (v^{-1/3}x)^2 c_1/6 + (v^{-1/3}x)c_2/3 \\ y = (c_1/f + c_2/f^2)x^2 + c_3 x \text{ and } v^{-1/3}x = f(u), \end{cases}$$

$$(20)\quad \begin{cases} y = 2c_1 v x + c_3 x + c_1 c_2 v^2 \\ y = -\, x^2 c_1/c_2 + u/c_2 + c_3 x, \end{cases}$$

$$(21)\quad \begin{cases} xy = v \\ P_2(xe^{-u},\ ye^{u}) = 0, \end{cases}$$

$$(22)\quad \begin{cases} y = 3Vx^2, \text{ where } P_2(1/V'^{1/3},\ 3V/V'^{2/3}) = 0 \\ P_2(x/u^{1/3},\ y/u^{2/3}) = 0, \end{cases}$$

$$(23)\quad \begin{cases} x = c_1 u + c_2 v \\ y = c_3 u + c_4 v + f_i(x), \text{ where } c_1 c_4 - c_2 c_3 = 1, \end{cases}$$

$$(24)\quad \begin{cases} x = (c_1 v + c_2)\sqrt{2u} \\ y = (c_3 v + c_4)\sqrt{2u} + f_i(x), \text{ where } c_2 c_3 - c_1 c_4 = 1, \end{cases}$$

$$(25)\quad \begin{cases} y = x^2 + v \\ y = f_i(x - u) + x^2 - (x - u)^2, \end{cases}$$

$$(26)\quad \begin{cases} y = \{c_3 - (3/2)c_1 v^{-2}\} x^2 + (c_4 - 2c_2 v^{-1})x \\ y = c_3 x^2 + c_4 x + (-\,3/2)c_1 f^2 - 2c_2 f, \text{ where } u = c_1(v^{-1}x)^3 + c_2(v^{-1}x)^2 \\ \qquad \text{and } f(u) = v^{-1}x, \end{cases}$$

$$(27)\quad \begin{cases} y = x^2 + 2Vx \\ y = x^2 + c_1 x^2/\sqrt{u} + c_2 x + c_3\sqrt{u}, \text{ where } 2V/\sqrt{V'} = c_1/V' + c_2/\sqrt{V'} + c_3. \end{cases}$$

This formulation, which is new, is given without proof that it is complete, the proof being extremely long and tedious. The nature of the families of curves in these transformations is not obvious by inspection, as was the case in each of the previous problems, and is not without interest. Every family, as a family, is completely determined by its envelope, including points (finite or infinite). In fact, it can be shown that in general, in *any* equiareal map, the envelope of one family is entirely composed of curves of the other family and/or the envelope of the other family. The parameter of distribution within the family is, in some cases, difficult of expression, the most complicated case being (22) where the determination of the function $V(v)$ requires the integration of

$$c_1 V'^{2/3} + 2c_2 V V'^{1/3} + c_3 V^2 + 2c_4 V' + 2c_5 V V'^{2/3} + c_6 V'^{4/3} = 0.$$

Despite its apparent complexity, this differential equation is integrable in elementary functions, but the resulting equation is transcendental in V, and it is not possible to solve for V explicitly in terms of v. The same difficulty arises in (16) and in (17), but in no case is there an integration which can not be effected

in elementary functions. Further, since our formulation is primarily analytic, some of these, notably (16), contain a large number of sub-cases which are special analytically, but limiting geometrically. The sharp distinction between central conics and parabolas is also noteworthy. A brief characterization of the families follows:

	u-curves	*v-curves*
(14)	Parallel lines,	Parallel lines,
(15)	Concurrent lines,	Parallel lines,
(16)	Parallel lines,	Conics through 2 points and tangent to 2 u-lines,
(17)	Concurrent lines,	Similar conics tangent to 2 u-lines,
(18)	Lines tangent to a (central) v-conic,	Central conics, similar and similarly placed,
(19)	Lines tangent to a v-parabola,	Parabolas tangent at a common finite and at a common infinite point,
(20)	Lines tangent to a v-parabola,	Parabolas with 4-point contact at infinity,
(21)	Central conics, similar and similarly placed,	Conics tangent to 4 (central) u-conics,
(22)	Parabolas tangent at a common finite and at a common infinite point,	Conics tangent to 4 u-parabolas,
(23)	Congruent conics tangent to 2 parallel lines,	Congruent conics tangent to the same 2 lines,
(24)	Congruent conics tangent to 2 parallel lines,	Conics tangent to the same 2 lines, passing through 2 points on a line parallel to those lines,
(25)	Conics tangent to 4 v-parabolas,	Parabolas with 4-point contact at infinity,
(26)	Parabolas with 4-point contact at infinity,	Parabolas with same infinite point as u-parabolas, and tangent to 2 u-parabolas,
(27)	Parabolas with same infinite point as v-parabolas, and tangent to 2 v-parabolas,	Parabolas with 3-point contact at infinity and passing through a common finite point.

In addition to the maps noted under paragraph 6, the Mollweide projection is a special case of (16), and the Goode homalographic projections (interrupted) for ocean units and land units are simple rearrangements of the Mollweide projection. Another special case of (16) has recently been employed by Deetz and Adams. The number of possible maps under these transformations is too large (well over 100) to permit of complete illustration. We content ourselves with showing a Mollweide projection (16), and an example of (21), the hyperbolas of the formulation being sheared to a family of concentric circles.

For convenience of reference, the illustrations may be classified as follows, the numbers referring to the equations in the text which define the maps:

Conformal (circles), Lagrange,
Fig. 1, 2, 2′, 3, 4, 4′.
Conformal (conics), Von der Mühll,
Fig. 5, 5′, 6, 7.
Equiareal (circles), Gravé,
Fig. 8, 8A, 9, 9′, 10, 10′, 11, 11′, 12, 12′, 13, 13′.
Equiareal (conics), the author,
Fig. 16, 21.

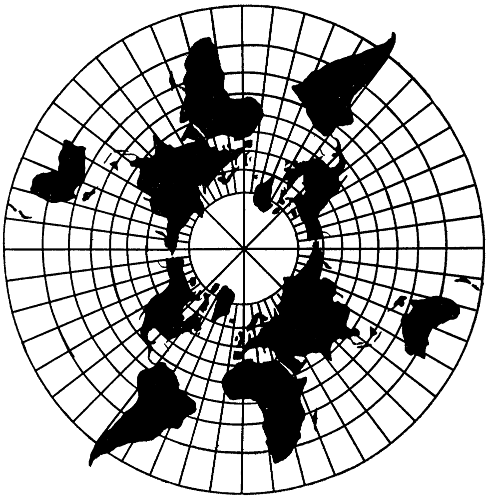

FIG. 2

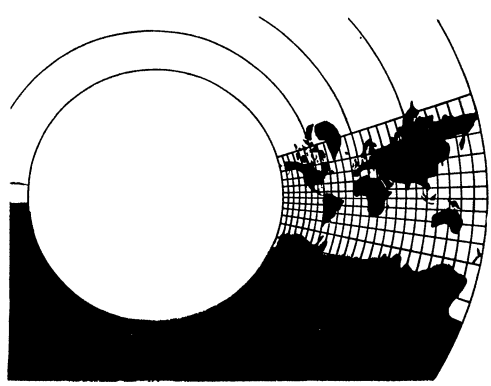

FIG. 2′

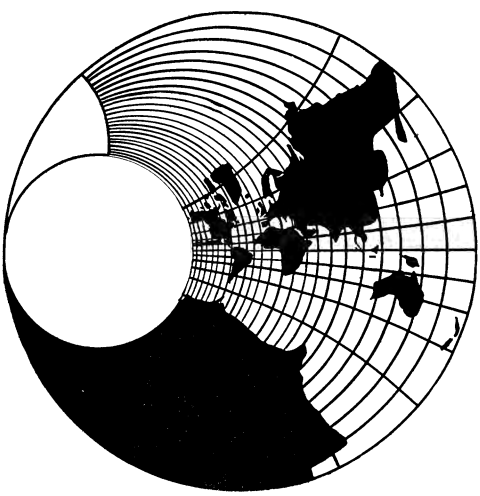

FIG. 3

FIG. 4

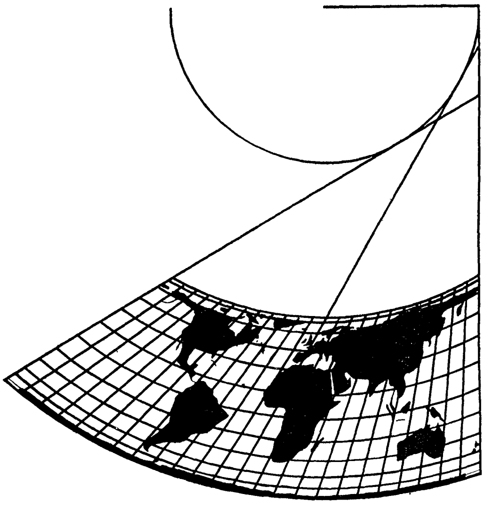

FIG. 12

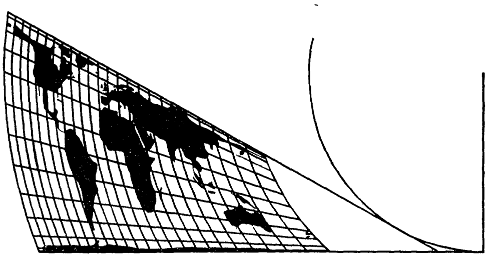

FIG. 12′

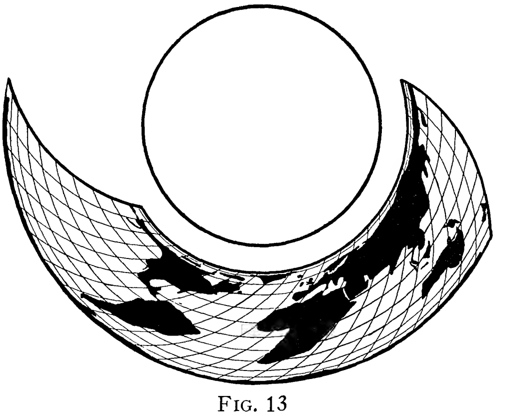

FIG. 13

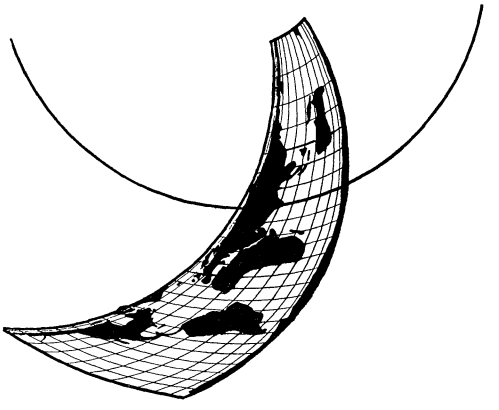

FIG. 13′

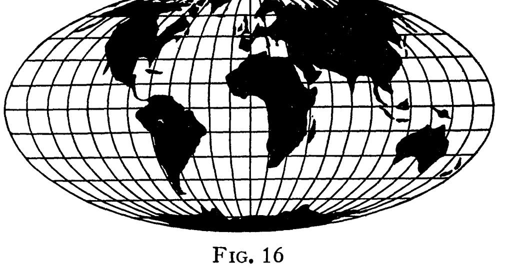

FIG. 16

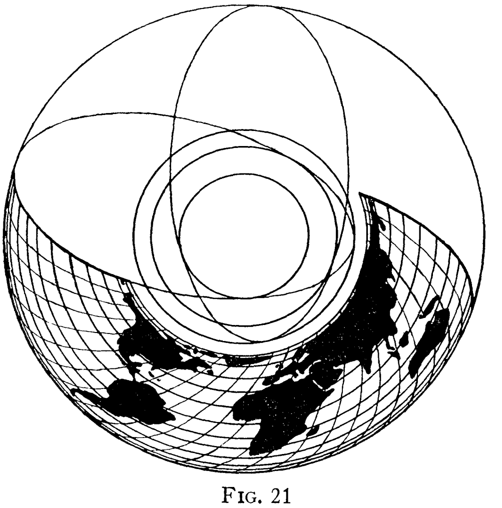

FIG. 21

MATHEMATICAL RELATION BETWEEN TURBULENCE AND DEPTH OF THE OCEAN

By G. A. LINHART, Junior College, Riverside, California

In recent articles* an equation is given which is applicable to a large variety of natural processes, and it is not surprising therefore to find that it applies also to the relation of turbulence and depth of the ocean. The equation may be derived from oceanodynamic principles entirely similar to those of thermodynamics.† However, as the science of oceanodynamics has yet to be developed, no attempt will be made at the present time to derive this useful equation, namely,

$$\theta = \theta_\infty k D^K/(1 + kD^K), \tag{1}$$

or

$$\log \left\{\theta/(\theta_\infty - \theta)\right\} = K \log D + \log k, \tag{2}$$

where θ denotes the drop in temperature with the increase in depth, D, of the ocean, and θ_∞ denotes the maximum drop in temperature for a chosen region of the ocean. K and k are constants.

There is at present no direct way of measuring the turbulence of the ocean. However, since the turbulence is due mainly to the convection of heat from the surface downward, and practically none from conduction, the drop in temperature, θ, may serve as an indicator or a proportional measure of the decrease in the turbulence.

Assembling of the Data

The data shown in table I are taken from a "Table showing Decrease of Mean Temperature with Increase of Depth for the whole Ocean," compiled by Murray and Hjort in their book entitled *The Depths of the Ocean*, page xvi; published by the Macmillan Company, 1912. The data shown in table II are taken from a paper, read before the Southern California section of the Mathematical Association of America, at Pomona College, March 4, 1933, by Professor George F. McEwen of the Scripps Institute of Oceanography of the University of California.‡ These are the averages of over a hundred sets of measurements, extending over a period of about ten years.

Method of Calculation

The values given in the tables under T and θ were plotted against the logarithms of the values given under D. The maximum drop in temperature, θ_∞,

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* Journal of Physical Chemistry, vol. 37, May, 1933; vol. 36, July, 1932.

† Journal of Chemical Physics, vol. 1, November, 1933.

‡ Mimeographed records of oceanodynamic data, on file at Scripps Institute of Oceanography, La Jolla, Calif.

TABLE I

$\theta_\infty = 21.0°C$; $K = 1.6072$; $\log k = -3.99937$

D (meters)	T°C	θ (obs.)	θ (calc.)
0	22.30	0	0
183	15.95	6.35	6.35
366	10.05	12.25	11.95
549	7.05	15.25	15.06
732	5.44	16.86	16.80
914	4.50	17.80	17.89
1097	3.89	18.41	18.59
1280	3.39	18.91	19.07
1463	2.95	19.35	19.40
1646	2.67	19.63	19.68
1829	2.50	19.80	19.87
2012	2.28	20.02	20.01
2195	2.11	20.19	20.14
2377	2.00	20.30	20.24
2560	1.89	20.41	20.33
2743	1.83	20.47	20.39
4023	1.78	20.52	20.66

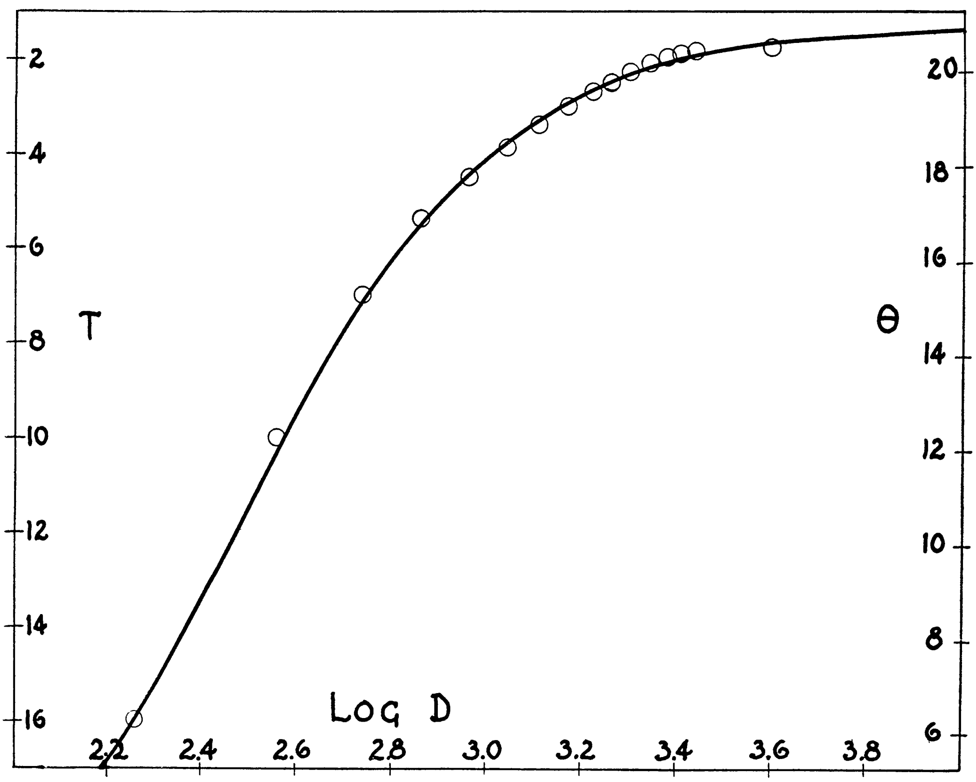
TABLE II

Station I $\theta_\infty=10.44$; $K=1.924$; $\log k=-2.5392$				Station II $\theta_\infty=10.44$; $K=1.6466$; $\log k=-1.8530$			
D (meters)	T°C	θ (obs.)	θ (calc.)	D (meters)	T°C	θ (obs.)	θ (calc.)
0	19.68	0.00	0.00	0	20.05	0.00	0.00
5	19.29	0.39	0.63	5	18.44	1.61	1.73
10	17.77	1.91	2.04	10	15.83	4.22	4.00
15	15.78	3.90	3.61	15	14.33	5.72	5.72
20	14.29	5.39	5.01	20	13.15	6.90	6.90
25	13.34	6.34	6.12	25	12.45	7.60	7.73
30	12.62	7.06	6.96	30	11.89	8.16	8.26
35	12.06	7.62	7.62	35	11.45	8.60	8.66
40	11.67	8.01	8.12	40	11.00	9.05	8.97
45	11.19	8.49	8.50	45	11.02	9.03	9.20
50	11.06	8.62	8.80	50	10.72	9.33	9.38
55	10.72	8.96	9.03	55	10.63	9.42	9.51
60	10.68	9.00	9.23	60	10.48	9.57	9.63
65	10.49	9.19	9.39	65	10.37	9.68	9.72
70	10.34	9.34	9.51	70	10.29	9.76	9.80
75	10.20	9.48	9.61	75	10.22	9.83	9.87
80	10.07	9.61	9.71	80	10.13	9.92	9.92
90	9.90	9.78	9.84	90	10.00	10.05	10.00
100	9.78	9.90	9.95	100	9.89	10.16	10.07
110	9.65	10.03	10.03	110			10.13
120	9.57	10.11	10.08	120			10.17
125	9.53	10.15	10.13	125			10.18
150	9.24	10.44	10.23	150			10.25

was then determined in each case from the inflection point on the curve, where the value of θ is one half of the value of θ_∞. This follows from equation 1 which is symmetrical with respect to the inflection point on the theta-log D curve. (That θ for the inflection point $=\frac{1}{2}\theta_\infty$ may readily be shown by taking the second derivative of theta of equations 1 or 2 with respect to log D and placing the resulting expression equal to zero.) The values for K and for log k were then obtained from the straight line plot of $\log\{\theta/(\theta_\infty-\theta)\}$ against log D, where K is the slope and log k is the intercept on the $\log\{\theta/(\theta_\infty-\theta)\}$ axis. With these values for K and for log k, the values given in the tables under θ (calculated) were then obtained by means of equation 2.

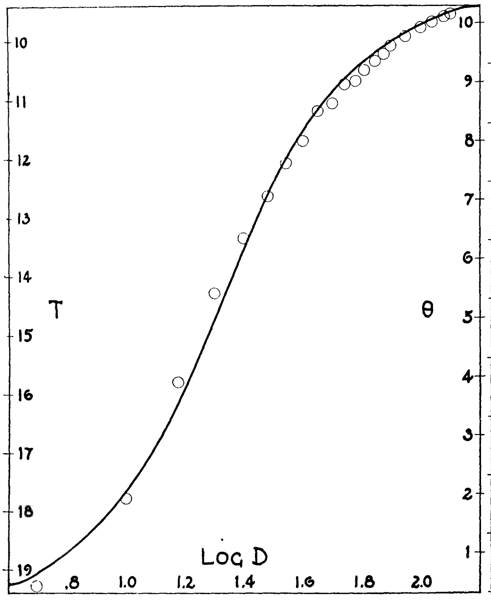
The Graphs

For the sake of clearness the observed temperatures, T, are plotted on the left hand side and the drop in temperatures, θ, are plotted on the right hand side of the curves, both to the same base, log D. Curve I corresponds to table I;

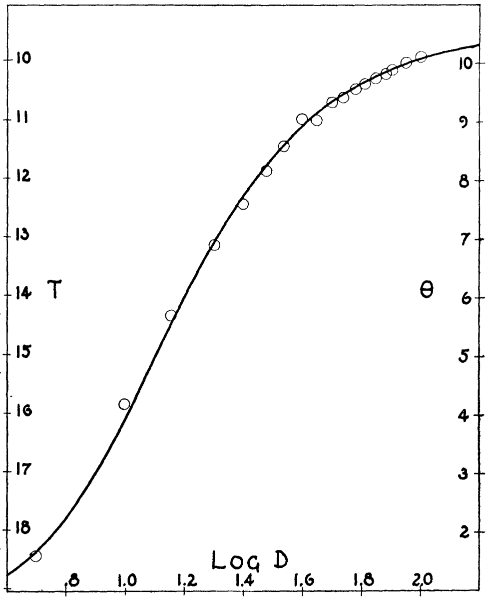


Curve I

curve IIa corresponds to table II, station I; curve IIb corresponds to table II, station II. (Stations I and II indicate the locations in the ocean at which the measurements were made.)



Curve IIa.



Curve IIb.

Conclusion

The agreement of the calculated and observed values for θ, which were obtained at different stations in the ocean and in different seasons of the year, is quite remarkable, and indicates that the general trend of the curves is the same regardless of season or location.

AN UNUSUAL NOMOGRAM

By O. E. BROWN, Case School of Applied Science

A single line nomogram may consist of three one-parameter scales as shown in Figure 1; of two one-parameter scales and a two-parameter net as in Figure 2; of one scale and two nets as in Figure 3; finally, of three nets as in Figure 4.

Charts of the type of Figure 1 occur very frequently in practice and may be found in abundance in the literature of applied science. On the other hand, charts with one or more nets are comparatively rare, the case of Figure 4 being, in fact, so rare that the writer does not recall ever having seen one. Concerning charts with three two-parameter nets Hewes and Seward* state, "These diagrams and indeed diagrams with two curve nets are largely of theoretic interest but there are special cases of practical value."

It has been the good fortune of the writer to encounter two closely related

* *Design of Diagrams for Engineering Formulas,* McGraw-Hill Book Co., p. 74.

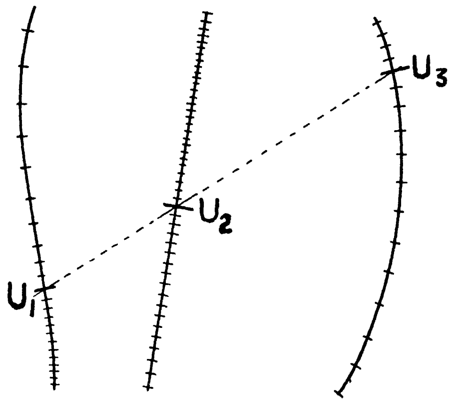


Figure 1

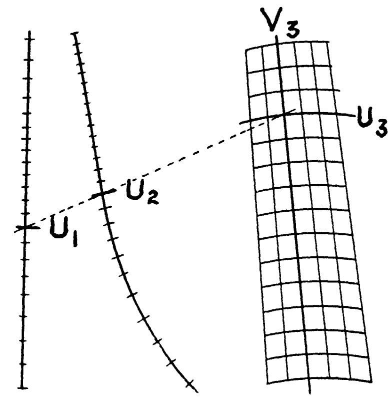


Figure 2

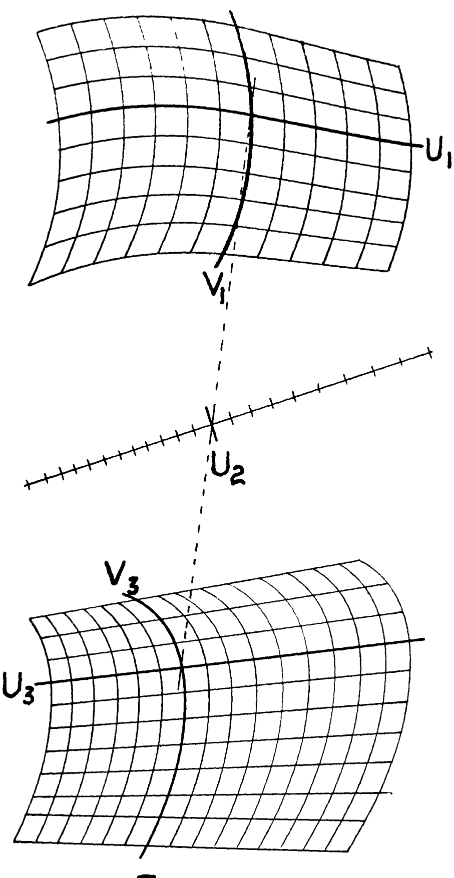


Figure 3

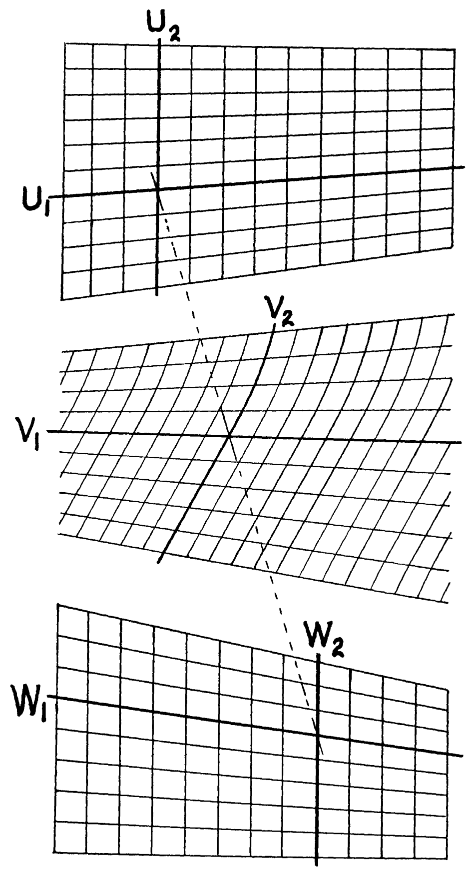


Figure 4

formulas, each in six variables, which lend themselves very readily to solution by such a chart. They are practical formulas in common use in connection with the design of thick cylinders. Moreover, in the construction of charts for these formulas, use is made of almost every idea fundamental to the subject of nomography.

The formulas referred to give the stress in the walls of a thick hollow cylinder of revolution subjected to internal or external pressure or both. If r_1 is the inside radius, r_2 the outside radius, w_1 the internal pressure, w_2 the external pressure, then the stress p at an intermediate radius r is given by Clavarino's* equation for cylinders with closed ends,

$$(1) \qquad p = \frac{r_1^2 w_1 - r_2^2 w_2 + 4r_1^2 r_2^2 (w_1 - w_2)/r^2}{3(r_2^2 - r_1^2)}$$

where the units in which p is given are the common units of w_1 and w_2 and where r, r_1, and r_2 are measured in any common units. If p is positive the stress is tensile while a negative value of p indicates compression at the point.

For cylinders with open ends, the Stress is given by Birnie's† equation

$$(2) \qquad p = \frac{2r_1^2 w_1 - 2r_2^2 w_2 + 4r_1^2 r_2^2 (w_1 - w_2)/r^2}{3(r_2^2 - r_1^2)}.$$

Equation (1) may be written in the form

$$(3) \qquad \begin{vmatrix} -3p & -4/r^2 & 1 \\ w_1 & 1/r_1^2 & 1 \\ w_2 & 1/r_2^2 & 1 \end{vmatrix} = 0$$

while equation (2) may be written as

$$(4) \qquad \begin{vmatrix} 3p & -4/r^2 & 1 \\ -2w_1 & 2/r_1^2 & 1 \\ -2w_2 & 2/r_2^2 & 1 \end{vmatrix} = 0.$$

Equation (3) shows clearly that the three points $(-3p, -4/r^2)$, $(w_1, 1/r_1^2)$, and $(w_2, 1/r_2^2)$ are collinear if, and only if, the equation is satisfied. Since p can be either positive or negative the net for locating the first of these points lies in the third and fourth quadrants. Since w_1 and w_2 are never negative the other two points are located by a common net in the first quadrant. By observing equation (4) we see that similar conditions obtain for the three points $(3p, -4/r^2)$, $(-2w_1, 2/r_1^2)$ and $(-2w_2, 2/r_2^2)$. In this case the common net for the last two points lies in the second quadrant. From the similarity of the first rows of the determinants in (3) and (4) it is possible to combine the charts for the two formulas, giving them a common net in the third and fourth quadrants for

* See Kimball and Barr, *Elements of Machine Design*, John Wiley and Sons, p. 217.

† Kimball and Barr, p. 290.

locating the points $(-3p, -4/r^2)$ and $(3p, -4/r^2)$. This necessitates reading p positive to the left for equation (3) and positive to the right for equation (4).

A chart made over this pattern, as appears in Figure 5, is almost useless because of the crowding of the lines in the first quadrant. To correct this condition, the writer used a projective transformation, transforming the points A, B, C,

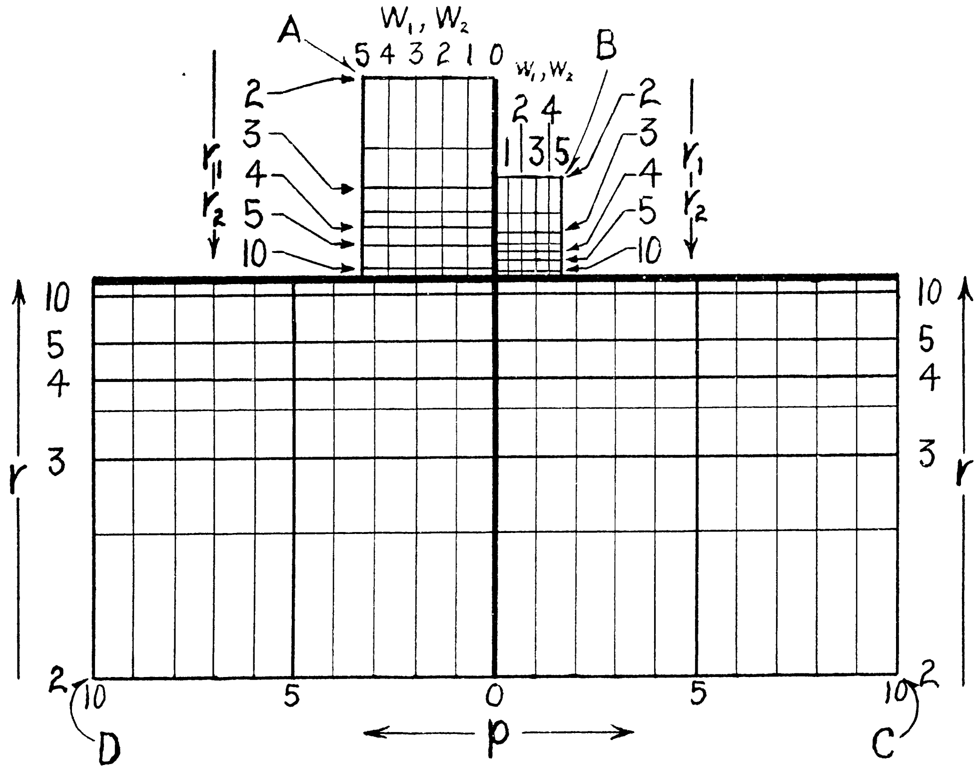
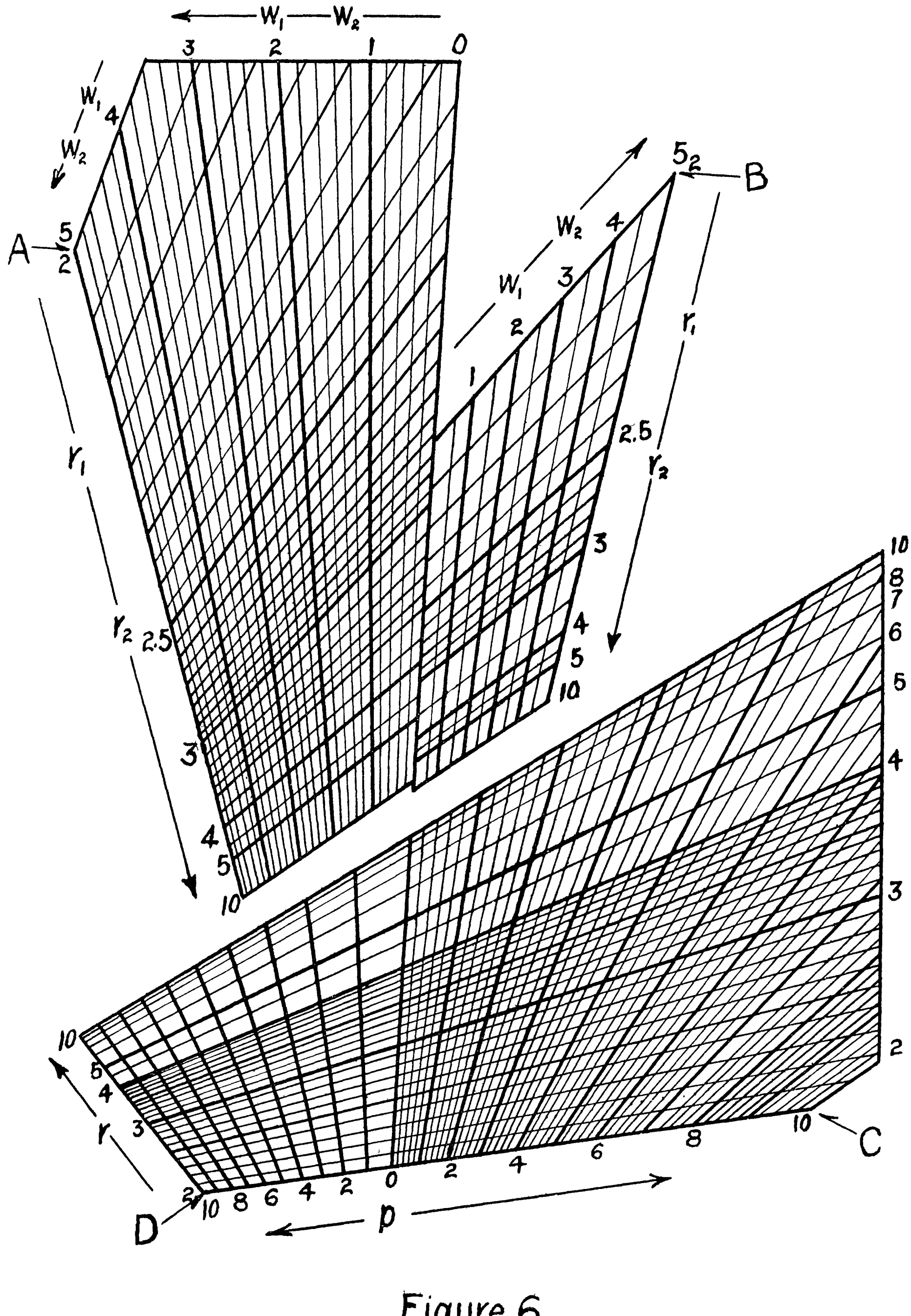


Figure 5

and D to the vertices of a rectangle, and the result appears in Figure 6. To keep the chart within a reasonable size two corners have been cut off. This does not limit the range of application since the equations are homogeneous in p, w_1, and w_2 and also in the r's. From this homogeneity, it is possible to multiply either of the sets (r, r_1, r_2) and (p, w_1, w_2) by an arbitrary constant. Unless the ratio of r_2 to r_1 is greater than 5, such constants may be found as to bring the points defined parametrically by (p, r), (w_1, r_1), and (w_2, r_2) within the boundaries of the chart.

A larger chart for these equations has been prepared by the writer and a print of it appears in "Machine Design" May, 1934.



W_1 — W_2
3 2 1 0
W_1
W_2
4
A
5
2
r_1
r_2
2.5
3
4
5
10
5_2
B
W_2
4
W_1
3
2
1
r_1
2.5
r_2
3
4
5
10
10
8
7
6
5
4
r
3
2
10
5
4
r
3
2
D
10 8 6 4 2 0 2 4 6 8 10
p
C

Figure 6

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

NOMOGRAPHIC SOLUTION OF A PROBLEM IN SPHERICAL TRIGONOMETRY

By J. B. FRIAUF, Milwaukee, Wisconsin

One of the problems in spherical trigonometry for which a solution is frequently desired is the determination of the great circle distance between two points of known latitudes and longitudes. This problem can be solved conveniently and rapidly and with a degree of precision which is sufficient for many purposes by means of the nomogram which is shown below.

The great circle distance between two points is given by the following formula:

$$\cos c = \cos a \cos b \cos L + \sin a \sin b$$

where c is the great circle distance between the two points, a and b, the latitudes of the two points, and L, the difference of longitude between them. For the purpose of nomographic representation this equation can be written in the first determinant form as follows:

$$\begin{vmatrix} 0 & \cos c & 1 \\ 1 & \cos L & 0 \\ -(\cos\alpha + \cos\beta) & (\cos\beta - \cos\alpha) & 2 \end{vmatrix} = 0$$

in which $\alpha = a + b$, the sum of the latitudes, and $\beta = a - b$, the difference of the latitudes.

The reduced determinant form of the equation is readily obtained from the first determinant form, and when subjected to a suitable projective transformation becomes:

$$\begin{vmatrix} 0 & 7\cos c & 1 \\ 6 & 10\cos L & 1 \\ \dfrac{84(\cos\alpha + \cos\beta)}{14(\cos\alpha + \cos\beta) - 40} & \dfrac{140(\cos\alpha - \cos\beta)}{14(\cos\alpha + \cos\beta) - 40} & 1 \end{vmatrix} = 0.$$

The corresponding nomogram is shown in Fig. 1. It consists of a network formed by two families of straight lines, one family being graduated according to the sum of the latitudes of the two points between which the great circle distance is to be found and the other according to the difference of the latitudes; and two linear scales, one for the great circle distance and the other for the difference in longitude. To use the nomogram, a point is found on the network

at the intersection of the lines corresponding to the sum and difference of the latitudes of the points between which the distance is to be found. From this point a straight line is drawn to the point on the right hand scale corresponding to the difference in longitude. This line cuts the distance scale at the great circle distance between the two points. The algebraic signs of the sum and difference of the latitudes are of no significance. This can be seen from the fact that the determinant given above involves only cosine functions which are not changed in value by a change in the sign of the argument; and is also apparent from the consideration that a change in the order of the two stations will change the sign of the difference in latitude, but will not alter the distance.

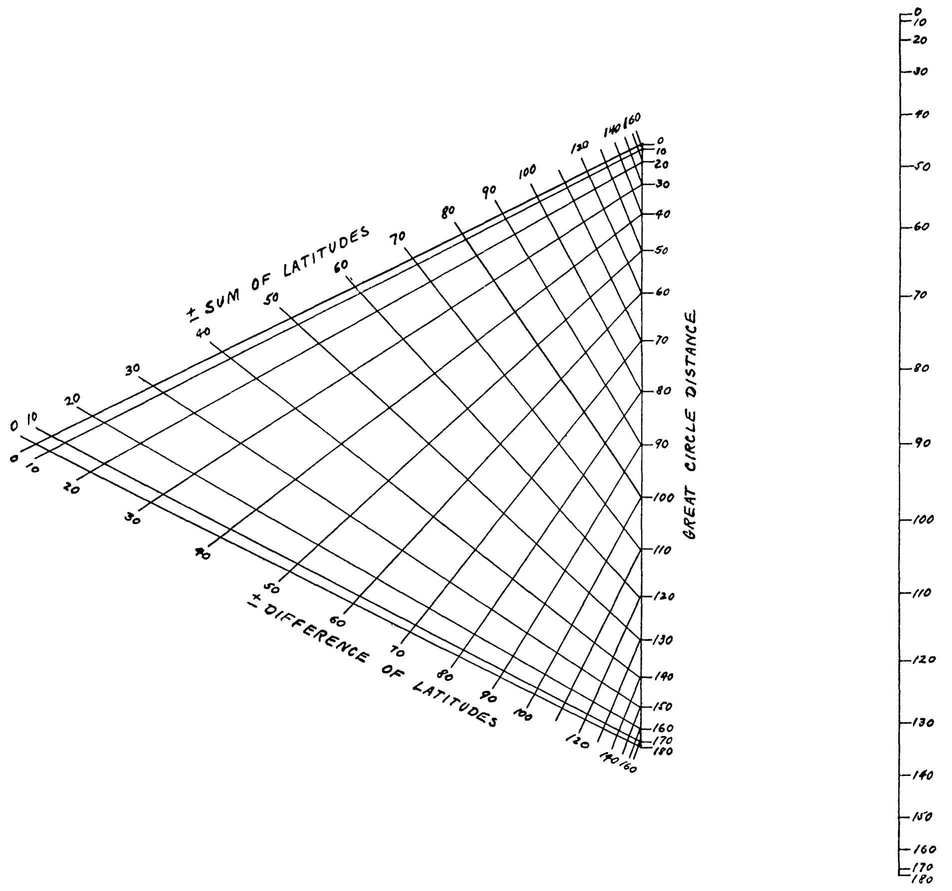


FIG. 1

Both the distance scale and the difference in longitude scale are cosine scales, the first being the projection of the second from the point at the left of the diagram which corresponds to the sum and difference of the latitudes both equal to zero. This illustrates one of the three special cases shown by the nomogram.

When both the sum and difference of the latitudes are equal to zero, both latitudes are zero, and the great circle distance, being measured along the equator, is equal to the difference in longitude. The two other special cases are illustrated by the fact that the lines for the difference in latitude radiate from the point which corresponds to zero difference in longitude, while the lines for the sum of the latitudes radiate from the point which corresponds to 180° difference in longitude. It is immediately apparent that when the difference in longitude is zero, the distance is equal to the difference of the latitudes irrespective of their sum; and that when the difference in longitude is 180°, the distance is equal to 180° minus the sum of the latitudes, irrespective of their difference.

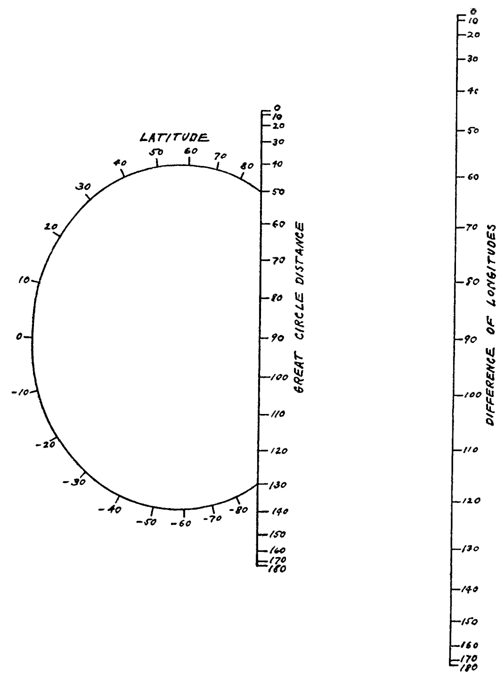


FIG. 2

The nomogram can be greatly altered in appearance by a projective transformation, and the network can be thrown between the distance and difference in longitude scales if desired.

The nomogram shown in Fig. 1 is suitable for the determination of the distance between any two points. In some cases it may be necessary only to determine the distance from a point of fixed latitude to any other point on the surface of the sphere. In this case, one of the latitudes being fixed, one less variable is involved and a simpler nomogram will suffice for the solution of the problem. This simplified nomogram can be readily constructed from the nomogram shown in Fig. 1. Suppose, for example, that the latitude of the fixed point

is 40°. When the latitude of the other point is 0° the sum and difference of the latitudes are 40°. The point on the network of the nomogram in Fig. 1 which corresponds to this sum and difference is found and is marked 0°, the latitude of the other point on the surface of the sphere. Points corresponding to other latitudes for the second point on the surface of the sphere are found in a similar way and marked with the latitude. In this way there is obtained the nomogram of Fig. 2 which is suitable for finding the distance from a point having a latitude equal to 40° to any other point on the surface of the sphere. To use this nomogram, a straight line is passed from the latitude of the second point as shown on the curved scale to the difference in longitude on the right hand straight scale. This line cuts the distance scale at the required distance.

It is apparent that the network of straight lines in the nomogram of Fig. 1 can be replaced by a network formed by two families of curved scales, the individual curves of one family being graduated with the latitude of one point on the surface of the sphere, and the curves of the other family being graduated with the latitude of the other point. The curved scale in the nomogram of Fig. 2 will be a curve in one of these families. Obviously a nomogram provided with a network formed of these curved scales could be used for finding the distance between any two points on a sphere without any necessity for calculating the sum and difference of the latitudes as must be done in using the nomogram shown in Fig. 1. However, if several other curves similar to the one in the nomogram of Fig. 2 are drawn and the curves of the other family are constructed, it will be seen that the resulting network is so complicated as to be of little practical value.

Remarks on the Geometry of the Triangle

By L. S. Johnston, University of Detroit

Let $P(x_i, y_i)$, $i=1, 2, 3$, be the vertices of a triangle, which we shall call the P triangle, $C(m, n)$ the circumcenter, $H(h, k)$ the orthocenter, and p_P the power of the origin with respect to the circumcircle. We shall find m, n, h, k, and p_P in terms of x_i and y_i.

If r be the radius of the circumcircle, we have

$$(1)\qquad \begin{aligned} &(m - x_i)^2 + (n - y_i)^2 = r^2, \qquad \text{or} \\ &2mx_i + 2ny_i - (m^2 + n^2 - r^2) = x_i^2 + y_i^2 \equiv \rho_i^2 . \end{aligned}$$

Since $m^2+n^2-r^2=p_P$, we have

$$(1')\qquad 2mx_i + 2ny_i - p_P = \rho_i^2 , \qquad i = 1, 2, 3.$$

Solving this system for m, n, and p_P we have

$$(2)\qquad m = \frac{[\rho^2, y, 1]}{2[x, y, 1]}, \qquad n = \frac{[x, \rho^2, 1]}{2[x, y, 1]}, \qquad p_P = -\frac{[x, y, \rho^2]}{[x, y, 1]},$$

where, here and hereafter,

$$[A(x, y),\ B(x, y),\ C(x, y)] \equiv \begin{vmatrix} A(x_1, y_1) & B(x_1, y_1) & C(x_1, y_1) \\ A(x_2, y_2) & B(x_2, y_2) & C(x_2, y_2) \\ A(x_3, y_3) & B(x_3, y_3) & C(x_3, y_3) \end{vmatrix}.$$

To find h and k we use the equation

$$(k - y_1)/(h - x_1) = -(x_2 - x_3)/(y_2 - y_3),$$

which with cyclic permutation of subscripts gives the system

$$\begin{aligned} h(x_1 - x_2) + k(y_1 - y_2) &= x_3(x_1 - x_2) + y_3(y_1 - y_2) \\ h(x_2 - x_3) + k(y_2 - y_3) &= x_1(x_2 - x_3) + y_1(y_2 - y_3) \\ h(x_3 - x_1) + k(y_3 - y_1) &= x_2(x_3 - x_1) + y_2(y_3 - y_1) \end{aligned} \tag{3}$$

which is a system of rank two. Now let ϕ be a function defined by the equation

$$hx_1 + ky_1 - \phi = -(x_2x_3 + y_2y_3) \equiv -\theta_1.$$

This equation together with system (3) gives the system

$$hx_i + ky_i - \phi = -\theta_i, \qquad i = 1, 2, 3, \tag{4}$$

where θ_2 and θ_3 are derived from θ_1 by cyclic permutation of subscripts. Solving system (4) we have

$$h = -\frac{[\theta, y, 1]}{[x, y, 1]}, \qquad k = -\frac{[x, \theta, 1]}{[x, y, 1]}, \qquad \phi = \frac{[x, y, \theta]}{[x, y, 1]}. \tag{5}$$

For purposes of computation the following forms, all readily verified, are perhaps more convenient:

$$\begin{aligned} h &= \frac{[x, xy, 1] - [y^2, y, 1]}{[x, y, 1]}, \qquad k = \frac{[xy, y, 1] - [x, x^2, 1]}{[x, y, 1]}, \\ \phi &= \frac{[x^2 - y^2, xy, 1]}{[x, y, 1]}. \end{aligned} \tag{6}$$

It will be noted that ϕ and p_P play somewhat similar roles in their respective environments. The geometric interpretation of ϕ is somewhat more involved, however, than that of p_P. Consider the triangle whose vertices are M_i, the midpoints of the sides opposite P_i respectively. It can be shown without much difficulty—in fact, with no more difficulty than that usually accompanying transformations on determinants—that $\phi = 2p_M$, where p_M is the power of the origin with respect to the circle through the points M_i—that is, the power of the origin with respect to the nine point circle of the P triangle.

Other interpretations of ϕ are interesting. Let the vertices P_i be given in vector form $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j}$, $\mathbf{i}$ and $\mathbf{j}$ being unit vectors along the X and Y axes respectively. Then we have

$$\theta_1 = \mathbf{r}_2 \cdot \mathbf{r}_3, \qquad \theta_2 = \mathbf{r}_3 \cdot \mathbf{r}_1, \qquad \theta_3 = \mathbf{r}_1 \cdot \mathbf{r}_2, \qquad \text{and}$$

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k} = \mathbf{r}_1 \times \mathbf{r}_2, \quad \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \mathbf{k} = \mathbf{r}_2 \times \mathbf{r}_3, \quad \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} \mathbf{k} = \mathbf{r}_3 \times \mathbf{r}_1$$

where $\mathbf{k}$ is the unit vector normal to the XY plane determined by the conventional right hand rule. Then the form of ϕ given in (5) can be written

$$\phi = \frac{(\mathbf{r}_1 \times \mathbf{r}_2)\mathbf{r}_1 \cdot \mathbf{r}_2 + (\mathbf{r}_2 \times \mathbf{r}_3)\mathbf{r}_2 \cdot \mathbf{r}_3 + (\mathbf{r}_3 \times \mathbf{r}_1)\mathbf{r}_3 \cdot \mathbf{r}_1}{\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1}$$

which is easily interpreted in terms of mechanics.

We can also express p_P vectorially. Since $\mathbf{r}_i^2 = \mathbf{r}_1 \cdot \mathbf{r}_1$, we can write

$$p_P = -\frac{(\mathbf{r}_1 \times \mathbf{r}_2)\mathbf{r}_3 \cdot \mathbf{r}_3 + (\mathbf{r}_2 \times \mathbf{r}_3)\mathbf{r}_1 \cdot \mathbf{r}_1 + (\mathbf{r}_3 \times \mathbf{r}_1)\mathbf{r}_2 \cdot \mathbf{r}_2}{\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1}.$$

From (6) we have

$$2\phi = \frac{[x^2 - y^2, 2xy, 1]}{[x, y, 1]}.$$

The numerator of this fraction is numerically equal to twice the area of the triangle $Q_i(X_i, Y_i)$, where $X_i = x_i^2 - y_i^2$, $Y_i = 2x_iy_i$, and the denominator is numerically twice the area of the P triangle. Hence the ratio of the area of the Q triangle to that of the P triangle is numerically 2ϕ. It is evident that the vertices of the P and the Q triangles are named in the same or opposite orders according as ϕ is positive or negative. Now if we let P_i be given in complex form $P_i(x_i + y_ij)$, $j^2 = -1$, we have $X_i + Y_ij = (x_i + y_ij)^2$. The construction by which the vertices of the Q triangle can be determined from those of the P triangle is well known, and the Q triangle is thus quite easily constructed.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematische Grundlagen der Quantenmechanik. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Vol. XXXVIII. By John von Neumann. Berlin, Julius Springer, 1932. 262 pages.

In the whirlwind of tremendous development of the modern physics during the last thirty years, in their eagerness to obtain quantitative results with which to work, the physicists became used to disregarding the requirements of logical

chapter VI (Der Messprozess, 15 pp.) is devoted to an axiomatic discussion of the general problem of measurements of physical quantities. This brief enumeration gives only an inadequate idea of the richness of material covered in the book, the appearance of which signifies an important step in the development of the most vital part of the structure of modern physics.

J. D. TAMARKIN

Physical Mechanics. An Intermediate Text for Students of the Physical Sciences. By R. B. Lindsay. New York, D. Van Nostrand Company, Inc., 1933.

The author in the preface makes the following remarks with respect to the scope of this text: "The present work is intended to serve as an intermediate text suitable for students who have had a year of general physics and the two-year course in general college mathematics .... The material presented includes not only particle dynamics and statics with an introduction to rigid bodies, but also enough of kinetic theory, elasticity, wave motion and the behavior of fluids to justify the title. ... the deliberate attempt has been made to emphasize the fundamental importance of mechanical principles in their application to all fields of physics. Thus the elementary kinetic theory of gases has been treated rather extensively as an illustration of the mechanics of aggregates ... in the discussion of motion problems there are applications to the Bohr theory of atomic structure and the motion of charged particles. ... In the chapter on oscillations there is a discussion of acoustical and electrical oscillations, as well as a section on the significance of the oscillator in atomic theory. ... In order that the student may get a unified point of view, waves of all kinds are considered, and a section has also been devoted to the new wave mechanics. ... The author feels that it is very important even at the intermediate stage for the student to realize that the fundamental method of mechanics is not 'cut and dried' but that there exist numerous alternative equally logical points of view which are very illuminating. ... Several of these have been introduced. ... No endeavor has been made to give the book an engineering slant. ... The importance of vector methods makes it desirable to use them considerably and this has been done with no assumption of previous knowledge of vector analysis."

Most existing texts on mechanics are written for engineers, and the applications are emphasized. This text is written for the physicist, and places more emphasis on the theoretical phases of the subject and on those fundamentals which are of supreme importance throughout all physics, pure and applied.

The text is a little difficult for students with one year of general physics and two years of college mathematics. The student should have had an elementary course in differential equations. Very careful teaching by a competent instructor will enable the average student to grasp the fundamentals of vector analysis which are demanded.

We call attention here to a few pedagogical slips. On page 16, seventh line from the top, a notation is used which is not explained until later on the page.

On page 23, line 12 from the bottom, the author states "if we finally neglect small quantities of the second order"—The student will not see any small quantities of the second order to neglect. On page 28 we find the equation $m\ddot{x} = F_0$. Why should the F be capitalized, and why use a subscript on the F? On page 159 the author writes "the condition for translational equilibrium, viz

$$\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0. \end{aligned} \tag{1}$$

These give respectively

$$\begin{aligned} W &= mq = F_2, \\ F_1 &= G." \end{aligned} \tag{2}$$

This change of order will momentarily confuse the student. Pedagogically the order used in equations (1) should be preserved in equations (2). For from $\Sigma F_x = 0$ one obtains $F_1 = G$ unless one takes the x-axis vertically, which is unusual.

The type is too small for comfortable reading.

The whole book is on a higher plane than the usual text on mechanics. The author expects the student to exert himself. Too many these days are serving their medicine in sugar coated pills with not too much medicine under the sugar coating. The text is up to date and interesting. For example the theory of central forces is developed and application made to planetary motion and to electron motions in the Bohr model of the hydrogen atom. For motion in a repulsive inverse square force field the illustration is that of alpha particle deflection. In chapter IX in the discussion of oscillations of a dynamical system with one degree of freedom we find as illustrations, the acoustic resonator, electrical oscillations, and the oscillator in electron and atomic theory.

W. V. LOVITT

The Calculus of Finite Differences. By L. M. Milne-Thomson. New York and London, The Macmillan Company, 1933. xxiii+558 pages. 30 sh.

This book satisfies a want which has been felt for a long time, a want for a modern treatment of Finite Differences.

To a person whose knowledge of the calculus of finite differences has been obtained from Boole's classic work, this book will come as a revelation of the great strides which have been made in the sixty years which have elapsed since the publication of the last edition of Boole's work. In this time the only works published in English have been several elementary works for the use of students of actuarial theory, Chrystal's Algebra, and treatises on interpolation and related subjects. While there is a great deal in this book which is of purely theoretical interest, there are many things which might be useful in an elementary book.

Some of the features of the book are:

Chapter 1. Divided differences; expression of divided differences by means of determinants; divided differences expressed by definite integrals; divided differences expressed by contour integrals; divided differences with repeated arguments.

Chapter 2. Difference quotients; introduction of Nörlund's operator; partial difference quotients; moments; partial summation.

Chapter 3. Interpolation; interpolation without differences—Jordan's method as improved by Aitken. This new method of interpolation is very rapid and has the great advantage of being suited to use with an arithmometer and is independent of tables of interpolation coefficients. Other important new methods are Aitken's quadratic process; Neville's process of iteration.

Chapter 4. Numerical applications of differences: differences when the interval is subdivided. Inverse interpolation; inverse interpolation by divided differences; inverse interpolation by iterated linear interpolation; inverse interpolation by successive approximation; inverse interpolation by reversal of series.

Chapter 5. Reciprocal differences. The usual interpolation methods considered are founded on the approximate representation of the function to be interpolated by a polynomial and the use of divided differences or the equivalent formula of Lagrange. Reciprocal differences, introduced by Thiele, lead to the approximate representation of the function by a rational function and consequently to a more general method of interpolation. Definition of reciprocal differences; Thiele's interpolation formula. Milne-Thomson's matrix notation for continued fractions; reciprocal differences expressed by determinants; reciprocal differences of a quotient; the remainder in Thiele's formula. Reciprocal derivates, the confluent case. Thiele's theorem: while Taylor's theorem gives the expansion of a function in a power-series, Thiele's theorem gives the development of a function as a continued fraction.

Chapter 6. The polynomials of Bernoulli and Euler. These play an important part in the finite calculus and have been generalized in a very elegant manner by Nörlund. The method used in treating these polynomials is a symbolic method by the author of the book. The principal articles have the titles: The ϕ polynomials; the β polynomials; definition of Bernoulli's polynomials; fundamental properties of Bernoulli's polynomials; complementary argument theorem; relation of Bernoulli's polynomials to factorials. Euler-Maclaurin theorem for polynomials.

Chapter 7. Numerical differentiation and integration.

Chapter 8. The summation problem.

Chapter 9. The psi function and the gamma function.

Chapter 10. Factorial series.

Chapter 11. The difference equation of the first order.

Chapter 12. General properties of the linear difference equation.

Chapter 13. The linear difference equation with constant coefficients.

Chapter 14. The linear difference equation with rational coefficients; operational methods.

Chapter 15. The linear difference equation with rational coefficients; Laplace's transformation.

Chapter 16. Equations whose coefficients are expressible by factorial series.

Chapter 17. The theorems of Poincaré and Perron.

Attention should be called to minor features of the book, but still important ones. There is an excellent collection of examples, both the ones worked out and those for original solution. This is an excellent feature of English books which is too often lacking in books in other languages.

The printing and proof-reading have been done with considerable care. No errors have been noted.

It would be captious to find fault with what has been left out in a work of this size, but it is to be hoped that in a future edition at least a brief account will be given of q-differences and the researches of Birkhoff.

E. B. Escott

Differential Equations. By H. B. Phillips. Third Edition. New York, John Wiley and Sons, Inc., 1934. vi+125 pages. $1.25.

The new edition of this well-known text covers the same ground as the previous editions: ordinary differential equations of the first and second orders, and linear equations with constant coefficients. The distinctive feature of this text is still the large and varied list of problems showing the application of differential equations to the most diverse branches of science. In the present edition the exposition has been improved in a number of particulars, some new topics are included (isobaric equations, the variation of parameters, symbolic methods), and many excellent problems have been added. A complete and correct list of answers to the problems is appended.

One error in the text still survives. The coefficient in the Torricelli formula for the velocity of efflux is given as 0.6 (actual value about 0.98); the value given is the coefficient of discharge. Also in dealing with problems involving the gas laws, the word density is used in the erroneous sense of weight per unit volume.

This text has proved to be admirable for students in chemistry, physics or engineering in a course to follow the calculus. If the book is again revised it could be further improved by more problems on the electric circuit and on chemical dynamics, and by the addition of an elementary treatment of partial differential equations.

Louis Brand

Differential and Integral Calculus. By R. Courant, Volume I, translated by E. J. McShane. London, Blackie and Son, 1934. xiii+568 pages. 20s.

The original German version of Courant's excellent text was reviewed in this Monthly, vol. 36 (1929), pages 96–98. While volume I of the original was de-

voted exclusively to functions of a single variable, the present volume has an added chapter on the differentiation and integration of functions of several variables, making it a better rounded treatise on the more elementary parts of the calculus. Besides a number of minor changes, the chapter on Fourier Series has been recast and very considerably improved. The main expansion theorem is now proved for functions $f(x)$, which with $f'(x)$ are sectionally continuous, the requirements on $f''(x)$ being abandoned.

Unlike the German edition, the present volume contains many lists of examples for the student to solve. The answers to these, together with hints for the solution of the more difficult ones, are given at the end of the book.

The translation itself is beautifully done. And with the various changes in form and in substance, the present version is even an improvement on its distinguished original.

LOUIS BRAND

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscripts should be typewritten with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Oklahoma

Greetings and best wishes to all chapters from the Oklahoma Alpha chapter of Pi Mu Epsilon. We are very glad to be able to report a most successful year under the direction of the following officers who were elected in May 1933: Mildred Dolezal, Director; Barbara Ellis, Vice Director; Mary Oella Tappan, Secretary; Henry G. Diebel, Treasurer; Professor Dora McFarland, Corresponding Secretary; Marian Mills, Librarian.

There are fifty-three members on the active roll of this chapter, fourteen of whom are faculty members, thirteen of whom are graduate students in mathematics or natural science, and twenty-six are undergraduate students in mathematics, engineering and the natural sciences. The Fall election of members was held on October 26, 1933 and at this time seventeen were elected to membership, sixteen of whom were initiated in the Spring. The Spring election was held on March 1, 1934. Five students were elected, three of whom were initiated. The annual banquet and initiation were held at the faculty club on March 20, 1934 at which Dr. F. W. Owens of Pennsylvania State College, Director General of Pi Mu Epsilon, was the principal speaker.

Regular meetings were held on the second and fourth Tuesdays of each month at 7:30 P.M. The meetings in both semesters were devoted primarily to the study of prominent mathematicians and their contributions to the calculus.

The meetings and programs were as follows:

September 28, 1933: Plans were made for the year.

October 12, 1933: "John Wallis, Sir Isaac Newton and the calculus" by Marian Mills and Henry Murble Pearson.

October 26, 1933: Election of members.

November 23, 1933: "Gottfried Wilhelm Leibniz and the calculus" by Mary Oella Tappan; "The lives and works of John and James Bernoulli" by Barbara Ellis.

November 26, 1933: Tea at the faculty club honoring the new members.

December 14, 1933: "Euler and Lagrange" by Margaret Morris; "A problem of Euler" by Jack Ernest Handley.

January 11, 1934: "Early development of the calculus" by Henry G. Diebel; "Laplace" by Florence Ganstine.

February 8, 1934: "Hyperbolic functions" by Thelma Sherry, "Demonstration of a geometric construction of a double cone and its various plane sections" by Henry Murble Pearson; "Demonstration of a construction of five regular solids and their relation to a circumscribed sphere" by Harold Feldstein.

March 1, 1934: Election of members.

March 8, 1934: "Projective coordinates and their applications" by Martha Davis and R. D. Dorsett.

March 20, 1934: Annual banquet and initiation.

March 22, 1934: "The mathematical theory of finance" by Kenneth Davis.

April 12, 1934: "An original problem in optics" by Paul Fine; "Galois" by Dot Jeannette Gifford.

April 26, 1934: "Lewis Carroll as a mathematician" by Jack Laudermilk; "Stereographic projections" by Edward Wedel.

May 10, 1934: Reports and election of officers.

May 19, 1934: Picnic.

After the business meeting and formal program, tea and cookies were generally served and informal discussions were encouraged.

The aim of the club is scholarship for the individual members in all subjects and especially in mathematics. The requirements for eligibility to membership are: a general average of "B" and an average between "A" and "B" in mathematics.

MARY OELLA TAPPAN, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of New Jersey College for Women

The officers for the academic year 1933–1934 were: Elizabeth Nolf, '34, President; Dorothy Venook, '34, Vice President; Juliette Marshall, '35, Secretary; Louise Rovner, '34, Treasurer; Richard Morris, Faculty Adviser.

Nine meetings in which fifteen different students participated, were held for the presentation of student papers. These papers were largely based on the book "Mathematical Excursions" by Helen A. Merrill. However, one meeting was devoted to solving problems presented by members and one to the presentation of papers submitted to the department of mathematics in partial fulfillment of the requirements for graduation with distinction in mathematics. These papers were: "The orthopole" by Louise Rovner and "Some aspects of the theory of functions of a complex variable" by Dorothy Venook.

Two joint meetings with the Mathematics Club of Rutgers University were held, one addressed by Doctor William H. Mitchell of the department of Zoology of Rutgers University on the topic "Mathematical basis of the Photo-Tropism theory of animal conduct" and the other by Mr. J. Whitney Colliton, head of the department of mathematics in the Trenton, N. J., High School on "Teaching geometry by opaque projection." Subsequently, a number of members of the club visited Mr. Colliton's classes.

Many members of the club attended the meeting of the Philadelphia Section of the Mathematical Association of America which was held at the New Jersey College for Women during the

Thanksgiving recess. The club entertained at tea after the sessions in honor of the visiting mathematicians.

The usual Christmas party and spring dance were held.

JULIETTE MARSHALL, *Secretary*

The Mathematics Club of the Cooper Union Institute of Technology

The officers for the year 1933–1934 were: L. Green, President; J. Gladstone, Vice President; K. Itkin, Treasurer; M. Greenbaum, Secretary; C. Wamser, Assistant Secretary; Professor F. H. Miller, Faculty Adviser.

The meetings and programs were as follows:

October 18, 1933: "Vector analysis" by Mr. Henry Walther of the Bell Telephone Laboratories.

November 8, 1933: "Theory and application of the slide rule" by Mr. M. Rubinsky, '34.

November 29, 1933: "Some interesting theorems in mathematical astronomy" by Mr. J. Magill, '24.

December 20, 1933: "Transfinite arithmetic" by Mr. J. Gladstone, '35.

January 31, 1934: "Ruler and compass constructions" by Mr. J. Maltz, '36.

February 21, 1934: "Various properties of infinite series" by Mr. C. Molloy, '35.

March 21, 1934: "A discussion and new solution of Malfatti's problem" by Mr. A. Gelbart, Cooper Union Night School, '35.

April 11, 1934: "A problem in the theory of numbers" by Professor H. W. Reddick. Election of officers for 1934–1935. A polyphase duplex slide rule was awarded by the club to Mr. Frank Hashmall, '37, for excellence in first year mathematics.

MORRIS GREENBAUM, *Secretary*

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 148. *Proposed by V. Thébault, Le Mans, France.*

Form two numbers, one of which is twice the other, using the ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 just once each. Is more than one solution possible? If instead of one of the two numbers being twice the other, it must be k times the other, what is the smallest positive integer value of k not a power of ten, such that the problem will have no solution?

E 149. *Proposed by J. A. Hurry, San Antonio Junior College, Texas.*

Show that the angle A has no value within the first quadrant which will satisfy the equation,

$$\sin\frac{A}{3}=\frac{\sin A}{2+\cos A}.$$

E 150. *Proposed by Maud Willey, Gulfport, Mississippi.*

Points M and N trisect side BC of triangle ABC, so that $BM = MN = NC$. A line parallel to AC meets lines AB, AM and AN in points D, E and F, respectively. Show that $EF = 3DE$.

E 151. *Proposed by W. R. Ransom, Tufts College, Massachusetts.*

Under what circumstances may the cube of an integer equal the difference of the squares of two non-consecutive, relatively prime, positive integers?

E 152. *Proposed by J. Rosenbaum, Hartford Federal College, Connecticut.*

Simplify the product

$$(2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1)(2^{2^3} + 1) \cdots (2^{2^n} + 1).$$

E 153. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Last summer in Arizona I overheard Wild Bill and Pesky Pete discussing different incidents which had occurred during the past half century. It seems that on one occasion Pete had asked Wild Bill the date, and in typical western fashion the latter had spun around, drawn his revolver, and shot a bullet into the calendar hanging at the far end of the barn. "Thar's your date, Pesky," Bill had exclaimed, and a closer inspection had proved he was right.

Out of curiosity, the two had then added up the various numbers Bill's bullet had punctured on the successive sheets of the calendar. (It was the usual kind, with a sheet for each month, and on each sheet a square for each date in the month, arranged in seven columns according to the days of the week.) When I left them, the two were arguing over the total, Bill claiming it had been 317, and Pete, 319. Which was right, and just when had the incident occurred?

SOLUTIONS

E 74 [1934, 45]. *Proposed by J. E. Trevor, Cornell University.*

A vertical sheet of horizontal rays of light falls upon the outside of a horizontal reflecting circular cylinder, the axis of which meets the incident sheet at an arbitrary angle. The reflected rays form an illuminated curve on a dark screen parallel to the incident sheet. Find the equation of this curve.

Solution by J. Hoekstra, Maastricht, Holland.

Let the horizontal axis of the cylinder C be the ξ axis. Let the vertical sheet ϕ of the horizontal rays of light pass through the ζ axis; meet the η axis at an angle α (fig. 1); and cut C along an ellipse of which HB is the top-front-quadrant. The plane ψ (parallel to ϕ) in which the illuminated curve lies, cuts the η axis in E. Let OE be p and the radius of the cylinder be r; then the coordinates of any point of the curve are functions of r, α, p, and a parameter.

To an arbitrary point of incidence R on HB corresponds a plane λ perpendicular to the ξ axis, which contains the normal PR, and cuts the $\xi\eta$ plane along the line PST, ϕ along RS, and ψ along TU. Let $\angle RPS = \delta$, and let $\cos\delta = a$ be a parameter. The plane μ, containing the incident ray MR, the reflected ray

RW, and the line PRU, cuts the $\zeta\xi$ plane along line PL. Let G be the point of intersection of the ξ axis and the plane ψ; then the intersection of ψ and the plane $\zeta\xi$ is the perpendicular at G. Let PL cut that perpendicular in K; then line KU (the intersection of μ and ψ) is parallel to GT and to the rays of light.

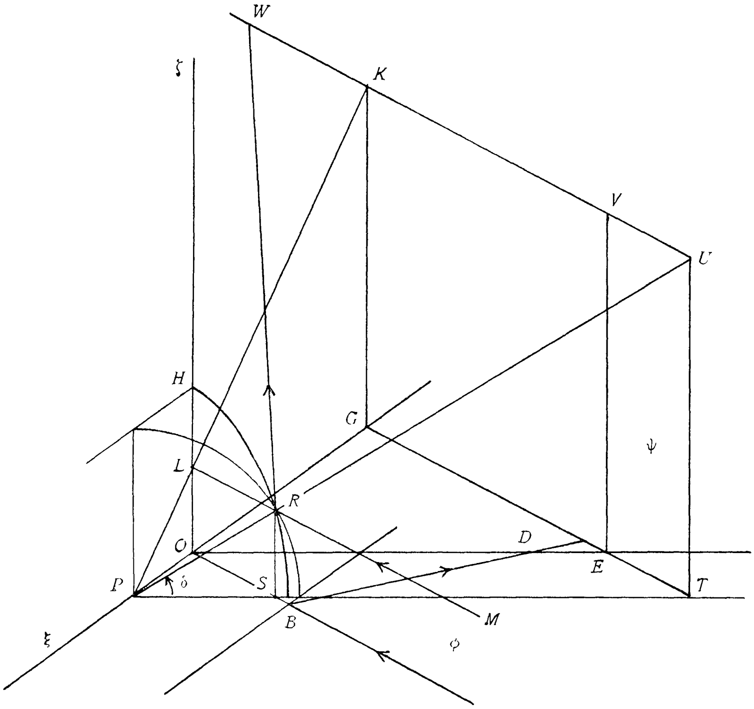


FIG. 1

Let the reflection of the ray incident at B cut the $\zeta\eta$ plane at D; then $OA=AD=r$. In the plane ψ we take EG and EV, the intersections with the $\xi\eta$ and $\zeta\eta$ planes, as the x and y axes respectively. Then we find the coordinates of point W in the following way:

From $UT:RS=PT:PS$ and $PR=r$, $PS=r\cos\delta=ra$, $ST=OE=p$, $RS=r\sin\delta=r\sqrt{1-a^2}$, we have

$$UT = y_w = \frac{ra+p}{a}\sqrt{1-a^2}.$$

Moreover, since $\angle URW=\angle MRU=\angle WUR$, we have $UW=RW$; and hence $UW=RU/(2\cos MRU)$. And since $RU=p/a$ and $\cos MRU=a\cos\alpha$, $UW=p/(2a^2\cos\alpha)$. Also $UV=SO=ra/\cos\alpha$, and therefore we have

$$VW = x_w = \frac{p}{2a^2\cos\alpha} - \frac{ra}{\cos\alpha} = \frac{p-2ra^3}{2a^2\cos\alpha}.$$

The consideration of the front half of the ellipse as possible points of incidence gives the parametric equations of the curve

$$y = \pm\, y_w \text{ and } x = x_w$$

with the parameter a in the range $0 \leqq a \leqq 1$. If we consider the back half, only, and look upon the *inside* of the cylinder as reflecting, the equations are the same with the parameter in the range $-1 \leqq a \leqq 0$.

In order to simplify the study of the character of the curve, we put $p = 2r$. Hence

$$\pm\, y_w = r \cdot \frac{a+2}{a} \cdot \sqrt{1-a^2} \quad \text{and} \quad x_w = \frac{r}{\cos\alpha}\,\frac{1-a^3}{a^2}.$$

We may, moreover, let $y = y_w/r$ and $x = x_w \cos\alpha/r$, when the equations take the form

$$(1) \qquad x = \frac{1-a^3}{a^2}, \qquad y^2 = \frac{(a+2)^2(1-a^2)}{a^2}, \qquad -1 \leqq a \leqq +1.$$

From (1): $x<0$ for $a>+1$ and $y^2<0$ for $a>+1$ or $a<-1$ with the *exception* of $a=-2$, which gives

$$y^2 = 0, \qquad x = 9/4,$$

an *isolated double point.*

All real points (except the isolated point) therefore correspond to a series of parameter values between -1 and $+1$.

The curve is symmetrical with respect to the x axis and cuts this axis in the points $x=0$, $x=2$, where the tangent is vertical.

For a fixed x there are 3 values for the parameter a, whence 6 values for y; this 6th degree curve has 2 parabolic branches, and a line $y=c\neq 0$ always cuts it in 2 real finite points (Fig. 2).

To eliminate the parameter a we set $u=y^2-4x+3$. Then from equations (1) we have

$$(2) \qquad u = \frac{4-a^3}{a}, \qquad x = \frac{1-a^3}{a^2},$$

and the elimination of a gives

$$u^3 + x^2u^2 - 18xu - 16x^3 - 27 = 0$$

or

$$(3)\qquad (y^2 - 4x + 3)^3 + x^2(y^2 - 4x + 3)^2 - 18(y^2 - 4x + 3) - 16x^3 - 27 = 0,$$

which is the required equation.

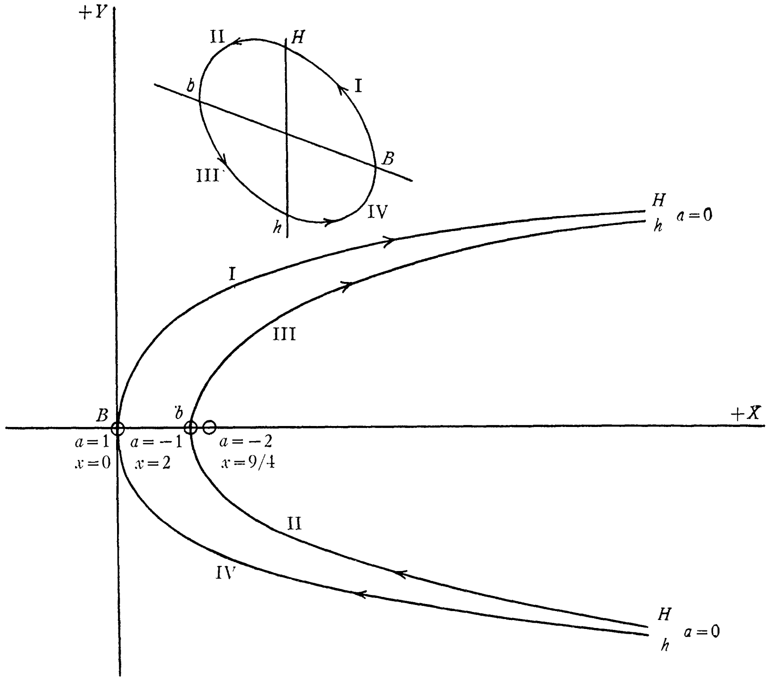


FIG. 2

Editorial Note. It should be noted that in the problem as stated by the proposer the light is reflected from the *outside* of the cylinder, and the required curve is therefore only the outer branch of the curve found in the above solution.

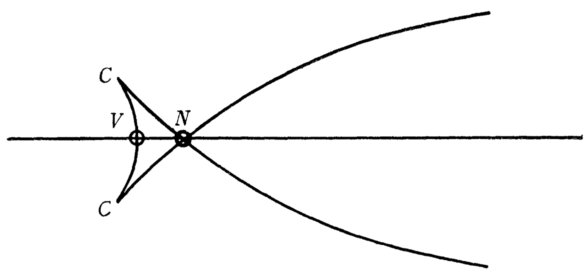


FIG. 3

By putting $p = 2r$ in the solution, the solver missed certain interesting facts about the inside branch of the curve when the screen is brought close to the sheet of incident rays. When $p < r$, the curve has a node (in place of the isolated

point) and two cusps, and has the appearance indicated in Fig. 3. As p approaches r the two cusps, the node, and the vertex approach coincidence; and for $p=r$ the curve is geometrically smooth but has algebraically a complicated multiple point at the vertex.

As p approaches zero, the part of the inside branch between the cusps approaches half of the elliptic section of the cylinder, while the outside branch approaches the other half of this ellipse.

The outside and inside branches of the curve do not approach each other asymptotically. As x increases, the difference between the y coordinates of points on the two branches approaches $2r$. This is true for any value of p.

Also solved by Daniel Finkel, Margaret Haspel, J. Rosenbaum and Simon Vatriquant.

E 118 [1934, 577]. *Proposed by V. Thébault, Le Mans, France.*

Determine the largest and smallest multiples of 63 which can be written with the ten digits, 0, 1, 2, $\cdots$ 9, used once each in each multiple.

Solution by Sidney Kaplan, Brooklyn, N. Y.

Since we are to use each digit once each in each multiple, it is evident that the multiples will be multiples of nine automatically, and hence we are concerned with making them also multiples of seven. A number is a multiple of seven if, and only if, when its digits are split into triads and the numbers formed by those triads are alternately added and subtracted, this result is a multiple of seven.

The largest and smallest numbers formable from the ten digits are 9,876,543,-210 and 1,023,456,789, but neither of these is a multiple of seven, and hence neither is a multiple of sixty-three. Now transposing some of the digits in a number will change the value of that number the least when the digits transposed are nearest to the right in the number. Hence we interchange digits at the extreme right in the two above numbers and examine them to see if either becomes a multiple of seven by the rule outlined above. Since each does, our problem is solved, and the largest and smallest multiples of sixty-three formable with ten different digits each, are 9,876,543,201 and 1,023,456,798 respectively.

However, a larger multiple of sixty-three may be formed from the ten different digits, if it is permissible to use some of them as exponents. In that case, the largest multiple of sixty-three would be

$$21^{3^{4^{5^{6^{7^{8^{90}}}}}}}$$

Also solved by W. E. Buker, M. L. Constable, A. J. Lewis, Roy MacKay, C. T. Oergel, B. D. Roberts, L. S. Shively, E. P. Starke, Woodrow Tichy, C. W. Trigg, Simon Vatriquant and the proposer.

E 119. [1934, 577]. *Proposed by Maud Willey, Long Beach, Mississippi.*

Nine equal squares, five marked with the letter X and four with the letter O, are arranged at random in a square array. What is the probability that some row, column or diagonal of the array contains only squares bearing the letter O?

Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

Suppose the top row contains only O's. Then the remaining O may occupy any of the six remaining spaces. Since there are $3+3+2=8$ rows, columns and diagonals, the number of favorable cases is 48. The total number of possible arrangements is ${}_9C_5=126$, so the probability sought is $48/126=8/21$.

Also solved by Roy MacKay, R. K. Morley, E. P. Starke, Woodrow Tichy, Simon Vatriquant and the proposer.

E 120. [1934, 577]. *Proposed by L. S. Johnston, University of Detroit.*

Given the perpendicular distances, a and b, from a point P to the arms of a known angle, θ, within which P lies; it is required to compute the lengths of the radii of the two circles, each of which passes through P and is tangent to both arms of θ.

Solution by R. K. Morley, Worcester Polytechnic Institute, Massachusetts.

Denote the vertex of the given angle by D, and the perpendiculars a and b by PA and PB. Suppose for definiteness $a<b$. Suppose the required circle drawn with center C (in either position). Let the point of tangency with DA be E. Draw DC, DP, AB, and the radii CE and CP. Denote the length of the radius by r.

Then $\angle ADC=\frac{1}{2}\theta$, $DC=r\csc\frac{1}{2}\theta$. The quadrilateral $DAPB$ is inscriptible, hence $\angle APB=180^\circ-\theta$, and $\angle ABP=\angle ADP$. Also $DP=a\csc ADP$.

From trigonometry,

$$\tan ABP = \frac{a\sin APB}{b-a\cos APB}, \quad \text{or } \tan ADP = \frac{a\sin\theta}{b+a\cos\theta},$$

and

$$\cot ADP = \frac{b+a\cos\theta}{a\sin\theta} \tag{1}$$

also

$$\cot\tfrac{1}{2}\theta = \frac{\sin\theta}{1-\cos\theta}. \tag{2}$$

Now in triangle DPC we have $r^2=DC^2+DP^2-2DC\cdot DP\cos PDC$, or

$$r^2 = r^2\csc^2\tfrac{1}{2}\theta + a^2\csc^2 ADP - 2r\csc\tfrac{1}{2}\theta\cdot a\csc ADP\cdot\cos\left(\tfrac{1}{2}\theta - ADP\right),$$

$$0 = r^2 \cot^2 \tfrac{1}{2}\theta + a^2(1 + \cot^2 ADP)$$
$$- 2ra \csc \tfrac{1}{2}\theta \csc ADP(\cos \tfrac{1}{2}\theta \cos ADP + \sin \tfrac{1}{2}\theta \sin ADP).$$

Substitution into this equation from (1) and (2) gives

$$0 = r^2 \frac{\sin^2 \theta}{(1 - \cos \theta)^2} + a^2\left[1 + \frac{(b + a \cos \theta)^2}{a^2 \sin^2 \theta}\right]$$
$$- 2ra\left[\left(\frac{\sin \theta}{1 - \cos \theta}\right)\left(\frac{b + a \cos \theta}{a \sin \theta}\right) + 1\right]$$

which reduces to

$$(1 + \cos \theta)^2 r^2 - 2r(a + b)(1 + \cos \theta) + (a + b)^2 = 2ab(1 - \cos \theta).$$

Whence finally,

$$r = \frac{a + b \pm \sqrt{2ab(1 - \cos \theta)}}{1 + \cos \theta} = \sec^2 \tfrac{1}{2}\theta\left(\frac{a + b}{2} \pm \sqrt{ab} \sin \tfrac{1}{2}\theta\right).$$

Also solved by Morris Lieblich, Roy Mackay, A. V. Richardson, E. P. Starke, C. W. Trigg, M. J. Turner, Simon Vatriquant, J. A. Ward and the proposer.

E 121 [1934, 577]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The sides of the real triangle, ABC, are three different positive integers, no two of which have a common factor. AD is tangent to the circumscribed circle at A, and meets BC produced at D. Prove that AD, BD and CD are each always rational, but that none of them can ever be an integer.

Solution by A. J. Lewis, University of Denver, Colorado.

Let $CD=x$, $AD=y$, $AB=c$, $BC=a$ and $CA=b$. Then

$$\frac{x}{y} = \frac{\sin CAD}{\sin ACD} = \frac{\sin CBA}{\sin ACB} = \frac{b}{c}, \qquad \text{so } y = cx/b.$$

Since AD is tangent to the circumcircle, $y^2=x(x+a)$. Solving these two equations simultaneously for x and y, we obtain

$$x = \frac{ab^2}{c^2 - b^2} \quad \text{and} \quad y = \frac{abc}{c^2 - b^2}, \quad \text{or else } x = y = 0.$$

The latter solution is eliminated by the given conditions. In the first solution, we note that b and c are each prime to c^2-b^2, and also that

$$|c^2 - b^2| = |c - b|(c + b) \geqq c + b > a.$$

(This follows because a, b and c are the integer sides of a real triangle.) Finally, observing that b^2 and c^2-b^2 have no common factor, and that a is not divisible by c^2-b^2 although they may have a common factor, we see that neither x nor y

can be reduced to an integer. That they are both rational is evident since a, b and c are integers.

Editorial Note. This theorem is true even though a may have some factors in common with b and others in common with c. It is necessary however, that b and c be relatively prime.

Also solved by W. B. Clarke, R. K. Morley, Roy MacKay, A. V. Richardson, J. Rosenbaum, E. P. Starke, C. W. Trigg, Simon Vatriquant, Maud Willey and the proposer.

E 122 [1934, 577]. *Proposed by C. A. Rasmussen, University of Alabama.*

The lines joining the three vertices of a given triangle, ABC, to a point O in its plane, cut the sides opposite the vertices A, B and C in the points K, L and M respectively. A line through M parallel to KL cuts BC at V and AK at W. Prove that $VM=MW$.

Solution by W. B. Clarke, San Jose, California.

Let KL cut AB at N. Then by Ceva's theorem,

$$\frac{AL\cdot CK\cdot BM}{LC\cdot KB\cdot AM} = 1 \tag{1}$$

and by Menelaus' theorem, using transversal KL in triangle ABC,

$$\frac{AL\cdot CK\cdot BN}{LC\cdot KB\cdot AN} = 1. \tag{2}$$

Combining (1) and (2) we see that

$$AN/AM = BN/BM. \tag{3}$$

Now since triangles NKA and MWA are similar, it follows that

$$NK/MW = AN/AM. \tag{4}$$

And since triangles NKB and MVB are similar, we also obtain

$$NK/VM = BN/BM. \tag{5}$$

A combination of (3), (4) and (5) now gives us

$$NK/VM = NK/MW, \qquad \text{whence} \quad VM = MW. \tag{6}$$

Also solved by W. E. Buker, Abe Gelbart, L. M. Kelly, Roy MacKay, A. V. Richardson, J. Rosenbaum, H. F. Schroeder, E. P. Starke, Simon Vatriquant and the pro-power.

E 123 [1934, 578]. *Proposed by W. R. Ransom, Tufts College, Massachusetts.*

Prove that if the integer, $1111 \cdots 12222 \cdots 24$, has one more 2 than **1's**. then it is the product of two factors whose digits are all 3's except for a terminal 4 in one factor and a terminal 6 in the other.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

Let n represent the number of 2's in the given number. Then

$$
\begin{aligned}
1111\cdots 12222\cdots 24 &= 1111\cdots 12222\cdots 22 + 2\\
&= \frac{10^{n+1}(10^{n-1}-1)}{10-1} + \frac{2(10^{n+1}-1)}{10-1} + 2\\
&= \frac{10^{2n} + 10^{n+1} - 2}{9} + 2\\
&= \frac{10^{2n} - 2\cdot 10^n + 12\cdot 10^n + 1 - 12}{9} + 3\\
&= \frac{9(10^n-1)^2}{9^2} + \frac{12(10^n-1)}{9} + 3\\
&= \left[\frac{3(10^n-1)}{10-1} + 1\right]\left[\frac{3(10^n-1)}{10-1} + 3\right]\\
&= [333\cdots 34][333\cdots 36].
\end{aligned}
$$

Also solved by W. E. Buker, M. L. Constable, A. J. Lewis, Roy MacKay, R. K. Morley, A. V. Richardson, E. P. Starke, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 124 [1934, 578]. *Proposed by B. W. Jones, Cornell University.*

Show that the volume generated by revolving a cube of edge a about one of its space diagonals is $\pi a^3/\sqrt{3}$.

Solution by Roy MacKay, Eastern New Mexico Junior College.

Let AB be the space diagonal used as an axis of rotation. The section of the cube by a plane through AB is a variable parallelogram $PARB$. The vertex P of the variable triangle ABP is either at a vertex E of the cube, or on an edge EF at a distance x, say, from the vertex E. The Pythagorean theorem applied to the right triangles AEP and PFB gives

$$AP = (a^2 + x^2)^{1/2}, \qquad PB = (a^2 + [a - x]^2)^{1/2},$$

and $AB = a\sqrt{3}$.

Drop PQ perpendicular to AB, and then as the cube rotates, x varies from 0 to a while Q moves along AB. The limiting positions of Q are found by setting $x=0$ and $x=a$. In both these cases the triangle ABP is right, and the legs are of lengths a and $a\sqrt{2}$. From the right triangle relationship that a leg is the mean proportional between the hypothenuse and its projection on the hypotenuse, we see that Q varies from a point C, one third of the distance from A to B, to a point D, two-thirds of the distance from A to B. In these limiting positions of P and Q, $PQ = y = a\sqrt{(2/3)}$. We next observe that Q divides DC in

the same ratio that P divides the edge FE. This readily follows after passing parallel planes through E, P and F perpendicular to AB at C, Q and D respectively. Hence $CQ=x/\sqrt{3}$, and from the right triangle APQ,

$$y^2 = \tfrac{2}{3}(x^2 - ax + a^2).$$

The volume of the required solid is equal to the volume of the two cones generated by the right triangles ACE and BDF, increased by the volume generated by rotating the area under the locus of P (in the cutting plane through AB) about CD. This latter volume is given by the integral

$$\pi \int_{v=0}^{a/\sqrt{3}} y^2 dv,$$

where $v=CQ=x/\sqrt{3}$, and $y^2=(\tfrac{2}{3})(x^2-ax+a^2)=(\tfrac{2}{3})(3v^2-av\sqrt{3}+a^2)$. Hence the volume of the required solid is

$$V = (2/3)\pi(a\sqrt{2/3})^2 a/\sqrt{3} + (2/3)\pi \int_0^{a/\sqrt{3}} (3v^2 - av\sqrt{3} + a^2)dv = \pi a^3/\sqrt{3}.$$

Also solved by Anthony Barra, A. J. Lewis, F. L. Manning, R. K. Morley, E. P. Starke, Woodrow Tichy, M. J. Turner, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3731. *Proposed by Raphael Robinson, University of California at Berkeley.*

In how many ways can a_1 1's, a_2 2's, $\cdots$, a_n n's be arranged, so that in reading from the beginning, none of the $(k+1)$'s are reached until at least one of the k's has been reached?

3732. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Given the focus and directrix of a parabola and a line anywhere in the plane of the parabola; derive a ruler-compass construction for determining the points in which the line cuts the parabola.

3733. *Proposed by E. P. Starke, Rutgers University.*

(a) For any two rational numbers, p and q, it is known that the triangle whose sides are the rational numbers $a=|p^2-q^2|$, $b=2pq$, $c=p^2+q^2$ is right

angled at C. Derive an analogous rule for a triangle with an angle of 60° or 120° and having rational sides.

(b) Show that, except for the cases noted in (a), no triangle with rational sides can contain an angle commensurate with a straight angle.

3734. *Proposed by A. A. Bennett, Brown University.*

A car with $n(n>2)$ passengers of different speeds of mental reaction passes through a tunnel and each passenger acquires unconsciously a smudge of soot upon his forehead. Suppose that each passenger

(1) laughs and continues to laugh as soon as and only so long as he sees a smudge upon the forehead of a fellow passenger;

(2) can see the foreheads of all his fellows;

(3) reasons correctly;

(4) will clean his own forehead when and only when his reasoning forces him to conclude that he has a smudge;

(5) knows that (1), (2), (3) and (4) hold for each of his fellows.

Show that each passenger will eventually wipe his own forehead. (For the case of $n=3$, this has been proposed in conversation by Dr. Church of Princeton.)

SOLUTIONS

3658 [1934, 50]. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

The Simson line of a point P on the circumcircle of a triangle ABC is the tangent at the vertex of a parabola tangent to the sides of ABC and having its focus at P. See the solution of 3535 [1933, 56].

I. *Solution by J. R. Musselman, Western Reserve University.*

Consider the equation

$$x = \frac{2}{(1-t)^2} \tag{1}$$

where t is a turn, i.e., a complex number whose absolute value is unity. This is the map equation of a parabola with focus at $x=0$ and having for the equation of the tangent at its vertex $x+\bar{x}=1$. The tangent at the point x_1 of the parabola is given by

$$x = \frac{2}{(1-t_1)(1-t)}.$$

The tangents at points x_1 and x_2 of the parabola meet at the point x_{12} whose coordinate is

$$x_{12} = \frac{2}{(1-t_1)(1-t_2)}; \tag{2}$$

consequently the equation of the circle on the points x_{12}, x_{23}, and x_{31}, is given by

$$x = \frac{2(1 - t)}{(1 - t_1)(1 - t_2)(1 - t_3)}. \tag{3}$$

When $t=1$ in (3), $x=0$; hence the focus of the parabola—as is well known—lies on the circumcircle of any three tangents of the parabola. The perpendiculars from the focus, $x=0$, to the tangents at x_1, x_2, and x_3 meet them respectively in the points

$$1/(1 - t_1), \qquad 1/(1 - t_2) \qquad \text{and} \qquad 1/(1 - t_3).$$

These lie on the line $x+\bar{x}=1$. Hence the Simson line of the focus as to the triangle x_{12}, x_{23} and x_{31} is the tangent at the vertex of the parabola.

Now the transformation

$$y = 1 - \frac{(1 - t_1)(1 - t_2)(1 - t_3)}{2} x$$

sends the equation of the circle (3) into that of the base circle $y=t$; sends the coordinates of the points x_{12}, x_{23}, x_{31} into t_3, t_1, t_2 respectively, and sends the coordinate of the focus $x=0$ into $y=1$, a point P on the circumcircle of the triangle $t_1t_2t_3$, which proves the theorem.

II. *Solution by Simon Vatriquant, Brussels, Belgium.*

The statement results immediately from the solution of problem 3535 given by Ethel I. Moody [1933, 56]. The negative pedal curve of the Simson line with respect to P is a parabola having P as focus. Since the Simson line is the locus of the feet of the normals from P to the tangents of this parabola, the three sides of the given triangle are such tangents.

The directrix of the parabola being parallel to the pedal line, this is the tangent at the vertex.

Solved also by L. M. Bauer, J. W. Clawson, and the proposer.

Editorial Note. The proposer, in addition to a statement somewhat similar to Solution II, noted that the pedal circle with respect to a triangle of a point not on the circumcircle is the pedal curve with respect to the point of an inscribed central conic with one focus at the point. See the above reference.

The following considerations for the triangle ABC, where P may or may not lie on the circumcircle, can be easily extended to any tetrahedron, and known results similar to those for the triangle may be obtained. But to the circumcircle in the case of the triangle there corresponds a surface of the third order and not a sphere. This surface appears in problem 3010, solved [1924, 208]; and in that problem it is called Steiner's cubic surface. The reader who is interested in these theorems will find entertainment in deriving in the following manner a number of important theorems for the tetrahedron.

Let P_a, P_b, P_c be the projections of P upon the sides of the triangle. Usually

these three points lie upon an actual circle, which is surely the case if P is inside the triangle. Let Q be the center of the circle, and produce PQ to P' so that $QP'=PQ$; then it is obvious that P_a', P_b', P_c', the projections of P' upon the sides, lie also on this circle. Let $P_b'P'$ cut the circle again in K_b', then $P_bP=P'K_b'$; and we have at once three equations such as

$$P_bP\cdot P_b'P' = r^2 - PQ^2, \tag{1}$$

where r is the radius of circle Q. These equations are true if P lies inside or outside the circle. Consider the system of homogeneous coordinates of a point, x_1, x_2, x_3, which are proportional to the lengths of the perpendiculars from P to the sides of ABC. In this system the coordinates x_1', x_2', x_3' of P' are inversely proportional to those of P, x_1, x_2, x_3. Hence we may speak of P and P' as an inverse pair with respect to the given triangle. They are also said to be isogonal conjugates, where this term means that if we join P and P' with any vertex, say A, then the angles BAP and $P'AC$ are equal. This fact easily follows by writing the equations of AP and AP' and by noting the inverse property above. We may regard P and P' as the foci of a central conic with the circle Q as auxiliary circle; and it then follows that this central conic is tangent to the sides of ABC at points which are easily constructed. If P' goes off to infinity, along a straight line l, then the conic has the limit form of a parabola tangent to the sides of ABC with its axis parallel to l. The limit of P is its focus and the straight line limit of the circle Q is the tangent at its vertex.

Returning to the case where P' is not at infinity we have from the figure

$$ax_2x_3 + bx_3x_1 + cx_1x_2 = 4Rs_p, \tag{2}$$

where a, b, c, R, s_p are the lengths of the sides of ABC, the radius of the circumcircle, and the area of the pedal triangle of P; and where x_1, x_2, x_3 are the coordinates of P such that $x_1=P_aP$. When this equation is made homogeneous and P varies so that the pedal area s_p is constant, the form of the equation shows that the locus of P is a circle concentric with the circumcircle. If P' is at infinity, $s_p=0$, and (2) becomes the equation of the circumcircle of ABC. Hence in the limit case of the parabola the focus lies upon the circumcircle. The statements in regard to (2) may be verified as follows: Consider the parallels AP', BP', CP' and discard the trivial case where a side lies upon a parallel. The triangle must lie within the strip bounded by two parallels, and we may suppose that AP' cuts BC within the segment. The reflection of AP' in the internal bisector at A cuts the reflection of BP' in the internal bisector at B in the point P, and therefore P must lie within the angle A on the side of BC opposite A. It now follows from the figure that angle APB is equal to C and therefore P lies on the circumcircle. It also follows that the corresponding reflection of CP' passes through P. Thus when P' is at infinity the projections of P on the sides lie in a straight line, $s_p=0$, and the coordinates of P make the right side of (2) zero.

It is now possible to trace the variations in the form of the inscribed conic

as P varies, and for this purpose it suffices to consider the region of the angle A. If P is within the triangle ABC it is obvious that the conic is an ellipse, which becomes a circle when P is at the incenter. If P is on BC and within this segment, it is also obvious that P' is at A; and in this case the foci P, P' lie on the auxiliary circle, and the ellipse has degenerated. Now suppose that P lies within the segment of the circumcircle bounded by BC and the arc of that circle subtended by angle A. Produce AP to cut this arc at P_0, then AP_0', BP_0', CP_0' are parallel. From the isogonal property the angles $P_0'CP'$, $P_0'BP'$ must be equal, respectively, to PCP_0, PBP_0, and P' must lie outside ABC on the side of BC opposite to P. Since P' lies also on the isogonal line to AP, it must lie within the angle A. Hence P' lies within the vertical angle to A formed by BA and CA produced, and each of its coordinates has a sign opposite to that of the corresponding one for P. The relation (1) shows that $PQ > r$ in this case; and the conic is an hyperbola. If P is on the arc BC, the conic is a parabola. If P is within the region bounded by the arc BC and the lines AC and AB produced, the same kind of argument shows that P' lies in the same region; the corresponding coordinates have the same sign and $r > PQ$. Thus in this region the conic is an ellipse, which becomes a circle when P is at the excenter for angle A. This suffices to describe all regions of angle A due to the inverse properties of P and P'.

If P describes a straight line l, its inverse P' describes a conic L through the vertices of the triangle, and conversely. The conic L and the line l intersect in two points, P_1 and P_1', which must be inverse points. The special cases where l passes through the incenter, an excenter, or a vertex are easily handled. The system of homogeneous coordinates makes these theorems obvious. Thus to each pair of inverse points there are associated a straight line and a conic through the vertices, the line and the conic intersecting in the pair of points. The general theorem may now be put in the following form: A pair of inverse points P_1, P_1' determine a conic L through the vertices and these two points, and, if P is any point on the conic L, its inverse P' is on the straight line l through P_1 and P_1'. The points P and P' are the foci of a conic inscribed in the triangle of reference and the six projections of P and P' on the sides of the triangle lie upon the auxiliary circle of the inscribed conic.

The circular points at infinity I, J, have the property that the angles BAI and JAC are equal for any triangle ABC. They are consequently inverse points for any triangle, and the conic L which is associated with these points and the line at infinity l must be a circle. If P is any point on this circle, its inverse P' must be on the line at infinity. In this case P is the focus of a parabola inscribed in the triangle with the axis PP', and the projections of P on the sides of the triangle lie upon the tangent at the vertex of the parabola.

When the conic L is not a circle, the variation in the form of the corresponding inscribed conic as P describes L may be traced by considering the movement of P' on the corresponding straight line l; and what has been said above about the regions of the plane suffices for this purpose.

3659 [1934, 50]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let H_1, H_2, H_3, H_4 be the feet of altitudes of a tetrahedron $A_1A_2A_3A_4$ and $A_{12}A_{13}A_{14}$, $A_{23}A_{24}A_{21}$, $A_{34}A_{31}A_{32}$, $A_{41}A_{42}A_{43}$ the antipedal triangles of $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ with respect to H_1, H_2, H_3, H_4 respectively.

Prove that the perpendiculars drawn from A_{12}, A_{13}, A_{14} on $A_1A_3A_4$, $A_1A_4A_2$, $A_1A_2A_3$ are concurrent at a point α_1: and that, if α_2, α_3, α_4 are the points similar to α_1, $A_1A_2A_3A_4$ and the tetrahedron formed by the mid-points of $H_1\alpha_1$, $H_2\alpha_2$, $H_3\alpha_3$, $H_4\alpha_4$ are orthologic.

Editorial Note. Since the only solution of this problem which has been received is rather long and by vector analysis, an elementary geometry proof will be given. The line $A_{ik}A_{il}$ is perpendicular to A_iA_j. The perpendicular from A_{il} to face $A_iA_jA_k$ is perpendicular to A_iA_j; and therefore the two perpendiculars from A_{ik} and A_{il} to their respective faces are each perpendicular to A_iA_j. They lie in the plane through $A_{ik}A_{il}$ perpendicular to A_iA_j. Hence the intersection of this plane and the other two similar planes determines α_i. We have the three pairs of perpendiculars

$$\alpha_iA_j, \quad A_iA_j; \qquad \alpha_iA_k, \quad A_iA_k; \qquad \alpha_iA_l, \quad A_iA_l;$$

and hence the points α_i, A_i, A_j, A_k, A_l lie on the circumsphere of $A_1A_2A_3A_4$ with center O and diameter α_iA_i. Thus $\alpha_i\alpha_j$ is parallel to and equal to A_jA_i. If B_i is the mid-point of α_iH_i, B_iO is perpendicular to face $A_jA_kA_l$ and has half the length of H_iA_i. Thus the tetrahedron $B_1B_2B_3B_4$ is such that the perpendiculars from its vertices to the corresponding faces of $A_1A_2A_3A_4$ meet in the point O; hence the two tetrahedrons are said to be orthologic. There is a general theorem which states that the perpendiculars from the vertices of the A tetrahedron to the corresponding faces of the B tetrahedron meet in a point provided the converse is true, as it is in this case. This theorem gives the complete significance of the term orthologic.

3660 [1934, 50]. *Proposed by Lulu Hofmann, Columbia University.*

Given a definite projective transformation T of a primitive one-dimensional form into itself, expressed with reference to a given particular anharmonic ratio coordinate system C. To characterize all other anharmonic ratio coordinate systems C' such that with reference to them T has the same analytic expression as with reference to C.

Solution by the Proposer.

The transformation T, referred to any system, is represented by ∞^2 (we consider the complex domain) ordered pairs of numbers (x, y), x and y respectively being the coordinates of the original and the transformed element. If T is to have the same analytic expression in two different coordinate systems C and C', it must determine the same pairing off of numbers in both. In particular, if with reference to C a definite number d is self-paired, i.e., determines

a self-paired element of the form, the element having the same coordinate d with reference to C' must also be self-paired.

Let T be non-parabolic; then if the two distinct self-corresponding elements D and D^* have the coordinates d and d^* with reference to C, they will upon passage to C' either both retain their coordinates: $D(d)$, $D^*(d^*)$, or they will interchange them: $D(d^*)$, $D^*(d)$. If the characteristic anharmonic ratio of T is α, where α has no one of the values 0, 1, ∞; and x, y are the coordinates of the original and the transformed element, the transformation with reference to C will read:

$$\frac{(x-d)(y-d^*)}{(x-d^*)(y-d)}=\alpha.$$

It follows that if $\alpha\neq -1$, D and D^* must separately have the same coordinates in C and in C'; if $\alpha=-1$, they may in addition upon passage from C to C' interchange their coordinates.

Now let T be a parabolic transformation. Then if the one self-corresponding element D has the coordinate d as referred to C, it must obviously have the same coordinate d as referred to C', and there must be no other element outside of D having the same coordinate in both systems. For if any other element P had the same coordinate in both systems, this would also be true of its homologue P' which by assumption must be different from P, so that three distinct elements, D, P, P', would each have the same coordinate in C and in C', and C and C' would be identical. It is easily seen that the necessary condition stated is also sufficient, by expressing T with reference to a coordinate system in which D lies at the origin: $y=ax/(ex+a)$, and then performing the permissible coordinate transformations: $x'=rx/(lx+r)$ which are all those for which the origin of the first system is the only element retaining its coordinate.

The complete answer to the question then reads as follows.

A coordinate system C' in which a definite projective transformation T of a primitive one-dimensional form into itself has the same analytic expression as in a particular given system C is characterized as follows:

(1) If T is a non-parabolic non-involutorial transformation, C' is any coordinate system such that the two self-corresponding elements of the transformation separately there have the same coordinates as in C;

(2) If T is an involution, C' is any system such that the two self-corresponding elements of the transformation together there have the same coordinates as in C (the coordinates separately remain the same or are interchanged);

(3) If T is parabolic, C' is any system such that the one self-corresponding element of the transformation, and no further element, there has the same coordinate as in C.

The same problem for a two-dimensional form or an arbitrary linear space is left as a useful exercise. It will be convenient to start out from the same argument as in the present case.

We close with the obvious remark that the stated problem is equivalent in any space to the following one: given a definite projective transformation T, to find all projective transformations commutative with it. For if the transformation T is expressed in C by $y=T(x)$ and the transition from C to C' by $x'=K(x)$, then the condition that the expression of T with reference to C' shall again read $y'=T(x')$ is: $KTK^{-1}(x')=T(x')$ or $KT(x')=TK(x')$. If now we think of K as of a second projective transformation expressed of course with reference to the same system as T, we have the interpretation as indicated.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

A new mathematical quarterly sponsored by Duke University has been established under the title "Duke Mathematical Journal." It will be devoted primarily to the publication of original research in mathematics, but expository papers of outstanding merit will be accepted. The editors are A. B. Coble, University of Illinois; D. V. Widder, Harvard University; and J. M. Thomas, Duke University, Managing Editor. The associate editors are H. E. Bray, L. W. Cohen, L. R. Ford, J. J. Gergen, R. E. Langer, C. C. MacDuffee, J. A. Shohat and G. T. Whyburn. The first number appears under the date of March 1935. Manuscripts and editorial correspondence may be addressed to the journal at 4785 Duke Station, Durham, N.C., and subscriptions to the Duke University Press, Durham, N.C. The subscription price is four dollars per volume. By reason of a subvention which the Trustees of the Mathematical Association voted for the new journal, individual members of the Association may subscribe at half price and should mention the fact of their membership when subscribing.

At the annual meeting of the British Mathematical Association held in London, January 7 and 8, Professor David Eugene Smith of Columbia was nominated for election as an honorary member.

Dr. Ellen Fitz Pendleton, formerly associate professor of mathematics at Wellesley College, and since 1911, its president, has tendered her resignation as president, to take effect in June 1936.

Dr. James Franck, formerly professor of physics at the University of Göttingen and now at Copenhagen, has been appointed professor of physics at the Johns Hopkins University.

Dr. H. P. Wirth of the College of the City of New York has been promoted to an assistant professorship.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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REPORT ON THE TRAINING OF TEACHERS OF MATHEMATICS

In June, 1933, the Trustees of the Mathematical Association of America directed the President of the Association to appoint a Commission on the Training and Utilization of Advanced Students of Mathematics. Subsequently, the President appointed the Commission as follows:

E. J. MOULTON, Chairman, Northwestern University;
WILLIAM BETZ, Rochester Public Schools and University of Rochester;
W. L. HART, University of Minnesota;
J. O. HASSLER, University of Oklahoma;
E. R. HEDRICK, University of California at Los Angeles;
E. V. HUNTINGTON, Harvard University;
M. H. INGRAHAM, University of Wisconsin;
R. G. D. RICHARDSON, Brown University;
H. E. SLAUGHT, University of Chicago;
E. B. STOUFFER, University of Kansas.

As a background for some of the work of this Commission it was recognized that with rare exceptions advanced students of mathematics have always looked to the teaching profession as their means of livelihood. Hence, it is appropriate from time to time to test the orientation of advanced instruction in mathematics when this training is considered as a means for preparing students to be teachers as apart from their possible service in research. In harmony with this point of view, the Commission decided to formulate suggestions for graduate training for teaching mathematics and to make recommendations with regard to the undergraduate training of those who plan to teach mathematics in the secondary schools. Accordingly, on vote of the Commission, the Chairman appointed a sub-committee to make a *Report on the Training of Teachers of Mathematics*. The membership of this sub-committee was as follows: Hassler, Hedrick, Ingraham, Moulton, Slaught, and Hart (Chairman).

This sub-committee, after numerous meetings and after having obtained the advice of many other mathematicians and of teachers in related fields at both the secondary and the university levels, submitted a preliminary report to the Commission as a whole which led to the report which is given below. It was found convenient to treat the problems of undergraduate and of graduate training separately in spite of natural connections between them.

At the Pittsburgh meeting of the Association in December, 1934, the report, after adoption by the Commission as a whole, was presented to the Trustees of the Association, by whom it was accepted and approved.

Part I. GRADUATE TRAINING FOR TEACHERS OF MATHEMATICS

1. *Introduction.* In the discussion of graduate training, we shall be thinking primarily of preparation for teaching mathematics in a college rather than at a more elementary level. However, the training which we shall outline will be

so rounded that it should likewise be recognized as furnishing a student with superior preparation for teaching secondary mathematics, provided that he also adds to his equipment whatever auxiliary knowledge is peculiarly necessary for elementary teaching. We shall assume that the graduate training for teachers should consist of three years of advanced study under the guidance of a department of mathematics, and not under the control of some outside agency such as a college of education. Any first-year graduate curriculum, perhaps leading to a master's degree, will be thought of principally as the first part of a complete program.

A three-year period is specified for the full training because, even for teaching at the secondary level, we believe that a knowledge of mathematics and related fields beyond what is currently represented by a master's degree, particularly if this knowledge is broad rather than specialized, is of assistance to the teacher of mathematics both as a teacher and as a member of his intellectual community. We agree that the level of intelligence demanded for successful completion of this three-year training should be at least as high as that required for the existing doctorate in mathematics. We specify this level regardless of any final decisions which may be reached concerning alterations in the training for the degree of Doctor of Philosophy or concerning the inauguration of a graduate curriculum leading to a new degree.

At such a level, the present facilities for graduate instruction and the available student material could be expected to produce annually about one hundred teachers of mathematics with a degree meriting at least as great respect as the present doctor's degree. To judge whether or not this rate of production would be appropriate, we note that there are more than three thousand persons engaged in teaching mathematics above the secondary level in the United States and Canada. Hence, after the present evil effects of reductions in college teaching staffs disappear, the normal replacement demand for teachers of mathematics in the best secondary schools and at higher levels should be sufficient to absorb the teachers whom our three-year curriculum would aim to produce.

In the preceding statements, the word *college* should be understood to refer to any institution of learning offering instruction in mathematics above the secondary level. Thus, we shall not consider preparation for teaching in a junior college as a problem distinct from that of a college or university which offers the full four years of undergraduate work. This attitude is taken because we believe that the teacher of mathematics in a junior college should have essentially the same background as a teacher of senior college courses. We recognize, however, that a candidate for teaching mathematics in certain junior colleges in order to satisfy existing legal requirements may have to make provision for a few items of training in addition to what we shall recommend.

2. *Characteristics of present doctoral training.* Before outlining our suggestions concerning graduate instruction, we wish to review certain facts about the present typical American training for the doctorate in mathematics. This train-

ing is primarily characterized by its demand for narrow specialization, even within the field of mathematics, leading to the preparation of a research thesis. The fundamental emphasis is on preparation for research, and the candidate spends a major portion of his three years of graduate study on work connected with his thesis. Even then the completed thesis is too often the result largely of industry and originality on the part of the directing professor.

One test of the efficiency of the training for research lies in the later productivity of the candidates. Although the amount of valuable research produced by American trained doctors in mathematics, considered as a whole, is increasing at a very creditable rate, we note that many of them produce no post-doctoral research. We find, for instance, that among those who received their degrees during the years 1920–24 there were 46 per cent who had no research publication of consequence* by the end of 1933, and only 25 per cent who had published more than two research papers by that time. Hence, it is no apparent overstatement to assert that, under present conditions, at least 80 per cent of those receiving the doctorate in mathematics will publish no useful research beyond their doctoral theses and closely associated results. The training for research must therefore be justified for most candidates, if at all, on other grounds than their contributions to the advancement of mathematical research, as that term is customarily understood.

We do not believe that these facts concerning published articles by doctors in mathematics reflect discredit on them as a group, in contrast with doctors in other fields where the per capita research output may seem greater. The peculiarities of our field and its traditions of long standing have created a very exacting standard for publishable research in mathematics. For instance, in our field, only infrequently would we attach the title of research to the mere collection of facts and an analysis of their relationships. Also, the experimental method has only limited application in mathematics, and hence we do not have available numerous simple variations of fundamental experiments as a field for research by those less gifted in the performance of original mathematical investigations. In certain fields other than mathematics, the mere intelligent use of an appropriate mathematical theory, as for instance in some statistical investigations, is frequently dignified by the name of research; in our field, as a rule, we demand that research should accomplish an advance in theory itself. In view of these facts, although we admit that there is very little publishable research performed by many of the doctors in mathematics, we retain the opinion that, as a body, they are fully the equals in general intellectual qualities of the doctors in any other field.

3. *Desirability of a second type of three-year training.* The failure of prevalent research training to make productive mathematicians out of the majority of the doctors in mathematics, which failure we have condoned, leads us to ques-

* This means that no article had been published in the journals devoted to research, according to a statistical study made under the direction of Dean Richardson.

tion the desirability of demanding a research thesis of the prevailing type from all candidates for the doctorate.

In the first place, the preparation of theses by those who will produce no subsequent research places a heavy tax on the productivity of the mature scholars who direct the work. A great many of these directors essentially perform much of their own research in an indirect and inefficient manner through the painful process of leading to a conclusion the investigations of students who possess no particular aptitude or desire for such work. Also, the production of research theses in large numbers creates undesirable pressure on the publication space available in our mathematical journals. These theses, which chronically exhibit a low standard of expository ability, are condensed and otherwise whipped into shape for publication only at the expense of great labor on the part of editors and referees, to the detriment of their own research. To require students not gifted in research ability to produce publishable research at the extravagant expense of time both of the students and of the most gifted research professors is a questionable procedure for the advancement of mathematical science.

These expensive consequences of the wholesale manufacture of research theses in mathematics lead us to question whether it is either necessary or desirable to demand doctors' theses of the current type from all candidates for a three-year degree in mathematics. We believe that in the case of a candidate who will probably publish no post-doctoral research, the preparation of a typical thesis would be of less value than other training which might be substituted for this research. For instance, in place of it, he might acquire a wider knowledge of the fields of mathematics and of their relations with each other; and, he might prepare an exposition of significant mathematical results which are new to him even if known or inferred without proof by maturer mathematicians, or a historical report on the development of some mathematical theory, or a critical review of work of another mathematician.

It may be argued that only by giving all candidates for the three-year degree the customary thesis training can one obtain the small number who will become successful at research. If this contention were true, the present system of thesis preparation would be for the best interests of mathematics, although perhaps not for the best interests of all graduate students involved. However, we believe that a reasonable diagnosis of the research ability of a student could be made before he should start on the preparation of a doctor's thesis. In the cases of a large number of candidates, the professor in charge should be able to make an early decision that any original investigation which the student might carry through in this thesis would probably be his last publishable research.

As a consequence of the preceding considerations, we are convinced that departments of mathematics as a whole should not demand research theses of the prevailing type from all students who wish to complete a three-year course of graduate training. However, we realize that certain departments may desire to welcome only students who wish to carry on such research and are fitted for the independent preparation of theses of the current type. In any case, we be-

lieve that a department is not giving the best possible training to a person who is to teach mathematics if, in spite of his lack of aptitude or desire for research, he is caused to spend a major portion of three years of graduate study on the preparation of a research thesis. Hence, we recommend that some departments of mathematics should offer a second type of three-year graduate training in addition to the present typical training for the doctorate.

4. *Synopses of two types of training.* In amplification of the preceding recommendation, the Commission offers the following synopses of two varieties of three-year training as a basis for further discussion. The first type, referred to hereafter as Type I, involves the preparation of a research thesis of the current variety. The second, referred to as Type II, might omit the customary kind of research thesis but includes various substitute features. Both types of training involve specialized preparation for teaching mathematics. Gradations between these two types might be used for adaptations to individual students.

Recommended Undergraduate Preparation for Candidacy

An undergraduate major or field of concentration in mathematics; introductory courses in physics, astronomy, psychology, economics, one other science, and one other social science; sufficient additional preparation in one field related to mathematics to meet the prerequisites for elementary graduate work in that field; training in English composition beyond the minimum required for a bachelor's degree; sufficient study of German and French to serve as a basis for learning to read mathematics in these languages.

Graduate Training Common to Types I and II

(1) Foundation courses in each major subdivision of advanced mathematics.

(2) Elementary graduate work in at least one outside field related to mathematics.

(3) Specialized preparation for teaching mathematics:

a. The equivalent of one year of observation and assisting three times a week in various college courses in mathematics which are taught by experienced members of the department of mathematics.

b. Practice teaching in college mathematics under the observation of, and with later criticism by, members of the department of mathematics. This teaching might advisably be done in different courses and should amount to the equivalent of at least a two semester-hour course. The practice teaching should involve participation in the construction and grading of examinations.

c. Guided reading in books and periodicals relating to the theory of teaching, testing methods, and educational research. This reading could be directed either by a member of the department of mathematics or, perhaps, by a person outside the department who appreciates the viewpoint of teachers of mathematics.

(4) A final examination, perhaps both oral and written.

Training Peculiar to Type I

(1) Intensive specialization beyond the foundation material in some large field of mathematics as the basis for later research.

(2) A research thesis of a caliber suitable for publication, where the topic and a minimum of general assistance may be given by a thesis director, but where the responsibility for the development of the topic rests strictly with the candidate.

Training Peculiar to Type II

(1) Additional course work in mathematics and in allied fields, with emphasis on breadth of training.

(2) One or more expository papers of the variety known as *minor theses* in the existing requirements for the doctorate in mathematics in some universities. A thesis of this variety would require the candidate to give evidence of his ability to learn independently and to present in good written form, in a brief period of time, some specified known mathematical results with which he was previously unfamiliar.

(3) A major thesis which would exhibit the candidate's mastery of some field of mathematics and expository ability of high order, although perhaps not research ability. The preparation of this thesis should be the independent responsibility of the candidate. This thesis might be of a historical nature, or it might involve material which, though original from the candidate's viewpoint, might not be acceptable for publication in a leading periodical.

5. *Discussion of the explicit training for teaching.* In the description of training common to Types I and II, item (3) is somewhat novel as judged by current methods for training teachers of college mathematics. In many universities it is customary to employ graduate students as instructors in elementary courses in mathematics. Such independent teaching by relatively inexperienced persons, even when they are under the outside direction of regular members of the staff, is not a suitable substitute for the training specified in (3a) and (3b), and frequently has ill effects on the mathematical futures of the elementary students who are being taught. We believe that it would be highly desirable for graduate students, however brilliant mathematically, to complete successfully the training described in (3a) and (3b) before they are given independent control of classes in college mathematics.

The guided reading described in (3c) is not intended to justify a requirement of course work in Education. In fact, we are convinced that the objective of (3c) would not be attained if the guided reading were replaced by typical general courses in educational theory. The reading described in (3c) should prepare the candidate to evaluate intelligently or criticize constructively conclusions and methods with which he may later be confronted in the teaching profession.

6. *Suggestions concerning names of degrees for Types I and II.* If we admit the desirability of the two types of training which have been described, it is neces-

sary to decide on the degree or degrees which would be granted for completion of the specified work. Before making a decision, it is proper to recall that the standards of achievement for work of Types I and II would be the same in all common features. It is assumed that the classes in graduate mathematics, aside from some which deal with specialized advanced content, would include on the same footing candidates for both types of training. Moreover, we must not infer that students desiring or advised to take Type II would be inferior, as a whole, to those of Type I from the standpoint of ability to learn, appreciate, and expound mathematics. In fact, some students, even though they possess a certain amount of aptitude for research, might desire the greater breadth of training presented by Type II in preference to the specialization and research of Type I. Also, it is conceivable that a student might produce publishable research of the current variety after obtaining a degree for training of Type II. Hence, on the basis of the quality and quantity of the work to be accomplished, it appears desirable that Types I and II should lead to the same degree or to degrees having equal respectability.

In deciding on the names for the degrees to be associated with Types I and II, three possibilities come to mind:

(A) The degree of Doctor of Philosophy might be retained for training of Type I, and a new degree might be associated with Type II, perhaps the degree of Doctor of Mathematics.

(B) The degree of Doctor of Philosophy might be assigned for both types of training. In such a case, some universities might wish to add to the description of the degrees the qualifying phrases "in research" for Type I and "in course" for Type II.

(C) The degree of Doctor of Philosophy might be assigned for training of Type II, and a new degree might be associated with Type I.

The Commission believes that (C) is impracticable principally because it would involve a name for our research degree out of harmony with practice in other fields. Hence, we shall restrict our consideration to (A) and (B).

If (A) is adopted without safeguards, student opinion and gradually developed usage might tend to assign much greater prestige to the research degree than to the new degree for Type II, regardless of our viewpoint that both degrees should be considered of essentially equal respectability. If such a situation should develop, graduate students of mathematics would automatically divide themselves into two groups, one considering itself distinctly superior to the other. Then, all students of the present doctoral caliber would seek the degree of Type I, and thus most of our present difficulties as to graduate training in mathematics would persist. Moreover, as an additional evil, there might be a resulting tendency to lower the intended standards for Type II and thus to cause an undesirable increase in the number of people possessing a stamp of approval for the teaching of college mathematics. Hence, if (A) is adopted, precautions should be taken to avoid the dangers to which we have just referred.

A major objection to (B) is that it would assign the degree of Doctor of

Philosophy for training of Type II in apparent violation of the tradition in all fields of learning that this degree implies the production of publishable research. One answer to this objection is that the program specified for Type II, including the major thesis, compares favorably with the existing requirements as to course work and thesis for the doctorate in other fields. While the major thesis of Type II might not qualify as publishable research under present mathematical standards, nevertheless this thesis would be on a par, from the standpoint of originality and prerequisite training, with a large body of the research theses in other fields. Hence, it is conceivable that some graduate faculties could adjust themselves to the acceptance of the major thesis of Type II as a basis for the degree of Doctor of Philosophy when this thesis is reinforced with the supplementary items included in Type II.

Another objection to (B) which may be advanced is that the absence of a requirement of a publishable thesis in Type II would encourage unqualified departments of mathematics to grant a flood of low grade doctorates of this kind. This is a real danger, but it must be admitted that a similar danger is faced at the present time in regard to the granting of the current doctor's degree. As a matter of fact, we believe that the demand for breadth of training in Type II would make it more difficult for a department to provide good instruction of this variety than to offer minimum training of the current narrow type which leads to a doctorate in mathematics.

An advantage of suggestion (B) is that it would make it simple to handle gradations between Types I and II. If it is agreed that training of Type II should not be given by an institution not qualified to provide training of Type I, which we recommend, the general standards for the degree of Doctor of Philosophy would seem to be safeguarded.

In Part I of this report, our major aim is to present a picture of a desirable new variety of training which we have referred to as Type II, in addition to our minor aim of urging specialized preparation for teaching in any three-year graduate training for teachers of mathematics. While there may be lack of agreement as to the degree to be awarded for completing the new variety of training, *we are convinced that this degree should be a doctor's degree*, preferably Doctor of Mathematics or Doctor of Philosophy. The decision as to the degree which is to be conferred must rest with the faculty of any graduate school which initiates training of the new type.

7. *Requirements for a master's degree.* We shall conclude our remarks with respect to graduate training by making recommendations in regard to requirements for a master's degree in mathematics to be given at the end of one year of graduate study. Under existing conditions in the teaching profession, such a degree by itself is most useful if it implies certification of exceptionally good preparation for teaching secondary mathematics. For it must be anticipated that in the future a mere master's degree will be of only slight use in a search for

placement on a college faculty, regardless of whether the college is a two-year or a four-year institution. However, in some cases the candidate for a master's degree will consider it only as the first milestone on the way to a doctor's degree, and will not be primarily interested in the use of the master's degree as an aid in obtaining a teaching position at the secondary level. Thus, in formulating requirements for a master's degree, we should have in mind the fact that some candidates for the degree will think of it as a goal in itself, while other candidates will consider the work for the degree only as a part of the three-year training for a doctor's degree.

Any candidate for a master's degree in mathematics should aim to obtain the undergraduate preparation which we outlined in the discussion of the three-year program, except that reading ability in either French or German alone would be satisfactory. Beyond this level of preparation the candidate should be given as broad training as is possible within the limits of (at least) twenty-four semester-hours of graduate work, of which approximately six hours might be devoted to a minor field related to mathematics. It might be desirable, although not absolutely essential, for the candidate to conclude his training by writing a major mathematical paper, although this thesis would be, in general, of an expository nature and not worthy of the name research.

The importance of breadth of training rather than specialization during a first year of graduate study for the usual student makes it advisable to recommend that only a minor fraction, say ten per cent, of his time should be devoted to preparation of a thesis, when one is required. In the assignment of subjects for the theses as well as in the choice of course work, the candidate for a master's degree who intends to continue his graduate study beyond this stage might be differentiated from the candidate who intends to enter immediately into secondary teaching. A thesis subject involving contacts with the secondary field might be assigned to the candidate with an interest in this direction, whereas the other candidate could be assigned a topic which prepares the way for later advanced work in some field of mathematics.

A candidate who desires a master's degree in preparation for secondary teaching should analyze his undergraduate preparation in fields outside of mathematics in the light of the detailed discussion of undergraduate preparation for teaching secondary mathematics which is to be presented in the second part of this report. Deficiencies in the candidate's undergraduate preparation in content fields and in work needed to meet the legal requirements in practice teaching at the secondary level and in the theory of education should be remedied while he proceeds, perhaps at reduced speed, with his graduate work for a master's degree. Even though a student may expect to go forward immediately with a three-year program of graduate study, he would be wise, under existing conditions, to prepare himself for teaching in the secondary field and follow the program which we have just recommended for the candidate who is pointing his preparation definitely toward that field.

Summary of Recommendations for the Graduate Training of Teachers of Mathematics

1. Departments of mathematics should offer three-year graduate training of Types I and II, which have been described, as appropriate preparation for teachers of mathematics at any level above the secondary field, and as exceptionally high grade training for teachers of secondary mathematics when supplemented by auxiliary knowledge necessary for elementary teaching. Any individual department might choose to offer only training of Type I, but no department should offer Type II without also offering Type I.

2. The degree of Doctor of Philosophy should be granted for training of Type I, and either Doctor of Mathematics or Doctor of Philosophy for Type II.

3. A department of mathematics should not feel qualified to provide training of either type described unless the department contains (a) people qualified to present advanced mathematics in each of its major fields; (b) people who themselves perform research and are qualified to direct the research of students; (c) one or more members with full advanced training who are noted for their interest and ability in teaching undergraduate mathematics and who are ready to direct the specialized training for teaching described in item (3) under Training Common to Types I and II.

4. A master's degree in mathematics should indicate that the recipient has completed satisfactorily at least twenty-four semester-hours of graduate study in advance of the level specified in this report as the preparation for candidacy for a three-year degree (except for the omission of one of the foreign languages mentioned in the outline). Outside of mathematics courses this work might include approximately six semester-hours of study in one field related to mathematics. In addition to the course work just mentioned, it might be desirable in some cases, although not necessary, to require the candidate for the master's degree to prepare a thesis, probably of expository nature, to which he would devote approximately ten per cent of his efforts during his study for the degree.

Part II. UNDERGRADUATE TRAINING FOR TEACHING SECONDARY MATHEMATICS

8. *Breadth of training.* A teacher, to be of maximum service to the community in which he lives, should be recognized as an educated man to whom adult members of the community may turn for consultation on intellectual matters. He should be able to participate in community activities, and assume his share of leadership. Certainly he cannot function satisfactorily if he is notably ignorant in what are commonly regarded as fundamentals of general culture. With these facts in mind we advocate a breadth of training for teachers of mathematics which will insure a degree of familiarity with language, literature, fine arts, natural science, and social science, as well as mathematics.

The practical necessity for a breadth of training is emphasized by considera-

tion of fundamental facts in the administration of secondary schools. In a large school it may be possible for a mathematics teacher to limit his teaching to mathematics, but under emergency conditions he may be called upon to give instruction in other fields, and in smaller schools he is almost certain to be asked to do so as a matter of necessity. Thus, it is highly important for the prospective teacher whose major interest is in mathematics to prepare himself for teaching other subjects as well.

Moreover it all too frequently happens that teachers whose major preparation has been in other fields are called on to give part time instruction in mathematics, particularly in the smaller high schools. For this reason it is important, that we should express ourselves in regard to the appropriate training and mathematical limitations of those secondary teachers who will perform work in mathematics.

In view of the preceding facts we shall discuss a training program in which mathematics is a minor field and another in which mathematics is the major field. As a guiding principle in regard to teaching in the secondary field, we wish to express the opinion that *no teacher should perform a majority of his teaching in a field for which he does not possess major preparation.*

In an effort to raise the general tone of teacher training, our recommendations will depart from what has been current practice. We believe that the higher standard should be obtainable in the near future in view of the prevailing high level of general education and the shrinkage of opportunities for satisfactory employment of college graduates.

9. *Mathematics as a minor field.* In considering possible suggestions for training in mathematics as a background for minor teaching in this subject, the Commission is impressed with the need for a distinct elevation in the minimum levels now in effect. In view of modern trends in the content of high school courses, a bare acquaintance with algebraic manipulation is no longer sufficient even for a deadly uninspiring presentation of eighth or ninth grade mathematics. In various courses in mathematics at these levels, it is indispensable for the teacher to have at his command a thorough knowledge of trigonometry, college algebra, and the typical methods of analytic geometry. We also find that eighth and ninth grade mathematics involve considerable content whose background is found in the college courses in elementary physics, statistics, economics, and the mathematics of investment. For a truly inspiring presentation of ninth grade mathematics, particularly to students who may plan to prepare for college entrance, the teacher should appreciate the significance of elementary mathematics in the light of important applications of more advanced mathematics. Such equipment requires far more than the mere minimum of training sufficient for a mechanical presentation of subject matter.

Hence, in our recommendations concerning minor preparation for teaching mathematics below the tenth grade, two levels of training are specified: the lower level may be considered as an *irreducible minimum* suitable under tempo-

rary emergency conditions, and the upper level as the *desirable training*. In addition to the preparation suitable for teaching ninth grade mathematics, the instructor of tenth grade mathematics, regardless of whether mathematics is his major or a minor field, should have a knowledge of Euclidean geometry beyond that involved in the typical high school course. In fact, we believe that people with only minor preparation for teaching secondary mathematics should not teach tenth grade mathematics except under temporary emergencies, and that these people should never instruct classes in mathematics above the tenth grade.

We make the following recommendations concerning the teaching of mathematics as a *minor* field at the secondary level:

1. A teacher with only minor preparation in mathematics should teach in this field only if he is associated with a teacher having major preparation for teaching mathematics.*

2. For teaching mathematics *below the tenth grade*, the minimum mathematical background (suitable only under emergency conditions) should include eight semester-hours of college mathematics above two and one-half units of secondary mathematics, completed with grades above the median level. A *desirable* mathematical background should include also: (a) college mathematics through six semester-hours of calculus; (b) introductory college physics and economics; and (c) an introduction to the mathematics of investment.

3. For teaching mathematics *through the tenth grade*, the minimum training should include all items mentioned in the previous paragraph, together with a college course in synthetic Euclidean geometry in advance of high school geometry. A teacher with this limited preparation, however, should teach mathematics in the tenth grade only under emergency conditions, and never above the tenth grade.

4. The prospective teacher with mathematics as a minor would derive profit from a course in methods of teaching secondary mathematics, even if he is able to include little or no associated practice teaching. It would be desirable to substitute such a course for some of the more general educational theory which otherwise might be studied by the candidate in satisfying legal requirements for a secondary teaching certificate.

10. *Mathematics the major field*. We have remarked that, in the preparation of a student for major teaching in secondary mathematics, provision should be made for rounded training outside of mathematics in order to provide him with teaching minors. Moreover, breadth of training is essential in giving the future teacher the necessary background for inspirational teaching in his major field. We believe that proper provision for background and cultural courses in the training of a teacher should take precedence over added acquaintance with his major field beyond a suitable level, and over elective courses in the theory of education.

* For legislative purposes, cities below a certain size might have to be excused from such a restriction.

Our recommendations will indicate two levels of preparation in content, one specified as a *minimum* and the other as a *desirable* level. This procedure is adopted because, in certain colleges, requirements in the theory of education and in other fields make it impossible for the desirable level to be attained within the four years of work leading to a bachelor's degree. In such colleges, the student should consider the advisability of completing at least that *desirable* preparation by additional study in a fifth year (which need not necessarily be concluded by the attainment of a master's degree).

In making a recommendation concerning training in educational theory and practice teaching, we shall not approve by name any of the variously labeled courses which appear in the requirements in Education for the high school teaching certificate in different universities. Among such course names we might mention Educational Psychology, History of Education, Adolescent Psychology, Educational Measurements, and so forth. In colleges and departments of education as a whole, and sometimes even within the limits of a single college or department, we find that courses with these names are not standardized as to content. Frequently these courses involve great duplication of material. Moreover, from the standpoint of a major student of mathematics, the statistical content and other theory of a mathematical nature involved in the courses is sometimes presented on such a low level that the student's time is inefficiently used.

The utility of these courses in Education probably depends on the qualifications of the instructors and their ideals more largely than is the case in fields where the subject matter is more definitely standardized. However, we believe that in the fields of pure psychology and educational theory there is much material which should be valuable as training for teachers of secondary mathematics. We are inclined to think that, outside of foundation work in psychology, all of the theory of education presented to the candidate for a secondary teaching certificate in mathematics could best be given in courses definitely oriented with respect to his major teaching field and containing only students whose major or minor interests are in this field.*

Recommendations Concerning Preparation for Major Teaching of Secondary Mathematics

(1) *Minimum* training in mathematics.

a. Courses in mathematics including complete treatments of trigonometry, college algebra, analytic geometry, and six semester-hours of calculus.
b. A college treatment of synthetic Euclidean geometry (or, possibly, descriptive geometry) (three semester-hours).
c. Advanced algebra, such as the theory of equations (three semester-hours).

* This method, though at variance with the common practice of presenting educational theory from a general viewpoint, is in successful operation at the University of Wisconsin in the training of teachers of secondary mathematics.

d. Either directed reading or a formal course in the history of mathematics and the fundamental concepts of mathematics.

(2) *Minimum* college training in fields related to mathematics.

a. Introductory courses in physics and in another science (twelve semester-hours).
b. A course in the mathematics of investment (three semester-hours).
c. An introductory course in economics.
d. A first course in statistics, with a mathematical viewpoint (three semester-hours).

(3) *Desirable* additional training in mathematics and related fields.

a. Advanced calculus and differential equations or mechanics (six semester-hours).
b. Additional work in geometry, such as projective geometry, solid analytic geometry, etc. (three semester-hours).
c. Additional study in algebra (three semester-hours).
d. Introduction to astronomy.
e. Additional study of physics and other sciences to complete a background in three or more sciences (nine semester-hours).

(4) Adequate training in English composition and cultural training outside of mathematics and related fields. Work in languages, literature, fine arts, and the social sciences in preference to increased specialization in mathematics and related fields, and in preference to elective work in the theory of education beyond the legal requirements.

(5) Training in the theory of education and practice teaching.

a. A one-year course in methods of teaching and practice teaching in secondary mathematics, together with any distinctly pertinent material concerning educational measurements and other content from educational theory (ten semester-hours). It is our belief that this essential part of the student's training should, if possible, be under the direction of professors who have had graduate mathematical training, who have taught mathematics at the secondary level, and who have maintained contacts with the secondary field.
b. Study of methods of teaching in the principal minor field selected by the student and any additional material relating to the history, psychology, or administration of education which can be objectively justified in the training of a teacher (not more than five semester-hours).

Note: In view of the debatable nature of certain features of our recommendation concerning training in the theory of education and practice teaching, the candidate for a teaching certificate in secondary mathematics is advised of the necessity for satisfying the legal requirements in educational theory as they exist in his locality. We believe that further work in the theory of education

beyond this legal minimum, by either an undergraduate or a graduate student, would not be as valuable in preparation for teaching mathematics as additional study of mathematics, related fields, and purely cultural subjects. The student is advised to continue study in these fields beyond the undergraduate level along the lines suggested in Part I of this report.

THE LOCATION OF THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL*

By MORRIS MARDEN, University of Wisconsin, Milwaukee

1. *Introduction.* When all the zeros of a polynomial are given, the polynomial together with its derivatives is, except for a constant factor, completely determined. This fact suggests that any restriction placed upon all or even some of the zeros of a polynomial would in one form or another be passed on to the zeros of the derivative. An early example of this sort dating from 1691 is Rolle's theorem as applied to polynomials.† *If a number of zeros of a real polynomial are required to lie on the real axis, between any pair of these zeros will lie at least one zero of the derivative.*

Ever since the Argand representation for complex numbers came into general use, various attempts have been made to supplement Rolle's theorem with information about the imaginary zeros of the derivative of a real polynomial or to construct analogous theorems for the derivative of an arbitrary polynomial. Indeed, these efforts were initiated by Carl Friedrich Gauss, the very man who helped most to bring about a universal acceptance of the Argand diagram. The principal results of these efforts extending from Gauss' time to the present day is what I propose to outline as my share in this symposium.

Before beginning my account, I wish particularly to remind you of the excellent addresses on our subject delivered in 1922 by Professor Curtiss and in 1929 by Professor Van Vleck, and of the survey contained in the exercises of Pólya-Szegö's "Aufgaben und Lehrsätze."‡

I also wish to point out that the derivative of the polynomial

$$f(z) = \prod_{j=1}^{p} (z - z_j)^{m_j}$$

may be written as

$$f'(z) = f(z)\left(\frac{f'(z)}{f(z)}\right)$$

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* Address delivered by invitation before the Association at Pittsburgh, Dec. 31, 1934.

† For history see F. Cajori, Bibliotheca Math. 11 (1910) 300–313.

‡ D. R. Curtiss, Science 55 (1922) 189–193.

E. B. Van Vleck, Bull. Amer. Math. Soc. 35 (1929) 643–683.

Pólya-Szegö, "Aufgaben und Lehrsätze aus der Analysis" vol. 2, pp. 55–65, Berlin (1925).

and that, hence, the zeros of $f'(z)$ fall into two classes. Either a zero of $f'(z)$ is also a zero z_j of $f(z)$, its multiplicity as a zero of $f'(z)$ being m_j-1; or, it is a root of the equation

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{p} \frac{m_j}{z - z_j} = 0. \tag{1}$$

Since we shall always suppose the positions of the zeros of $f(z)$ to be known, we may limit our discussion to the roots of equation (1)—that is to say, to the zeros of the logarithmic derivative of $f(z)$.

The study of the roots of equation (1) is strictly speaking an algebraic and function-theoretic problem. Yet, as we shall see, a large number of results have been derived or can be obtained by regarding our problems from certain mechanical or geometrical points of view.

2. *Gauss-Lucas Theorem.* The first to give the problem of locating the zeros of $f'(z)$ a mechanical interpretation was Gauss himself.* According to his correspondence he did a considerable amount of work on the imaginary zeros of equations, but of this work unfortunately only two fragments remain. Both were written between 1836 and 1846 and both are to be found among his "Nachlass" in a memorandum book otherwise devoted to astronomy. Only the first of these concern us at the moment. It is the remark that *the roots of equation* (1) *may be regarded as the positions of equilibrium in a certain field of force, that field being due to particles of mass m_j placed at the points z_j and attracting a unit particle at point z according to the inverse distance law.*

Unaware of Gauss' remark, Lucas about 1870 stated the same theorem.† His proof was approximately the following. If z is a root of equation (1), it is likewise a root of the negative conjugate of equation (1),

$$\sum_{j=1}^{p} \frac{m_j}{\bar{z}_j - \bar{z}} = 0. \tag{$-\bar{1}$}$$

The j-th term of equation $(-\bar{1})$ is however a vector with a magnitude of $m_j/|z_j-z|$ and with a direction the same as that of z_j-z. Hence, it may be regarded as the force due to the mass m_j at z_j attracting a unit particle at z according to the inverse distance law. The point z will therefore be a root of (1) if and only if the sum of these forces vanishes; in short, if and only if z is a point of equilibrium.

From Gauss's theorem Lucas drew an immediate corollary, now known as the Gauss-Lucas theorem. As a point of equilibrium, z must be situated within any convex polygon which encloses the attracting masses z_j. *The zeros of the logarithmic derivative of a polynomial must therefore lie within any convex region*

* C. F. Gauss, Collected Works, vol. 3, p. 112 and p. 492; vol. 8, p. 32, and vol. 9, p. 187.

† F. Lucas, Comptes Rendus 77 (1874) 431–433; 78 (1874) 140–144, 180–183, 271–274; 89 (1879) 224–226; Journ. de l'Ecole Poly. 46 (1879) 1–33; Bull, Soc. Math. de France 17 (1888) 2–69.

containing all of the zeros of the polynomial. In particular, this convex region may by chosen as the closed interior of a circle.

Since Lucas' time, at least thirteen proofs of the Gauss-Lucas theorem have been published,* but of these about ten more or less duplicate Lucas' reasoning.

The Gauss-Lucas theorem holds when the m_j in equation (1) are any positive numbers, not necessarily integers. It was extended by Porter to entire functions of genre 0 or 1, by Nagy to equations involving more general partial fractions than (1) and by myself to these general equations with the m_j as complex numbers.†

3. *The curves* $|f(z)|$, *arg* $f(z)$, $\Re(f(z))$, $\Im(f(z))$ *constant.* The potential of the field of force given by equation $(-\bar{1})$ is log $|f(z)|$ and hence the equipotential lines are the lemniscates $|f(z)|=$const. As Professor Walsh has treated these lemniscates in detail,‡ it will suffice for us merely to sketch the earlier results from an historical point of view.

Several mathematicians from 1864 to 1904 studied these curves and their orthogonal trajectories, arg $f=$const., with the object of generalizing Rolle's theorem to complex variables. Among the properties investigated by Lucas, Stieltjes and others§ were the location of the singular foci and multiple points and the determination of the number of closed branches comprising the lemniscates and of the number of zeros of $f(z)$ and $f'(z)$ in any one of these branches.

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* G. J. Legebeke, Archives Néerlandaises 16 (1881) 273–278.
F. De Boer, ibid. 19 (1884) 207–240.
Berlothy, Comptes Rendus 99 (1884) 745–747.
M. E. Césaro, Nouvelles Annales de Math. 4 (1885) 328–330.
M. Bôcher, Annals of Math. 7 (1892) 70.
J. H. Grace, Proc. Camb. Phil. Soc. 11 (1901) 352–357.
T. Hayashi, Annals of Math. 15 (1914) 112–113.
F. Irwin, ibid. 16 (1915) 138.
Y. Uchida, Tôhoku Math. Journ. 10 (1916) 139–141.
B. Conggryp, Liouville Journ. 1 (1915) 353–365.
M. P. Porter, Proc. Nat'l Acad. Science 2 (1916) 247–248, 335–336.
M. Krawtchouck, L'Enseignement de Math. 25 (1926) 74–77.
J. V. Sz. Nagy, Jahresb. d.d. Math.-Ver. 27 (1918) 44–48 and Tôhoku Math. Journ. 35 (1932) 126–135.

† J. v. Sz. Nagy, Acta Univ. Hung. 1 (1923) 127–141.
M. Marden, Bull. Amer. Math. Soc. 35 (1929) 363–370 and Trans. Amer. Math. Soc. 32 (1930) 658–668.

‡ J. L. Walsh, Bull. Amer. Math. Soc. 39 (1933) 775–782; Proc. Nat'l Acad. Sci. 20 (1934) 551–554; this Monthly, January 1935.

§ See Lucas and De Boer, loc. cit.
M. J. Liouville, Liouville Journ. 9 (1864) 84–88.
T. J. Stieltjes, Archives Néerlandaises 18 (1883) 1–19.
H. M. Macdonald, Proc. London Math. Soc. 29 (1898) 575–584.
C. De la Vallée Poussin, Mathesis 2 (1902) 1–11 (supplement).
Dall'Agnola, Rendi. d. R. A. dei Lincei 13 (1904) 337–339.
Fr. Lange-Nielsen, Comptes Rendus 170 (1920) 922–924.
W. J. Tritzinsky, Bull. Amer. Math. Soc. 33 (1927) 693–5.

Generally along with these curves were also treated the loci $\Re(f) = \text{const.}$ and $\Im(f) = \text{const.}$ In those respects, however, the earliest results seem to have been due to Gauss. In fact, the second of the notes which we mentioned at the beginning embodies the following theorem. The plane may be divided into n regions, each containing one zero of $f(z)$, by means of $n-1$ curves upon each of which $\Im(f)$ is constant and upon each of which lies one and only one zero of $f'(z)$.

4. *Jensen's Theorem*. So far we have placed no restriction upon the coefficients of the polynomial $f(z)$. If all the coefficients are real, the Gauss-Lucas polygon and the equipotential curves will be symmetrical in the real axis.

More specific information about the zeros of $f'(z)$ may in this case be obtained by examining the resultant of the forces due to any pair of conjugate imaginary zeros of $f(z)$. Let the circles having as diameters the line-segments joining in pairs the conjugate imaginary zeros of $f(z)$ be called the Jensen circles of the polynomial. At any point z not on the real axis and not in any Jensen circle of the polynomial, the resultant force due to any pair of conjugate zeros turns out to be always directed towards the real axis. This implies that at such a point z the total force as given by the left-hand side of equation $(-\overline{1})$ cannot vanish. Accordingly Walsh was able to give in 1920 the first published proof of the following theorem stated in 1913 by Jensen.* *There are no imaginary zeros of the derivative of a real polynomial outside all of the Jensen circles of the polynomial.*

Without the language of statics, essentially the same method appears in Echols's proof of Jensen's theorem and in Nagy's proof of a similar theorem due to Jensen regarding the zeros of a certain linear combination of the derivatives of a real entire function.†

In a recent paper‡ Walsh discussed the distribution of the zeros of $f'(z)$ when the zeros of $f(z)$ were situated symmetrically, not with respect to a line, but with respect to a point 0. He concluded, for example, that, if all the zeros of $f(z)$ were interior to or on any equilateral hyperbola with 0 as center, then the zeros of $f'(z)$ also were interior to or on the same hyperbola, except perhaps for a zero at 0.

5. *Laguerre's Theorem*. Jensen's theorem and the Gauss-Lucas theorem both pertain to the roots of equation (1). Equation (1) forms, however, only the special case $\zeta = \infty$ of the equation

$$\sum_{j=1}^{p} \frac{m_j}{z - z_j} - \frac{n}{z - \zeta} = 0 \tag{2}$$

where $n = \sum_{j=1}^{p} m_j$. The left-hand side of the negative conjugate of this equation

* J. L. W. Jensen, Acta Math. 36 (1913) 181–195.
J. L. Walsh, Annals of Math. 22 (1920) 128–141.

† W. E. Echols, Amer. Math. Monthly 27 (1920) 299–300.
J. v. S. Nagy, Jahresb. d.d. Math.-Ver. 31 (1922) 238–251.

‡ J. L. Walsh, Bull. Soc. Sci. de Cluy (Roumania) 7 (1934) 521–526.

$$(-\bar{2})\qquad \sum_{j=1}^{p}\frac{m_j}{\bar{z}_j-\bar{z}}-\frac{n}{\bar{\zeta}-\bar{z}}=0$$

represents the total force due to the p attractive particles at z_j and the single repulsive particle at ζ. The new equation has the advantage over the old of remaining unchanged when all the quantities z, z_j and ζ are simultaneously subjected to the same arbitrary linear transformation.

If into equation $(-\bar{2})$ we set $w_j-z=1/(\bar{z}_j-\bar{z})$ and $\sigma-z=1/(\bar{\zeta}-\bar{z})$ we shall be replacing points z and ζ by their inverses with respect to point z and equation $(-\bar{2})$ by $\sigma=(\sum_{j=1}^{p}m_jw_j)/n$. We are thus led to a second mechanical interpretation—namely that of centers of gravity. As point σ is the centroid of the points w_j, any line through σ must either pass through all the points w_j or separate some of the points w_j from the remaining points w_j. On inversion, this straight line transforms into a circle through z and ζ. *Any circle through z and ζ must therefore either pass through all the points z or separate some of the points z_j from the remaining z_j.* In particular let us suppose all the points z_j lie interior to a circular region C, by which we mean the closed interior or exterior of a circle, a closed half plane or the entire plane. The theorem then implies that, if ζ lies exterior to region C, all of the corresponding roots z of equation (2) lie interior to C; on the other hand, if a root z of equation (2) lies exterior to C, the corresponding ζ lies interior to C.

From these facts, which were discovered by Laguerre* in 1878, flow two lines of thought. Along the first the objectives are reached through applying Laguerre's theorem to a kind of generalized higher derivative. Along the second the results are attained by applying the theorem to further equations of the type $\sum m_{ik}(z-z_{jk})^{-1}=0$.

6. *Grace's Theorem.* Following the first, let us consider the polynomial of degree at most $n-1$

$$f_\zeta=(\zeta-z)f'(z)+nf(z)$$

which consists merely of the left-hand side of equation (2) multiplied by $(\zeta-z)f(z)$. This polynomial, called by Laguerre the "émanant" of $f(z)$ with respect to ζ, may be written in the more symmetric form $\zeta_1\partial F/\partial z_1+\zeta_2\partial F/\partial z_2$ on introducing the homogeneous coordinates $z=z_1/z_2$, $\zeta=\zeta_1/\zeta_2$ and $F(z_1, z_2)=z_2^nf(z_1/z_2)$. Let us denote by $f_{\zeta\lambda}$, the émanant of the polynomial f_ζ with respect to λ; by $f_{\zeta\lambda\mu}$, the émanant of $f_{\zeta\lambda}$ with respect to μ, and so on.

The zeros of f_ζ are essentially the roots of equation (2). By Laguerre's theorem, therefore, if the point ζ lies exterior to any circular region C containing all of the zeros of $f(z)$, all of the zeros of f_ζ lie interior to C. For the same reason, if λ, μ, etc., also lie exterior to C, all of the zeros of the successive émanants $f_{\zeta\lambda}$, $f_{\zeta\lambda\mu}$, etc. also lie interior to C.

Choosing the points ζ, λ, μ, etc., as the zeros of a certain polynomial $g(z)$, Szegö† showed the assumption that all of the zeros of $g(z)$ were situated out-

* Laguerre, Oeuvres vol. 1, pp. 48–50, 51, 52–55, 56–63, 64–66, 133–143.

† G. Szegö, Math. Zeitschrift 13 (1922) 28–55.

side of C leads to a contradiction. Accordingly, he was able to prove the following theorem originally due to Grace: *If the coefficients of the polynomial* $f(z) = \sum_{j=0}^{n} a_j z^j$ *satisfy a linear relation*

$$L[f(t)] = \sum_{j=0}^{n} l_j a_j = 0,$$

and if $g(z)$ *is the polynomial obtained by applying the linear operation upon the function* $f(t) = (t-z)^n$—i.e.,

$$g(z) \equiv L[(t-z)^n],$$

than any circular region containing all of the zeros of one of the polynomials $f(z)$ *and* $g(z)$ *encloses at least one zero of the other.*

Other proofs of this theorem have been given by Grace, Kakeya, Walsh, Curtiss, Cohn and Dieudonné.*

To show how the theorem applies, let us give a new proof of the Gauss-Lucas theorem. Let us suppose that all of the zeros of a polynomial $f(z)$ lie in a circle C and that β is any zero of $f'(z)$. We wish to show that β also lies in the circle C. Now the expression

$$f'(\beta) = \sum_{j=0}^{n} j\beta^{j-1} a_j = 0$$

defines a linear relation among the coefficients of $f(z)$. Applied to $f(t) = (t-z)^n$, the linear operation yields a polynomial $g(z)$ of degree n,

$$g(z) = 0 \cdot z^n + n(\beta - z)^{n-1},$$

whose only zeros are ∞ and β. As at least one zero of $g(z)$ must be in the circle C, it follows that the point β must lie in circle C.

As a second application, we shall prove the following analogue of Rolle's theorem due to Grace and Heawood.† *If* $z = \pm 1$ *are zeros of a polynomial of degree* n, *at least one zero of the derivative lies in the circle* K *with center at the origin and radius* $\cot \pi/(2n)$.

By hypothesis,

$$\int_{-1}^{1} f'(t)dt = 0.$$

* J. H. Grace, loc. cit.
S. Kakeya, Tokyo Math.-Phys. Soc. Proc. 3 (1921) 94–100.
J. L. Walsh, Trans. Amer. Math. Soc. 24 (1922) 163–180.
D. R. Curtiss, ibid. 24 (1922) 181–184.
M. Cohn, Math. Zeitschrift 14 (1922) 110–148.
J. Egerváry, Acta Univ. Hung. 1 (1922) 39–45.
M. J. Dieudonné, Bull. Soc. Math. de France, 1–24.
† J. H. Grace, loc. cit.
P. J. Heawood, Quarterly Jour. Math 38 (1907) 84–107.

This integral defines a linear relationship among the coefficients of $f'(z)$. The polynomial $g(z)$ obtained by applying the operation to $f'(t) \equiv (t-z)^{n-1}$

$$g(z) = \int_{-1}^{1} (t - z)^{n-1} dt = \frac{1}{n}[(1 - z)^n - (-1 - z)^n].$$

has the zeros

$$z = i \cot \frac{k\pi}{2n} \qquad (k = 1, 2, \cdots, n - 1).$$

As all of these lie in the circle k, at least one zero of $f'(z)$ must also lie in circle K.*

Our time is too short, however, to give any further applications of Grace's theorem. To gain an appreciation of the power and beauty of the theorem, I recommend the reading of the articles of the subject by Szegö and Walsh.

7. *Bôcher's and Walsh's Theorems.* The second current of thought issuing from Laguerre's theorem takes a different direction from that which led to Grace's theorem. It concerns the roots of the equation

$$\sum_{j=0}^{q} \sum_{k=1}^{p_j} \frac{m_{jk}}{z - z_{jk}} = 0, \tag{3}$$

where for a given j all the m_{jk} have the same sign,

$$\sum_{k=1}^{p_j} m_{jk} = n_j, \qquad n = \sum_{0}^{q} n_j,$$

and all of the points z_{jk} lie in a given circular region C_j. If the point ζ_j is defined by the relation

$$\frac{n_j}{z - \zeta_j} = \sum_{k=1}^{p_j} \frac{m_{jk}}{z - z_{jk}}, \tag{4}$$

and if root z of equation (3) lies exterior to all of the regions C_j, the point ζ_j will lie interior to the region C_j—by Laguerre's theorem—and the root z will also satisfy the simpler equation

$$\sum_{j=0}^{q} \frac{n_j}{z - \zeta_j} = 0. \tag{3'}$$

* For other analogies to Rolle's theorem see M. Fekete, Acta Univ. Hung. 1 (1923) 98–100 and 4 (1929) 234–243; Jahresb d.d. Math.-Ver. 32 (1923) 299–306 and 34 (1926) 220–233; Math. Zeitschrift 22 (1926) 1–7.

J. v. Sz. Nagy, Jahresb. d.d. Math.-Ver. 32 (1923) 307–309.

M. Marden, Bull. Amer. Math. Soc. 35 (1929) 363–370 and 39 (1933) 750–754.

S. Kakeya, Tôhoku Math. Journ. 11 (1917) 5–16.

M. Biernacki, Bull. Acad. Polonaise (1927) 660–670

M. J. Dieudonné, Annals of Math. 31 (1930) 79–116 and Comptes Rendus 198 (1934) 1966–1967.

What can hence be said about the location of those roots of (3) which fall outside of all of the given regions C_j?

First, let us consider the case that $q=1$ and $n_0=-n_1>0$. The left hand side of equation (3) is then the logarithmic derivative of the rational function

$$f(z)=\frac{\prod_{k=1}^{p_0}(z-z_{0k})^{m_{0k}}}{\prod_{k=1}^{p_1}(z-z_{1k})^{-m_{jk}}}$$

or, if homogeneous co-ordinates are used, the Jacobian of two binary forms—an invariant under linear transformation of the variables z and z_{jk}. The negative conjugate of this left-hand side represents the total force due to p_0 attractive particles m_{0k} at the points z_{0k} and the p_1 repulsive particles $(-m_{1k})$ at the points z_{1k}, the algebraic sum of the masses being zero. From equation (3′), it follows that $\zeta_0=\zeta_1$. Hence, *if the regions C_0 and C_1 have no points in common, no root z of* (3) *may lie exterior to both C_0 and C_1*—which is Bôcher's generalization of Laguerre's theorem.*

Secondly, let us take the case that $q=1$, $n_0>0$ and $n_1>0$ and that the regions C_0 and C_1 are the closed interiors of circles having centers α_0 and α_1 and radii r_0 and r_1 respectively. From equation (3′) it follows then that

$$z=\frac{n_0\zeta_1+n_1\zeta_0}{n_0+n_1}$$

and hence that any root z of (3) exterior to both C_0 and C_1 lies in the closed interior of a circle with center and radius respectively

$$\frac{n_0\alpha_1+n_1\alpha_0}{n_0+n_1}\quad\text{and}\quad\frac{n_0r_1+n_1r_0}{n_0+n_1}.$$

This theorem, due to Walsh,† may also be stated as follows: *If all the zeros of a polynomial $f_j(z)$ of degree n_j lie in the closed interior of a circle C_j, any zero of the logarithmic derivative of the product $f_0(z)f_1(z)$ lies in at least one of the circles C_0, C_1 and C, the latter being the "Mittelbereich"‡ of the circles C_0 and C_1.*

Thirdly, let us study with Walsh§ the case that $q=2$ and $n_0+n_1+n_2=0$. As equation (3′) then becomes identical with the cross-ratio

* M. Bôcher, Proc. Amer. Acad. Sci. 40 (1904) 469–484.

See also L. R. Ford, Proc. Edingburgh Math. Soc. 33 (1915) 103–106.

† J. L. Walsh, Congrès Intern. Math. Strasbourg (1920) pp. 1–4.

‡ H. Minkowski, Collected Works, vol. 2, p. 177.

§ J. L. Walsh, Trans. Amer. Math. Soc. 19 (1918) 291–298 and 22 (1921) 106–116; Rendi. de Palermo 46 (1922) 1–13.

See also A. B. Coble, Bull. Amer. Math. Soc. 27 (1921) 434–437 and T. Nakahara, Tôhoku Math. Journ. 23 (1924) 97.

$$(\zeta_0\zeta_1\zeta_2 z) = \frac{(\zeta_0 - \zeta_2)(\zeta_1 - z)}{(\zeta_0 - z)(\zeta_1 - \zeta_2)} = -\frac{n_1}{n_0},$$

any root z exterior to all of the regions C_j must lie interior to a circular region C defined by the following lemma. *If the points ζ_0, ζ_1 and ζ_2 independently describe circular regions C_0, C_1 and C_2 respectively, the point z as defined by the constant* (real or complex) *cross-ratio $(\zeta_0\zeta_1\zeta_2 z) = \lambda$ also describes a circular region C.*

Lastly, let us examine the general case, the study of which was suggested by the above and other theorems due to Walsh,* and was carried out in my doctoral thesis† and, greatly simplified, in my September 1934 paper to the Society. I found for example that *if all the $m_{jk} > 0$ and all the circles C_j are the closed interiors of the circles $C_j \equiv |z - \alpha_j|^2 - r_j^2 = 0$, then any root of* (3) *exterior to all the C lies within one of the simply-connected closed regions bounded by the ovals of a certain q-circular $2q$-ic curve. This curve has the equation*

$$\sum_{j=0}^{q} \frac{nn_j}{C_j} - \sum_{j=0,k=j+1}^{q} \frac{n_j n_k \tau_{jk}}{C_j C_k} = 0,$$

where $\tau_{jk} \equiv |\alpha_j - \alpha_k|^2 - (r_j - r_k)^2$, the square of the common external tangent of the circles C_j and C_k. The singular foci of the curve coincide with the roots of (3′) *with $\zeta_j = \alpha_j$, all j.*

8. *Van den Berg's Theorem.* The geometrical character of these as well as of all the preceding results seems to suggest the possibility of studying the zeros of the derivative by giving our equations geometrical interpretations in place of the mechanical which we have used up to this point. Such a geometrical interpretation was given in 1882 by Van den Berg.‡ According to his theorem the two roots of the equation (1) with $p = 3$ coincide with the foci of the ellipse inscribed in triangle $z_1z_2z_3$ and touching the sides z_1z_2, z_2z_3 and z_3z_1 in points dividing these sides in the ratios m_2/m_1, m_3/m_2 and m_1/m_3 respectively. More generally, *the roots of equation* (1) *are the foci of the curve of class $p-1$ which touches each line-segment z_jz_k at a point dividing the segment in the ratio m_k/m_j.*

Because Van den Berg's theorem involves the class and foci of a curve, its natural proof is by means of line-coordinates. We shall outline the proof only for the special case $p = 3$. For $p = 3$ equation (1) may be rewritten

$$\sum_{j=1}^{3} \frac{m_j}{\left(\dfrac{1}{x+iy}\right)x_j + \left(\dfrac{i}{x+iy}\right)y_j - 1} = 0 \tag{1′}$$

with $x + iy = z$ and $x_j + iy_j = z_j$. Let us compare (1′) with the equation

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* J. L. Walsh, Proc. Nat'l Acad. Sci. 8 (1922) 139–144; Trans. Amer. Math. 24 (1922) 31–69.

† M. Marden, Trans. Amer. Math. Soc. 32 (1930) 81–109.

‡ F. J. Van den Berg, Nieuw Archief voor Wiskunde 9 (1882) 1–14, 60; 11(1884) 153–186; 15 (1888) 100–164, 190.

$$(6) \qquad \sum_{j=1}^{3} \frac{m_j}{\mathcal{L}_j} = 0, \qquad \mathcal{L}_j = \lambda x_j + \mu y_j - 1$$

in the line coordinates λ and μ, or the equivalent equation

$$(6') \qquad F(\mu, \lambda) = m_1\mathcal{L}_2\mathcal{L}_3 + m_2\mathcal{L}_3\mathcal{L}_1 + m_3\mathcal{L}_1\mathcal{L}_2 = 0.$$

We see that the line $\lambda = 1/(x+iy)$, $\mu = i/(x+iy)$ satisfies equation (6) and hence is a tangent to the curve of class two represented by equation (6). As this tangent is an isotropic line through the point (x, y), an arbitrary root of (1′) must be a focus of the curve. We also see that the line connecting the points (x_j, y_j) and (x_k, y_k) satisfies simultaneously the two equations $\mathcal{L}_j=0$ and $\mathcal{L}_k=0$, and hence satisfies equation (6′). This line consequently also is tangent to the curve (6′), its point of contact having the equation

$$(7) \qquad m_j\mathcal{L}_k + m_k\mathcal{L}_j = 0.$$

Its point of tangency therefore divides the segment z_jz_k in the ratio $m_k{:}m_j$. As the m_j are all positive, the curve of class two in question is an ellipse.

Van den Berg's theorem was proved in the special case $p=3$ by Grace and Bôcher and in the general case by Van den Berg, Heawood, Juhel-Renjoy and Linfield. Furthermore, Fujiwara and Linfield extended the theorem to the zeros of the k-th derivative of a rational function.*

It seems to me, however, that much greater use could be made of Van den Berg's geometrical interpretation and possibly by means of it additional results could be discovered which might be interesting from the standpoint of line geometry as well as from the standpoint of the zeros of the derivative.

ON THE COMPUTATION OF THE PROBABLE ERROR OF A WEIGHTED MEAN

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INTRODUCTION

The formulas for estimating the precision of measurements, both direct and indirect, have been known and used for a long time, and it would seem that the proper application of the different formulas would have been a settled matter long before now. Since this is not the case, however, one may perhaps be excused for offering a further paper on an old subject. The investigations described in the present paper were undertaken in an effort to clear up one or two controversial points concerning the applicability of a well-known formula to a certain type of problem.

* See J. H. Grace, M. Bôcher, and P. J. Heawood, loc. cit.

J. Juhel-Rénoy, Comptes Rendus 192 (1906) 700.

M. Fujiwara, Tôhoku Math. Journ. 9 (1916) 102–108.

B. Z. Linfield, Bull. Amer. Math. Soc. 27 (1920) 17–21 and Trans. Amer. Math. Soc. 25 (1923) 239–258.

In order to see just what are the points at issue, let

$$M_1 \pm r_1, M_2 \pm r_2, \cdots M_n \pm r_n$$

denote the results of several sets of measurements of some physical magnitude, where $M_1, M_2, \cdots M_n$ are the means (simple or weighted) of the sets of measurements and $r_1, r_2, \cdots r_n$ are the respective probable errors of these means. Since the weights of measurements are inversely proportional to the squares of the probable errors, the weighted or general mean of the several sets is given by the formula

$$M_0 = \frac{\Sigma wM}{\Sigma w} = \frac{M_1(1/r_1)^2 + M_2(1/r_2)^2 + \cdots + M_n(1/r_n)^2}{(1/r_1)^2 + (1/r_2)^2 + \cdots + (1/r_n)^2},$$

where the w's denote the weights and $w_1r_1^2 = w_2r_2^2 = \cdots w_nr_n^2$.

The probable error of this general mean has been computed by any one of the following three formulas, presumably on the assumption that they are all equally appropriate for this case and will give practically the same result:

$$(A) \qquad r_a = \frac{r}{\sqrt{\Sigma w}} = \frac{1}{\sqrt{\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \cdots + \dfrac{1}{r_n^2}}},$$

where r denotes the probable error of a set of unit weight;

$$(B) \qquad r_b = 0.6745\sqrt{\frac{\Sigma wV^2}{(n-1)\Sigma w}},$$

where $V_1 = M_0 - M_1$, etc.;

$$(\mathrm{C}) \qquad R = \sqrt{\left(\frac{\partial M_0}{\partial M_1}\right)^2 r_1^2 + \left(\frac{\partial M_0}{\partial M_2}\right)^2 r_2^2 + \cdots + \left(\frac{\partial M_0}{\partial M_n}\right)^2 r_n^2}.$$

Formulas (A) and (C) give identical results in any problem, but (B) rarely or never gives the same result as the other two.

In a previous paper* the writer expressed the view that the probable error of the general mean in a problem of this type, where the weights are found from the given probable errors, is properly found only by (A) or (C), and not by (B); that (B) should be used only when the weights are assigned arbitrarily, since in that case no other formula is available. With this view, as regards the use of (A) or (C) to the exclusion of (B), Professor Raymond T. Birge† did not agree, and stated that (A) and (B) were equally applicable to this case and would give the same result, except for statistical fluctuations, if no systematic errors were present. He emphasized the hypothesis that any considerable disagreement be-

* Proceedings of the National Academy of Sciences, Vol. 15, No. 8 (August, 1929), pp. 665–668.

† Physical Review, Vol. 40, No. 2 (April 15, 1932), pp. 207–227.

tween the results given by (A) and (B) is an indication of the presence of systematic errors.

The purpose of the present paper is to set forth the results of three investigations that were designed to throw further light on the points at issue by (1) determining under what conditions, if any, the two formulas (A) and (B) will give the same result and (2) testing Birge's hypothesis that a considerable disagreement between the results of the two formulas is an indication of systematic errors.

To determine whether or not and under what conditions (A) and (B) will give the same result for the probable error of the general mean of several sets of measurements, it is only necessary to assume sets of measurements expressed as literal quantities, find the means, residuals, probable errors, weights, etc., and substitute them in (A) and (B). The probable errors given by the two formulas can then be compared. The complete solution of the most general case has not been carried through by this method of procedure and appears to be practically impossible on account of the intractable algebra that must be handled, but the writer has succeeded in solving two special cases that are sufficient to indicate what may be expected in the general case.

To test the hypothesis that a discrepancy in the results obtained by (A) and (B) is an indication of systematic errors it is obviously necessary to apply the formulas to sets of measurements that are known to be free from systematic errors and yet follow the Gaussian law of accidental errors. Two kinds of measurements that would meet these requirements were finally devised and carried out. They are described in Parts II and III.

PART I. *ANALYTICAL*

1. *Special Case of Two General Sets.* In order to compare formulas (A) and (B) analytically let us first consider two sets of measurements, one the result of p measurements of equal weight and the other the result of q measurements of equal weight. If $m_1, m_2, \cdots m_p$ denote the individual measurements of the first set and $n_1, n_2, \cdots n_q$ denote those of the second set, then the arithmetic means of the two sets are

$$M_1 = \Sigma m/p, \qquad M_2 = \Sigma n/q.$$

Let $v_1, v_2, \cdots v_p$ and $u_1, u_2, \cdots u_q$ denote the residuals of the individual measurements of the two sets. Then the probable errors of M_1 and M_2 are

$$r_1 = \rho\sqrt{\frac{\Sigma v^2}{p(p-1)}}, \qquad r_2 = \rho\sqrt{\frac{\Sigma u^2}{q(q-1)}}, \qquad \text{where } \rho = 0.6745.$$

For the weights we have $w_1/w_2 = (r_2/r_1)^2$, or

$$w_1 = \frac{r_2^2}{r_1^2} w_2 = \left(\frac{\rho^2\Sigma u^2}{q(q-1)} \Big/ \frac{\rho^2\Sigma v^2}{p(p-1)}\right) w_2 = \frac{Ap(p-1)}{Bq(q-1)} w_2,$$

where $A = \Sigma u^2$, $B = \Sigma v^2$; and the weighted mean of the two sets is

$$M_0 = \frac{w_1 M_1 + w_2 M_2}{w_1 + w_2} = \frac{Ap(p-1)M_1 + Bq(q-1)M_2}{Ap(p-1) + Bq(q-1)}.$$

The residuals of the two sets are therefore

$$V_1 = M_0 - M_1 = \frac{Bq(q-1)(M_2 - M_1)}{Ap(p-1) + Bq(q-1)},$$

$$V_2 = M_0 - M_2 = \frac{Ap(p-1)(M_1 - M_2)}{Ap(p-1) + Bq(q-1)}.$$

Substituting these and the weights in (B), we get

$$(1) \qquad r_b = \rho\sqrt{\frac{w_1 V_1^2 + w_2 V_2^2}{w_1 + w_2}} = \frac{\rho(M_1 - M_2)\sqrt{AB}\sqrt{pq(p-1)(q-1)}}{Ap(p-1) + Bq(q-1)}.$$

Likewise, on substituting in (A) the values of r_1 and r_2 we have

$$(2) \qquad r_a = \frac{\rho\sqrt{AB}}{\sqrt{Ap(p-1) + Bq(q-1)}}.$$

The reader will observe that the formula for r_b contains the factor $M_1 - M_2$, whereas r_a does not. On account of the fact that instrumental readings can be obtained to only a few figures, the results of all physical measurements are necessarily limited to a few figures. Hence it may happen in some cases that M_1 and M_2, the arithmetic means of two sets of measurements, will be numerically equal. In such cases r_b will be zero, regardless of the probable errors of M_1 and M_2. On the other hand, r_a could not be zero, but would be of the same order of magnitude as r_1 and r_2. Two examples of the kind just mentioned are given in Part III of this paper.

In the case of two sets of measurements where M_1 and M_2 are not equal, it is possible to find the conditions under which r_a and r_b will be equal. To this end, let a denote the true value of the measured quantity; let $\epsilon_1, \epsilon_2, \cdots \epsilon_p$ denote the errors of the original measurements in the first set; and let $\delta_1, \delta_2, \cdots \delta_q$ denote the corresponding errors in the second set. Then

$$M_1 = a + \Sigma\epsilon/p, \quad M_2 = a + \Sigma\delta/q, \quad \text{and} \quad M_1 - M_2 = (q\Sigma\epsilon - p\Sigma\delta)/pq.$$

Hence

$$v_1 = M_1 - m_1 = a + \Sigma\epsilon/p - (a + \epsilon_1) = \Sigma\epsilon/p - \epsilon_1, \ v_2 = \Sigma\epsilon/p - \epsilon_2, \ \text{etc.};$$

and

$$u_1 = M_2 - n_1 = a + \Sigma\delta/q - (a + \delta_1) = \Sigma\delta/q - \delta_1, \ u_2 = \Sigma\delta/q - \delta_2, \ \text{etc.}$$

Therefore

$$v_1^2 = (\Sigma\epsilon)^2/p^2 - 2\epsilon_1\Sigma\epsilon/p + \epsilon_1^2, \text{ etc.};$$
$$u_1^2 = (\Sigma\delta)^2/q^2 - 2\delta_1\Sigma\delta/q + \delta_1^2, \text{ etc.}$$

Hence

$$A = \Sigma u^2 = \Sigma\delta^2 - (\Sigma\delta)^2/q, \quad B = \Sigma v^2 = \Sigma\epsilon^2 - (\Sigma\epsilon)^2/p.$$

Now forming the ratio r_b/r_a we have, by (1) and (2),

$$r_b/r_a = (M_1 - M_2)\sqrt{\frac{pq(p-1)(q-1)}{Ap(p-1) + Bq(q-1)}}$$
$$= \sqrt{\frac{pq(p-1)(q-1)(M_1-M_2)^2}{Ap(p-1) + Bq(q-1)}}.$$

Replacing A, B, and $M_1 - M_2$ by their values in terms of the ϵ's, δ's, p, and q, and noting that

$$(\Sigma\epsilon)^2 = \Sigma\epsilon^2 + 2[\epsilon_1(\epsilon_2 + \epsilon_3 + \cdots + \epsilon_p) + \epsilon_2(\epsilon_3 + \epsilon_4 + \cdots + \epsilon_p) + \cdots + \epsilon_{p-1}\epsilon_p]$$
$$= \Sigma\epsilon^2 + 2E_p,$$
$$(\Sigma\delta)^2 = \Sigma\delta^2 + 2[\delta_1(\delta_2 + \delta_3 + \cdots + \delta_q) + \delta_2(\delta_3 + \delta_4 + \cdots + \delta_q) + \cdots + \delta_{q-1}\delta_q]$$
$$= \Sigma\delta^2 + 2D_q,$$

we find, after a little reduction,

$$r_b/r_a$$
$$(3) \quad = \left[1 + \frac{2pq[q(q-1)E_p + p(p-1)D_q - (p-1)(q-1)\Sigma\epsilon\Sigma\delta]}{q^2(p-1)(q-1)\Sigma\epsilon^2 + p^2(p-1)(q-1)\Sigma\delta^2 - 2q^2(q-1)E_p - 2p^2(p-1)D_q}\right]^{1/2}.$$

In order that r_a and r_b may be equal, the fractional term under the square-root sign in (3) must be zero. It will be shown below that the limiting value of this fraction will be zero when p and q become very large and are of the same order of magnitude, but it will not be zero for small values of p and q except by accident.

The number of product terms of the form $\epsilon_i\epsilon_j (i \neq j)$ in the expression represented by E_p is found by the formula for the sum of an arithmetical progression to be $p(p-1)/2$. The number of products of the form $\delta_i\delta_j(i \neq j)$ in D_q is likewise found to be $q(q-1)/2$. The number of terms in the product $\Sigma\epsilon\Sigma\delta$ is evidently pq. Hence within the brackets of the numerator of the fraction in (3) there are

$$q(q-1)[p(p-1)/2] + p(p-1)[q(q-1)/2] = pq(p-1)(q-1)$$

product terms preceded by the plus sign, and there are evidently the same number of product terms preceded by the minus sign.

When the number of measurements in each set is very large (p and q large), the errors of these measurements will be about half positive and half negative. But the terms in the expanded form of the numerator of the fraction in (3) will

still be about half positive and half negative, and they will be, on the average, about the same size. Hence they will cancel one another, and the numerator will be reduced to zero.

The denominator cannot be zero when the number of measurements in each set is large, because in that case E_p and D_q will be negligible and the first two terms will still be large and positive. The fraction in (3) will therefore reduce to zero when p and q become very large and are of the same order of magnitude. Hence in that case r_a will be equal to r_b.

The preceding investigation shows that in the case of two sets of measurements the results given by formulas (A) and (B) will never be equal except by accident, because the number of individual measurements in actual practice is never large enough to reduce the fractional term in (3) to zero.

2. *Special Case of Three Sets.* Let us next consider the case of three sets of measurements, each set being made up of two original measurements of equal weight. Representing them literally, we have

$$M_1 = \frac{m_1^{(1)} + m_2^{(1)}}{2} = \frac{(a + \epsilon_1) + (a + \epsilon_2)}{2} = a + \frac{\epsilon_1 + \epsilon_2}{2},$$

$$M_2 = \frac{m_1^{(2)} + m_2^{(2)}}{2} = a + \frac{\delta_1 + \delta_2}{2},$$

$$M_3 = \frac{m_1^{(3)} + m_2^{(3)}}{2} = a + \frac{\sigma_1 + \sigma_2}{2},$$

where a represents the true value of the magnitude and the ϵ's, δ's, and σ's represent the errors of measurement. Here it will be noted that there are only six errors involved, and there is no reason for supposing that they will all be nearly of the same size or that they will be half positive and half negative.

By proceeding as in the case of two sets and forming the ratio r_b/r_a, it will be found that r_b will be equal to r_a only when a certain homogeneous function of the errors reduces to zero. This function, when reduced to its simplest form, consists of 117 terms of the eighth degree, some positive, some negative, and all having numerical coefficients that are divisible by 2. If these 117 terms are replaced by the simple products for which they stand, the homogeneous function of the errors will consist of 9216 simple terms (coefficients unity) of the eighth degree, such as $\epsilon_1^2 \epsilon_2^2 \delta_1 \delta_2 \sigma_1 \sigma_2$, etc., half the terms being preceded by plus signs and half by minus signs.

Since only six different quantities (errors) enter into these product terms, and since there is no reason for supposing that half of such a small number of terms will be positive and half negative, it is unreasonable to suppose that half the 9216 terms will be positive and half negative when the signs of the individual errors are substituted in these terms. Hence not all these terms will cancel out, in general, and the function will not reduce to zero. In that case r_a and r_b will not be equal.

3. *The General Case.* The general case of any number of sets, each the result of any number of original measurements, can be handled for a few steps just as the case of two sets was handled; but the algebra soon becomes intractable, and it may not be possible to solve the problem by such a procedure or by any other. It seems reasonable to suppose that in the case of more than two sets the ratio r_b/r_a will approach unity as the number of measurements in each set is increased, and the more rapidly as the number of sets is increased.

PART II. *EXPERIMENTAL*

1. *Introductory.* If an urn contains a large number of black balls and an equal number of white balls, if all the balls are thoroughly mixed up, and if a blind-folded person draws a ball from the urn, the chances are even that the ball may be either white or black. If the ball be returned to the urn, all the balls be shuffled, and a second drawing be made by the blind-folded person, the chances are again even that the second ball may be either white or black. It is evident that the percentage of white balls in any such urn could be measured closely by making a large number of drawings in the manner just described and then finding the ratio of white balls to the total number drawn. It is further evident that if such a measurement be carefully and honestly carried out, the errors of measurement will be entirely accidental. There will be no systematic errors.

With these ideas in mind, the writer planned a series of measurements to test the hypothesis that a considerable discrepancy between the results given by formulas (A) and (B) is an indication of systematic error. One hundred and eight toy pool balls, $\frac{3}{4}$ inch in diameter and each bearing a large clearly-printed number, were placed in an urn. Half the balls bore even numbers and half odd numbers. The problem to be solved with this outfit was to determine by repeated drawings the percentage of even-numbered balls in the urn. The thing to be measured was the ratio of even-numbered balls to the total number in the urn.

2. *Description of the Experiment.* The measurements were made by repetitions of the following process: A ball was drawn, its number recorded, the ball returned to the urn, and then all the balls shuffled. The same process was repeated with the next and succeeding drawings. The chance of drawing an even-numbered ball was therefore always $\frac{1}{2}$. In carrying out the series of measurements a total of 7200 drawings were made, the number of even-numbered balls drawn being 3595 and the number of odd-numbered being 3605.

The drawings were recorded on strips of paper containing three columns: the first for the number of the drawing, the second for even-numbered balls, and the third for odd-numbered balls. As each ball was drawn, the number it bore was recorded in its proper column and place. This method of tabulation enabled all entries to be checked at a glance.

An effort was made to perform all drawings with the utmost fairness, so that each number drawn would be as nearly as possible the result of pure chance.

Every drawing was made, recorded, and checked by the writer himself; and the balls were hidden from view while the drawings were being made. All drawings and counts were carefully checked in one or more ways, and it is certain that no mistake was made.

A single measurement of the ratio in question consisted in taking a certain number of consecutive drawings, varying from 50 to 600, counting the even-numbered balls drawn (and also the odd-numbered as a check), and then finding the ratio of the number of even-numbered balls to the total number drawn in that measurement. Some of the sets of drawings constitute rather rough measurements of the ratio sought, but they possess the merit of being free from systematic error and of following the Gaussian law fairly well. The accuracy of such measurements could be increased by having a larger number of balls in the urn, keeping them shuffled continuously in an irregular manner, and using 1000 or more drawings as a single measurement.

In order to carry out the purpose of this investigation as well as possible, the entire number of drawings have been grouped in five different ways into measurements and sets of different degrees of accuracy. An effort was made to make the number of individual measurements in a set and the number of sets in a group correspond roughly to the usual measurements made in the physical sciences. The drawings in each individual measurement and in each set of a group were taken in consecutive order, just as they were drawn from the urn. There has been no interchange of order anywhere.

In computing the arithmetic means, residuals, probable errors, weights, etc., all computations were carried to six or seven significant figures, in order to guard against errors of computation due to rounding off numbers, etc. This procedure is justifiable on the grounds that the data are exact numbers. The results are given to several more figures than would ordinarily be the case with physical measurements, but they are given in this form to let the reader see just what was found. All computed results have been carefully checked for gross errors, and the most of them have been checked throughout by methods different from those used in obtaining them.

In order to see how well the residuals of the entire set of drawings conformed to the Gaussian law, and also to have a standard of comparison for all the different groupings, the 7200 drawings were first divided into 72 measurements of 100 drawings each. The arithmetic mean, residuals, probable error of a single measurement, and index of precision were computed. Then, with the index of precision and a table of the probability integral, it was an easy matter to compute the number of residuals that should lie between any given limits in order to conform to the Gaussian law. The computed and the actual distributions of residuals are given in the table below.

The table shows that the residuals follow the Gaussian law pretty closely, the greatest deviation being in the case of small residuals. In a smaller number of measurements the residuals would still follow the Gaussian law, but not as closely in general.

Magnitude of residuals	Number calculated	Number found
0–0.01	13.3	16
0–0.02	25.9	25
0–0.03	37.2	35
0–0.04	46.8	46
0–0.05	54.5	52
0–0.06	60.4	62
0–0.08	67.6	69
0–0.10	70.6	70
0–0.13	71.8	72

3. *Results.* The results of the drawings and groupings are given below.
Group I. One set of 72 measurements of 100 drawings each.

$$M_0 = 0.4993056 \pm 0.0034048.$$

Group II. Seven sets.

$M_1 = 0.4892857 \pm 0.0117464$,	from	7	measurements	of	120	drawings	each.
$M_2 = 0.5090000 \pm 0.0083792$,	"	10	"	"	100	"	"
$M_3 = 0.5055556 \pm 0.0075977$,	"	10	"	"	90	"	"
$M_4 = 0.4770833 \pm 0.0102124$,	"	12	"	"	80	"	"
$M_5 = 0.4857143 \pm 0.0097931$,	"	14	"	"	70	"	"
$M_6 = 0.5196078 \pm 0.0079041$,	"	17	"	"	60	"	"
$M_7 = 0.5110000 \pm 0.0114454$,	"	20	"	"	50	"	"

Weighted or general mean, $M_0 = 0.5021330$

Probable error, $\begin{cases} r_a = 0.0034822 \\ r_b = 0.0038814 \end{cases}$

Ratio of probable errors, $r_a/r_b = 0.897.$

Group III. Seven sets.

$M_1 = 0.5025000 \pm 0.0105386$,	from	12	measurements	of	100	drawings	each.
$M_2 = 0.5022222 \pm 0.0095240$,	"	9	"	"	"	"	"
$M_3 = 0.4853333 \pm 0.0059192$,	"	15	"	"	"	"	"
$M_4 = 0.4840000 \pm 0.0075377$,	"	10	"	"	"	"	"
$M_5 = 0.5300000 \pm 0.0036053$,	"	8	"	"	"	"	"
$M_6 = 0.5136364 \pm 0.0114057$,	"	11	"	"	"	"	"
$M_7 = 0.4842857 \pm 0.0083075$,	"	7	"	"	"	"	"

Weighted mean, $M_0 = 0.5092688$

Probable error, $\begin{cases} r_a = 0.0024597 \\ r_b = 0.0056992 \end{cases}$

Ratio of probable errors, $r_a/r_b = 0.432.$

Group IV. Two sets.

$M_1 = 0.4987097 \pm 0.0052502$,	from	31	measurements	of	100	drawings	each.
$M_2 = 0.4997561 \pm 0.0045276$,	"	41	"	"	"	"	"

Weighted mean,	$M_0 = 0.4993098$
Probable error,	$\begin{cases} r_a = 0.0034287 \\ r_b = 0.00034906 \end{cases}$
Ratio of probable errors,	$r_a/r_b = 9.82$

Group V. Two sets.

$M_1 = 0.4952778 \pm 0.0047376$, from 6 measurements of 600 drawings each.
$M_2 = 0.5033333 \pm 0.0057937$, " 9 " " 400 " "

Weighted mean,	$M_0 = 0.4985058$
Probable error,	$\begin{cases} r_a = 0.0036676 \\ r_b = 0.0026626 \end{cases}$
Ratio of probable errors,	$r_a/r_b = 1.38$.

4. *Discussion*. The results of Group I may be regarded as the most reliable of all, because they were obtained from a large number of direct measurements. The probable error of this set should perhaps be taken as the standard of comparison for all the groups.

Group II contains the greatest variation in the number of drawings constituting the individual measurements, but the total number of drawings in each set is about the same. There is no great variation in the weights of the several means, the weights (to four figures) being 1, 1.965, 2.390, 1.323, 1.439, 2.209, 1.053. The values of r_a and r_b are in close agreement.

The sets in Group III vary considerably as to number of measurements in a set and consequently as to number of drawings in a set. The outstanding feature of this group is the discordant set 5. This mean is widely different from the others, and its probable error is the smallest in the group. The small probable error came from the fact that all the measurements of the set were nearly equal and therefore did not differ much from their arithmetic mean. This gave small residuals and therefore a small probable error.

It is instructive to notice the individual measurements of this set. They are 0.54, 0.52, 0.55, 0.53, 0.50, 0.54, 0.53, 0.53.

The small probable error gave this set a controlling weight over the whole group, the weights for this group being 1.171, 1.434, 3.713, 2.290, 10.01, 1, 1.885. One effect of the preponderant weight of this set was to give r_a a much smaller value than it would have had without this set. Such an effect is perfectly proper and as it should be when the set of greatest weight is nearest the true value, as is usually the case, but here the weightiest set is the poorest in the group.

The discordant character of set 5 and its great weight had an opposite effect on r_b. They swayed the weighted mean in the direction of M_5 and thereby caused it to differ considerably from the means of the sets. This resulted in large residuals for the sets. The large residuals, in conjunction with the great weight of set 5, gave a large value for the quantity $\Sigma w V^2$. This explains the large value for r_b in this group.

The occurrence of this discordant mean and its small probable error must be ascribed to chance and not to the presence of systematic error, as it is the result of 800 consecutive drawings blindly made while the balls were shuffled and stirred in every possible manner. It is easy to explain the discrepancy between r_a and r_b in this group, but it is not due to systematic error.

The results in Group IV show in a striking manner that formulas (A) and (B) will not always give results of the same order of magnitude in the absence of systematic errors. Here r_b is but little more than one-tenth the size of r_a. The two sets are each comparable with Group I as to number of measurements, and their residuals follow the Gaussian law equally as well as do the residuals of that group, as may be seen from the tables below.

	Set 1			Set 2	
Residuals	Calculated	Found	Residuals	Calculated	Found
0 − 0.01	5.7	9	0 − 0.01	7.5	7
0 − 0.02	11.0	11	0 − 0.02	14.7	14
0 − 0.03	15.8	15	0 − 0.03	21.1	20
0 − 0.04	20.0	20	0 − 0.04	26.6	26
0 − 0.05	23.3	22	0 − 0.05	31.0	30
0 − 0.06	25.8	27	0 − 0.06	34.3	35
0 − 0.08	29.0	30	0 − 0.08	38.4	39
0 − 0.10	30.3	30	0 − 0.10	40.2	40
0 − 0.13	30.9	31	0 − 0.12	40.8	41

The probable errors of these two sets are reasonable, the smaller belonging to the set containing the greater number of measurements and both agreeing in a reasonable way with that in Group I. The weighted mean in Group IV is almost exactly the same as the arithmetic mean in Group I, and r_a in this group (Group IV) is nearly the same as the probable error of the mean in Group I. These facts should be noted.

The disagreement between r_a and r_b in this group is not due to systematic errors nor to statistical fluctuation;* it is due to the fact that the means of the two sets happened to be nearly equal. This compelled the residuals of the two sets to be small and consequently gave a small value for r_b. Such must always be the case when only two sets having nearly the same mean are combined.

Before leaving this group it is well to note the significance of the results found. The group was formed by taking 72 direct measurements and dividing

* The term "statistical fluctuation" is too indefinite to figure in this discussion unless defined quantitatively. Birge proposes to use the quantity $u=0.4769/\sqrt{n}$ as a measure of statistical fluctuation, where n denotes the number of measurements. He accordingly gives $1\pm u$ as the expected value of the ratio r_b/r_a. Whenever this ratio falls outside the limits $1\pm 5u$, he ascribes the discrepancy between r_a and r_b not to statistical fluctuation but to some other cause or causes—usually to systematic errors.

The quantity $0.4769r/\sqrt{n}$ is the probable error of the probable error r of a single observation and is derived in Merriman's "Method of Least Squares" (eighth edition), pp. 206–208. The limits $1\pm 5u$ correspond to the rule for the rejection of doubtful observations.

them into two sets. The two sets were then combined into one, giving a weighted mean practically the same as the arithmetic mean of the original undivided set. The probable error of this weighted mean was also practically the same as the probable error of the arithmetic mean of the original set when computed by formula (A), but *by the mere trick of using formula* (B) *we were able to reduce it to about one-tenth the size of the original probable error.* Formula (B) thus lends itself to scientific jugglery and enables one to vary his probable errors to suit his taste.

In Group V there is no striking difference between the values of r_a and r_b, but it is to be noticed that the latter is much less than the probable error in Group I.

On looking back over the results of this experiment, the reader will notice that r_a is always of the same order of magnitude as the probable errors of the sets that were combined into a single measurement, but is less than any of them. This is not the case with r_b. It has fluctuated widely, from nearly twice the probable error in Group I down to about one-tenth of that quantity. Frequently r_b bears very little relation to the given probable errors of the sets. Moreover, a small value for r_b does not necessarily indicate that the weighted mean is very near the true value of the quantity measured. It merely indicates that the means of the sets were nearly equal and therefore gave small residuals for those means. If two "wild" sets like M_5 in Group III happened to be nearly equal, r_b for their weighted mean would be very small and yet this mean would be far from the true value of the quantity measured.

PART III. *ARTIFICIAL MEASUREMENTS*

1. *Introductory.* In an effort to improve upon the urn experiment as a means of testing the hypothesis that a disagreement between r_a and r_b is an indication of systematic errors, the writer conceived the idea of constructing artificial measurements whose errors could be made to follow the Gaussian law as closely as desired. Systematic errors can be kept out of such measurements. By assuming a true magnitude for a measurable quantity, the precision index of a set of measurements, and the number of individual measurements in the set, it is possible to construct, by means of known formulas and a table of the probability integral, any set of measurements desired.

Let

m = assumed true value of the magnitude to be measured,
m_0 = most probable value of this magnitude
= arithmetic mean of the individual measurements.

Also let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ denote the errors of the n measurements and $v_1, v_2, \cdots, v_n$ the residuals. Then the necessary formulas are*

$$m_0 = m - \Sigma\epsilon/n \tag{4}$$

$$v_i = \epsilon_i - \Sigma\epsilon/n \qquad (i = 1, 2, \cdots, n). \tag{5}$$

* These formulas are derived in the writer's "*Numerical Mathematical Analysis,*" page 310.

On assuming the probability equation, $y = he^{-h^2x^2}/\sqrt{\pi}$, which the errors of the n measurements are to follow, and using a table of the probability integral, one can compute the number of errors of different sizes that must lie between any given limits; for the number of errors is equal to the total number of measurements (in the set) multiplied by the probability of an error lying within the limits. A sample of such a computation is shown in the following table:

$h = 72,\ n = 12$

x	hx	P	$N(=nP)$
0.0030	0.2160	0.23999	2.88
0.0034	0.2448	0.27081	3.25
0.0038	0.2736	0.30119	3.61
0.0042	0.3024	0.33110	3.97
0.0046	0.3312	0.36049	4.33
0.0050	0.3600	0.38933	4.67

In this table the last column gives the number of errors whose magnitude is between 0 and x. About half of these errors are positive and half negative.

Since the limits between which the n errors (one for each measurement) lie are known, and since about half the errors are positive and half negative, it is possible to assign approximate values and appropriate signs to the n errors of the set. The following table shows how the errors were assigned in one set of artificial measurements:

$h = 72,\ n = 12$

Individual measurements	ϵ	v
1	+ 0.0012	+ 0.0027
2	− 0.0022	− 0.0007
3	+ 0.0034	+ 0.0049
4	− 0.0044	− 0.0029
5	+ 0.0056	+ 0.0071
6	− 0.0068	− 0.0053
7	+ 0.0082	+ 0.0097
8	− 0.010	− 0.0085
9	+ 0.012	+ 0.0135
10	− 0.014	− 0.0125
11	+ 0.018	+ 0.0195
12	− 0.029	− 0.0275
	$\Sigma\epsilon = -0.0180$	$\Sigma v = 0$

Chance played very little part in the assignment of these errors. They were assigned carefully and deliberately, so as to conform to the given probability equation as closely as possible. They were given alternate signs so as to make the distribution as nearly symmetrical as possible.

Inasmuch as the largest error of a set controls the sign of $\Sigma\epsilon$, the signs of the

errors in the several sets were so assigned as to make the signs of the largest errors alternate from set to set. This procedure made the sums $\Sigma\epsilon$ alternate in sign and consequently gave a symmetrical distribution of the means of the sets, by formula (4).

After the errors (ϵ's) had been assigned as above described, the means of the several sets and the residuals in those sets were computed by formulas (4) and (5). Then the probable errors of the means were computed by the formula

$$r_0 = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}.$$

By proceeding as outlined above, the writer constructed several sets of artificial measurements of a quantity of magnitude 50 and combined them into single measurements, as was done with the drawings from the urn. The groupings and results are given below.

2. *Results. Case I.* Seven sets.

$$\begin{aligned}
M_1 &= 50.0024 \pm 0.0048682. & h &= 20, & n &= 30\\
M_2 &= 49.9968 \pm 0.0055456. & h &= 24, & n &= 18\\
M_3 &= 50.0022 \pm 0.0049622. & h &= 18, & n &= 35\\
M_4 &= 49.9956 \pm 0.0082980. & h &= 15, & n &= 20\\
M_5 &= 50.0030 \pm 0.0048431. & h &= 30, & n &= 16\\
M_6 &= 49.99675 \pm 0.0072998. & h &= 12, & n &= 36\\
M_7 &= 50.00325 \pm 0.0048376. & h &= 36, & n &= 12
\end{aligned}$$

Weighted mean, $M_0 = 50.00102.$

Probable error, $\begin{cases} r_a = 0.0020774. \\ r_b = 0.0007722. \end{cases}$

Ratio of r_a to r_b, $r_a/r_b = 2.69.$

Case II. Seven sets having twice the precision of those in Case I.

$$\begin{aligned}
M_1 &= 50.00110 \pm 0.0024963. & h &= 40, & n &= 30\\
M_2 &= 49.99845 \pm 0.0028381. & h &= 48, & n &= 18\\
M_3 &= 50.00110 \pm 0.0025310. & h &= 36, & n &= 35\\
M_4 &= 49.997825 \pm 0.0042684. & h &= 30, & n &= 20\\
M_5 &= 50.00140 \pm 0.0024530. & h &= 60, & n &= 16\\
M_6 &= 49.998375 \pm 0.0037141. & h &= 24, & n &= 36\\
M_7 &= 50.00150 \pm 0.0024629. & h &= 72. & n &= 12
\end{aligned}$$

Weighted mean, $M_0 = 50.000443.$

Probable error, $\begin{cases} r_a = 0.0010546. \\ r_b = 0.00037328. \end{cases}$

Ratio of r_a to r_b, $r_a/r_b = 2.83.$

Case III. Two sets having equal means.

$$\begin{aligned}
M_1 &= 50.0024 \pm 0.0048682. & h &= 20, & n &= 30\\
M_2 &= 50.0024 \pm 0.0045522. & h &= 24, & n &= 25
\end{aligned}$$

$$\begin{array}{ll} \text{Weighted mean,} & M_0 = 50.00240. \\ \text{Probable error,} & \begin{cases} r_a = 0.0033250. \\ r_b = 0. \end{cases} \end{array}$$

Case IV. Two sets having nearly equal weights.

$$M_1 = 50.00300 \pm 0.0048431. \quad h = 30, \quad n = 16$$
$$M_2 = 50.00325 \pm 0.0048376. \quad h = 36, \quad n = 12$$
$$\begin{array}{ll} \text{Weighted mean,} & M_0 = 50.003125. \\ \text{Probable error,} & \begin{cases} r_a = 0.00342225. \\ r_b = 0.0000843125. \end{cases} \\ \text{Ratio of } r_a \text{ to } r_b, & r_a/r_b = 40.6. \end{array}$$

3. *Discussion.* The measurements in Case I are typical physical measurements of fair precision. The weights of the means are fairly uniform, being 2.905, 2.239, 2.796, 1, 2.936, 1.137, 2.942. The errors of the original measurements follow the Gaussian law, there is no systematic error, and there is very little statistical fluctuation. Yet r_b is but little more than one-third as large as r_a. This result does not support the hypothesis that a considerable disagreement between r_a and r_b is an indication of systematic error.

The measurements in Case II are more precise than in Case I. The weights of the means are 2.924, 2.262, 2.844, 1, 3.029, 1.321, 3.004, and are therefore slightly more varied than in the preceding case. The ratio r_a/r_b is a little larger. The results are quite similar to those of Case I, as might have been expected.

The two sets of measurements in Case III were computed from different probability equations and different numbers of measurements. They were not planned to turn out with equal arithmetic means; they just happened to do so without any foreknowledge on the part of the writer as to what the results were going to be. A zero result for r_b is what should have been expected in the light of the nature of the formula by which it was found. The result simply corroborates the analytical results found in Part I.

The sets in Case IV are the fifth and seventh of Case I. They were combined because their probable errors are nearly equal. It is to be observed that r_a is of the same order of magnitude as the given probable errors of the sets, whereas r_b is less than one-fortieth as large as r_a. This wide difference is not due to systematic errors nor to statistical fluctuations, not even when measured by the liberal criterion proposed by Birge, but is due to the fact that the means of the two sets are nearly equal.

CONCLUSION

The analytical part of this investigation indicates that formulas (A) and (B) will not give the same result, except by accident, unless the sets to be combined are the results of a great number of individual direct measurements. When only two sets are combined, the probable error of the weighted mean is liable to be anything from zero up to a value greater than the given probable errors if computed by (B), whereas if computed by (A) it will always be of the same order

of magnitude as the given probable errors and somewhat smaller than either of them. For the determination of the probable error of the weighted mean of two sets of measurements, formula (B) is not reliable. This statement holds whether the weights are computed from the given probable errors or are assigned arbitrarily. *No one would think of trying to find the probable error of the mean of only two direct individual measurements by means of the formula*

$$r = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}.$$

It seems equally absurd to try to find the probable error of the mean of two sets by means of the corresponding formula

$$r = 0.6745\sqrt{\frac{\Sigma w V^2}{(n-1)\Sigma w}}.$$

No results in this whole investigation support the hypothesis that a serious disagreement between the results given by (A) and (B) is proof of the existence of systematic errors. On the contrary, it has been shown that the two formulas sometimes give widely different results in cases where it is positively known that no systematic errors exist.

THE TYPE OF MATHEMATICAL TRAINING NEEDED BY ELECTRICAL ENGINEERS*

BY A. M. DUDLEY, Westinghouse Electric and Manufacturing Company

By way of introduction I should like to comment on one or two questions which might naturally arise when a practicing engineer speaks on mathematics. These are:

(1) I am not terribly disturbed about the preparation in mathematics evidenced by recent graduates in electrical engineering. I do not think this "all wrong" and I am not obsessed by the idea that I know how it should be made right. However, I am sincerely concerned, as I know you all are, with the fact that engineers in general do not use mathematical analyses as often and with the freedom they should in their regular work. So I take it we are met together for the purpose of exchanging ideas on how engineers might acquire greater facility in the exercise of the most useful tool they have.

(2) Answering the question as to what opportunity I may recently have had to observe the methods and results of up-to-date training in mathematics, it is part of my work to help induct into our engineering organization graduates from any and all of the engineering schools in the country. In normal times this group would run from 80 to 100 men per year. In addition, I have within

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* An address presented at the Mathematics Conference of the S.P.E.E. at Ithaca, N. Y., June 20, 1934.

the last three years taken two courses in ordinary differential equations for a total of six credits under Dr. Culver of the University of Pittsburgh and one course in symmetrical components for two credits under Dr. W. A. Lewis of the Westinghouse Electric & Mfg. Co., who is a properly accredited member of the Graduate Faculty of the University of Pittsburgh. I say these things, not as evidence of mental agility in a man of my age, but simply that you may know that I am fairly close to what is being done and in sympathy with the idea of offering constructive criticism.

(3) As to the time-honored question often asked by students, "How much mathematics does an engineer need in his professional work?" my answer has always been, "The more mathematics that an engineer has *that he can use as a tool*, the greater is likely to be his accomplishment." This has a bearing on the point I make a little later that mathematics as a means for intellectual development or as a mental recreation is entirely aside from what we are talking about.

In order that we may have a common idea, the curriculum as to specific branches of mathematics which I have in mind would assume that differential and integral calculus are included in the standard undergraduate program and also in most cases an introductory course in ordinary differential equations. Carrying this over into graduate work would assume a study of the functions of the complex variable, and linear partial differential equations, coupled with such courses in so-called engineering mathematics as operational calculus, symmetrical components, etc. From this point of view, I am perfectly regular and orthodox, and, so far as I am aware, in line with accepted courses and ideas at the present time. What I do wish to do is to support vigorously and heartily an idea, not at all new, which has to do with pedagogical methods rather than subject matter. You have all been urged to carry out this idea and have tried to do so, but we are still so far from realizing its possibilities that I want to repeat it and in so doing see if I can add anything to its appeal which will fire your imagination with some new means for bringing it about. I refer to the necessity for helping the student realize the application of calculus to countless problems of nature and every day life and actually showing him how to set up differential equations for their solution. Let us take such a simple problem as a cylindrical tank full of water which is being emptied by natural flow through an orifice located near the bottom. Or the case of a ladder leaning against a building which starts to slide and eventually lands flat on the ground. The curve traced in space by a point on that ladder is a most intriguing problem if properly approached and the student helped to imagine it and work it out himself. It may be a compliment to us as engineers that many of you teachers in mathematics have given us credit for being a lot more intelligent than we are. To you who have some vision of mathematics as such, it seems evident that if you give a man a useful tool he will use it. I am sorry to say that as engineers we can be a lot "dumber" than you would think possible and I have seen many bright boys whose equipment in mathematics and ability in solving set exercises stopped far short of realizing that these same formulae and exercises had a practical bearing

on a practical problem before them. Yes, this is really the whole story. The imagination that carries to the specific problem the proper form of mathematical analysis required, and the ability to set up differential equations which may allow a chance for their solution, is the thing we need most and the thing that would be of the greatest value in engineering. Those of you who are in my generation and who finished your undergraduate work early in the present century will remember what a feeling, almost of awe, overcame you when one of your own classmates dared to analyze a matter-of-fact problem in concepts of the calculus and to set up a differential equation for its solution. At first we were openly skeptical and rated it merely a piece of grandstand play to catch the eye of the instructor and impress him with the erudition of the adventurer. And then there dawned on some of the other members of the class that these laws and these formulae, to which were attached the names of great mathematicians, were not a species of mental gymnastics or mathematical recreations for the gratification of the mentally agile or the embellishment of the mathematically erudite, but were a sincerely honest and exceedingly clever shortcut that might be used to "adapt the forces of nature to the uses of man." And when that basic idea took root and really grew, the ensuing intellectual development of the individual and the broadening and deepening of his power of analysis and increased ability to use all his technical, factual knowledge was tremendous and most inspiring to witness. But on the dark side of the picture, if the student graduates without that vision and without that facility, he very seldom acquires it later. There are some phases of engineering which are distinctly pathetic. By the nature of his work, the engineer must be a thoroughly practical man and common sense must be of the essence of his conclusions. He deals with immutable physical laws and he dare not get his decimal point in the wrong place. At the same time, he should have at his command tools for the solution of his most intricate problems which border on fairyland. You may be "hard boiled" enough to feel that some of the cleverest devices are only mathematical tricks and the successful engineer who uses them a kind of glorified circus performer on a flying trapeze, but at the same moment inside yourselves, you know that the inspiration of some of the world's great mathematicians was of the same order and the same substance that actuated Michael Angelo or Leonardo da Vinci. And in passing, let us pay tribute to the latter who had one of the greatest minds for sheer intellectual breadth and grasp that the world has ever known. He was an engineer who knew how to use mathematical analysis even when it carried through to the whimsical smile on the lips of Mona Lisa.

Not to give the student a chance at some of this creative imagination through lack of what the instructor could readily do but feels is too elementary, is to cut down greatly the enjoyment of life by the individual and correspondingly impoverish the engineering profession.

How are you going to do it? I don't know. I have a notion that when the student first tackles calculus is the time to start the seed but certainly when he comes into the beginnings of differential equations he can begin to see the light.

Take him mentally by the hand, slowly and carefully, with some fairly elementary conception and make sure that he grasps it. You will find him missing it from many angles: (1) He will be mentally too agile and imagine he sees it when he does not. (2) He will be mentally too immature and lacking in preparation to be able to grasp it. (3) He will be too matter-of-fact to realize that here is something beautiful and something very useful. So that finally you will conclude that the individual who can really get it and put it to use is rare. He may be, but I think not so rare as the present product would indicate.

And at a risk of giving offense to you as pure mathematicians, may I say with explosive violence, don't ever teach mathematics as an end in itself. As concerns engineering, it is not an end, it is only a means. A very beautiful means, it is true, but the engineer looks only for ends and pays scant respect to his means. But always present it through a practical solution and always with the idea of showing how a natural process or action can be expressed by the idea of one quantity varying in a certain way relatively to another.

Having given you an old idea and some would-be poetry, lest you feel cheated of any real tangible ideas, I will quote in some rough notes the opinions or ideas of two men who combine to a high degree ability in mathematical analysis and attainment in engineering. I refer to Dr. Charles Fortescue, known for his development of symmetrical components as a useful tool in the solution of transmission problems, and Dr. Joseph Slepian, whose ideas of deionizing the path of arcs have almost remade the art of circuit interruption.

Apropos of the specific subject of this memorandum, Dr. Slepian says:

"I would list the following subjects in *pure* mathematics as valuable to the electrical engineer and in order of their value:

(1) Differential equations.
(2) Potential theory and linear partial differential equations.
(3) Functions of a complex variable.

Number 2 might be covered in a course on mathematical hydrodynamics, or mathematical theory of electricity and magnetism.

I am supposing that courses of a mathematical nature are taken by the student also in the department of electrical engineering such as operational calculus and symmetrical components, etc."

In notes of a slightly different form, Dr. Fortescue says: "Mathematical training should permit him to read Maxwell and Jeans. It should be broad enough so that he can readily read treatises on thermodynamics, electron theory (conduction in gases, etc.), and physics.

Specifically he should have knowledge of:

(1) Differential equations (including Heaviside's operational calculus).
(2) Mathematical analysis and theory of functions.
(3) Algebraic forms (theory of determinants—complex numbers).
(4) Engineering mathematics (symmetrical coordinates).
(5) Mathematics of physics (Lagrange's equations, etc.) Legendre's functions,

Bessel's functions, spherical harmonics, etc. (Not to be an expert in these, but to have some appreciation of their usefulness).
(6) Mathematics of choice and chance. Probabilities.

The essential qualities a scientific student should have are curiosity and imagination. With these qualities he will largely educate himself and that is what is really wanted."

In closing, I should simply like to repeat the next to the last sentence—"The essential qualities a scientific student should have are curiosity and imagination." I would ask you men who teach mathematics of any sort to present it so that it arouses curiosity and imagination. If you can do this with any increase in results over the past, we shall show a correspondingly bettered performance in electrical engineering.

Discussion

In order to check and make sure that the foregoing is not entirely foreign to the present day student point of view, the manuscript was presented to a young man graduated from an eastern college who is just completing two years of graduate study in engineering mechanics at a mid-western university. His comment is as follows:

"You note that most of the students fail to grasp the concept of expressing physical relations in terms of mathematical forms, or even do not realize that mathematics is intended to be useful in present as well as past investigations. My own experience confirms this. You do well to advise the teachers of mathematics to emphasize this point by every means at their disposal, and to point out the artistic as well as the utilitarian aspects of their subjects; but even if this were perfectly done, I doubt whether more than a very small percentage of the students would assimilate it. To do so would require, first, a strongly developed character, and second, some ability to philosophize: to see the general purpose of a scheme of things without being confused by the details of it; in a word, to have a real sense of perspective. These qualities are rare enough by themselves; the combination is rarer still. A student without them cannot be made into a successful man of science by any curriculum; on the other hand, a man who does possess them will, as Dr. Fortescue suggests, largely educate himself. In reading a life of Frederick W. Taylor recently, I found that he held similar views; he listed the qualities necessary to make a successful engineer, in order of decreasing importance, as (a) character, (b) common sense ("perspective"), (c) intellectual ability. I think it may be inferred from this that Taylor would have concurred in the self-education idea.

Finally, it seems to me that self-education, at least in part, is the only practicable way in which most students can acquire higher mathematics. I have converted Dr. Fortescue's list into the approximate number of credit hours which would be required to get these subjects in the academic way. It comes to about twenty-two hours, and as the courses could not all run concurrently, such an undertaking would absorb the major part of at least two years of gradu-

ate work. This would certainly be out of the question for most students. It is much more feasible to let the student dig out the material he needs as he finds he needs it, and after he has matured. He will then have both the motive for such study and the will power to carry through."

A NOTE ON POLYNOMIAL CURVES

By J. R. Britton, University of Colorado

The polynomials are probably our most familiar and simplest class of curves. This note is a brief study of the restrictions under which coordinates of the extremes (maxima and minima) may be assigned.

We restrict ourselves to the case where the maximum number of extremes (one less than the degree of the curve) is present.

I. Let $x_1, x_2, \cdots, x_n$, all distinct and in any order, be the abscissas of the n extremes of

$$y = a_0x^{n+1} + \cdots + a_{n+1}.$$

Then

$$y' = f'(x) = A\prod_{i=1}^{n}(x - x_i) = A\sum_{i=0}^{n}(-1)^iS_ix^{n-i},$$

where A is a constant, $\neq 0$, to be determined, and S_i is the ith symmetric sum over the x_i; $S_i = \sum x_1 x_2 \cdots x_i$, $S_0 = 1$. ($A = 0$ is obviously trivial.)

Integrating,

$$y = A\sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x^{n+1-i} + C,$$

where C is a second constant to be determined.

We are now free to impose two more conditions on $f(x)$. Let y_1, y_2 be the ordinates of the two extremes corresponding to x_1, x_2, respectively. Then

$$A\sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x_k^{n+1-i} + C = y_k, \qquad k = 1, 2,$$

will be the two equations for the determination of A and C.

For a unique solution, we must have

$$\Delta = \begin{vmatrix} \sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x_1^{n+1-i} & 1 \\ \sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x_2^{n+1-i} & 1 \end{vmatrix}$$

$$= \sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}(x_1^{n+1-i} - x_2^{n+1-i}) \neq 0.$$

In case $\Delta \neq 0$ and $y_1 = y_2$, we should have $C = y_1$, $A = 0$, which has already been excluded as trivial. We shall, therefore, assume $y_1 \neq y_2$.

Since only a change in scale and a shift in origin is involved, we may, without loss of generality, place $x_1 = +1$, $x_2 = -1$;

$$\Delta = \sum_{i=0}^{n} (-1)^i \frac{S_i}{n+1-i} \left[(1)^{n+1-i} - (-1)^{n+1-i}\right].$$

For n even,

$$\Delta = 2\left\{\frac{S_0}{n+1} + \frac{S_2}{n-1} + \cdots + \frac{S_{n-2}}{3} + \frac{S_n}{1}\right\}.$$

For n odd,

$$\Delta = -2\left\{\frac{S_1}{n} + \frac{S_3}{n-2} + \cdots + \frac{S_{n-2}}{3} + \frac{S_n}{1}\right\}.$$

In either case, $\Delta \neq 0$ when and only when

$$\frac{S_n}{1} + \frac{S_{n-2}}{3} + \frac{S_{n-4}}{5} + \cdots = \sum_{j=0}^{[n/2]} \frac{S_{n-2j}}{2j+1} \neq 0,$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$.

Since $x_1 = 1$ and $x_2 = -1$, we are able to express this last condition entirely in terms of $x_3, x_4, \cdots, x_n$ as follows:

Let S_i' be the ith symmetric sum over the elements $x_3, x_4, \cdots, x_n$.

$$\begin{aligned}
S_n &= x_1x_2S'_{n-2} = -S'_{n-2},\\
S_{n-1} &= x_1x_2S'_{n-3} + (x_1 + x_2)S'_{n-2} + S'_{n-1} = -S'_{n-3} + S'_{n-1},\\
S_{n-2} &= x_1x_2S'_{n-4} + (x_1 + x_2)S'_{n-3} + S'_{n-2} = -S'_{n-4} + S'_{n-2},\\
&\cdots\\
S_{n-k} &= -S'_{n-k-2} + S'_{n-k},\\
&\cdots\\
S_1 &= (x_1 + x_2) + S_1' = S_1',\\
S_0 &= S_0' = 1.
\end{aligned}$$

With the convention that $S_i' = 0$ for $i < 0$ or $i > n-2$, the formula

$$S_{n-k} = -S'_{n-k-2} + S'_{n-k}$$

holds throughout.

Now,

$$\begin{aligned}\frac{S_n}{1}+\frac{S_{n-2}}{3}+\frac{S_{n-4}}{5}+\cdots\\ &= -\frac{S'_{n-2}}{1}+\left(\frac{-S'_{n-4}+S'_{n-2}}{3}\right)+\left(\frac{-S'_{n-6}+S'_{n-4}}{5}\right)+\cdots\\ &= -2\left\{\frac{S'_{n-2}}{1\cdot 3}+\frac{S'_{n-4}}{3\cdot 5}+\frac{S'_{n-6}}{5\cdot 7}+\cdots\right\}.\end{aligned}$$

$\therefore \Delta\neq 0$ if and only if

$$\frac{S'_{n-2}}{1\cdot 3}+\frac{S'_{n-4}}{3\cdot 5}+\frac{S'_{n-6}}{5\cdot 7}+\cdots\neq 0.$$

We notice that if $\Delta=0$, the left hand members of our equations for the determination of A and C will be identical. If, in addition, $y_1\neq y_2$, there will obviously be no solution for A and C, i.e., no polynomial curve satisfying the given condition. On the other hand, if $y_1=y_2$, we shall have a one-parameter family of curves

$$y = A\sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x^{n+1-i}+y_1-A\sum_{i=0}^{n}(-1)^i\frac{S_i}{n+1-i}x_1^{n+1-i},$$

where A is arbitrary, and the ordinates for $x=x_1$ and $x=x_2$ will be equal.

We may sum up in the following:

Theorem: *If $x_1=1$, $x_2=-1$, x_3, x_4, $\cdots$, x_n are assigned as the abscissas of the extremes, and $y_1\neq y_2$, as the ordinates corresponding to the abscissas $+1$ and -1, respectively; then the necessary and sufficient condition for the existence of a unique polynomial curve of degree $n+1$ with these x_i is*

$$\frac{S'_{n-2}}{1\cdot 3}+\frac{S'_{n-4}}{3\cdot 5}+\frac{S'_{n-6}}{5\cdot 7}+\cdots\neq 0,\ \textit{where}\ S'_i = \sum x_3x_4\cdots x_{i+2}.$$

If the condition is not satisfied, there will be a one-parameter solution for $y_1=y_2$, and no solution for $y_1\neq y_2$. (If the condition is satisfied, but $y_1=y_2$, only the trivial solution $y=y_1$ is obtained.)

II. It is possible to obtain a few more results by the use of a slightly different notation.

Let

$$y' = A(x-x_1)(x-x_2)\cdots(x-x_n) = A\phi(x),$$

where the x_i are, as before, the abscissas of the extremes.

Integrating,

$$y = A\int_0^x \phi(x)dx + C,$$

where A and C are to be determined by the equations:

$$A\int_0^{x_k} \phi(x)dx + C = y_k, \qquad k = 1, 2.$$

If we consider only unique solutions, then

$$\Delta = \begin{vmatrix} \int_0^{x_1} \phi(x)dx & 1 \\ \\ \int_0^{x_2} \phi(x)dx & 1 \end{vmatrix} = \int_{x_2}^{x_1} \phi(x)dx \equiv J \neq 0,$$

and $y_1 \neq y_2$, as previously,

Since $\phi(x)$ is a polynomial with zeros $x_1, x_2, \cdots, x_n$, the integral J represents, in absolute value, the algebraic sum of the areas between $\phi(x)$ and the x axis. Our condition says that this sum is not to vanish. We may at once draw the conclusion: *if x_1 and x_2 are two adjacent zeros, J will never vanish, and if $y_1 \neq y_2$, we shall always have a unique polynomial curve of degree $n+1$ with the assigned abscissas for its extremes.*

III. Now let

$$\phi(x) = \psi(x)\cdot(x - x_n), \ \psi(x) = (x - x_1)(x - x_2)\cdots(x - x_{n-1}).$$

$$\Delta = \int_{x_2}^{x_1} \phi(x)dx = \int_{x_2}^{x_1} x\psi(x)dx - x_n\int_x^{x_1} \psi(x)dx \neq 0,$$

or

$$x_n \neq \frac{\int_{x_2}^{x_1} x\psi(x)dx}{\int_{x_2}^{x_2} \psi(x)dx} = \frac{P}{Q}.$$

If we select $n-1$ of the x_i, then the value P/Q must not be assigned to x_n if we are to have a unique curve.

There are two cases of interest:

1. $P\neq 0$, $Q=0$: J will vanish for no finite value of x_n. We shall have a unique solution for any x_n and any $y_1\neq y_2$. Example: $x_1=+1$, $x_2=-1$, $x_3=0$, and $x_4=x_n$ may have any value (distinct from the others). (No solution for $y_1=y_2$.)

2. $P=0$, $Q=0$: J will vanish for any x_n whatever, and there will never be a unique solution. Example: $x_1=+1$, $x_2=-1$, $x_3=-1/\sqrt{5}$, $x_4=+1\sqrt{5}$, $x_5=x_n$. (One-parameter solution for $y_1=y_2$.)

Between the "exceptional" value P/Q and any other x_k, $k\neq 1, 2$, there is a reciprocal relation which may be exhibited as follows:

Let

$$G(x) = (x - x_1)(x - x_2) \cdots (x - x_{n-2}) \text{ and } x_{n-1} = v; \frac{P}{Q} = u.$$

$$u = \frac{\int_{x_2}^{x_1} x\psi(x)dx}{\int_{x_2}^{x_1} \psi(x)dx} = \frac{\int_{x_2}^{x_1} x^2G(x)dx - v\int_{x_2}^{x_1} xG(x)dx}{\int_{x_2}^{x_1} xG(x)dx - v\int_{x_2}^{x_1} G(x)dx} = \frac{a - bv}{b - cv},$$

and

$$v = \frac{a - bu}{b - cu}.$$

That is, if in our set of x_i $(i=1, 2, \cdots, n-1)$ we substitute P/Q for any x_k different from x_1 and x_2, then the new "exceptional" value will be x_k. Example: For $x_1=1$, $x_2=3$, $x_3=4$, the "exceptional" value is 1.9, while for $x_1=1$, $x_2=3$, $x_3=1.9$, the "exceptional" value is 4.

ON A CERTAIN IDENTITY DUE TO HERMITE

By M. A. BASOCO, University of Nebraska

1. *Introduction.* In a letter to Stieltjes,* Hermite states, without proof, the following identity:
Let U, V, be two functions of x for which the first n derivatives exist, and define the function

$$F(U, V) \equiv \sum_{j=0}^{n} (-1)^j \binom{n}{j} U^{(j)}V^{(n-j)}, \tag{1}$$

where $U^{(1)}, U^{(2)}, \cdots, U^{(n)}$; $V^{(1)}, V^{(2)}, \cdots, V^{(n)}$, are the successive derivatives, and $U^{(0)}$, $V^{(0)}$ are U and V respectively. Then, if a, b, α are arbitrary constants

$$F(ae^{\alpha x}U, be^{\alpha x}V) = abe^{2\alpha x}F(U, V). \tag{2}$$

In this paper we note a generalization of this result which yields an identity involving an arbitrary even number of functions. The proofs of these identities are, of course, elementary exercises in the application of Leibniz's theorem for the pth derivative of a product and of the symbolic differential operator. However, the identities themselves are not without interest and find an immediate application to the doubly periodic functions of the second and third kinds, the intermediary functions of Hermite,† the theta functions, etc. In particular, the application to these last gives rise to a series of simple differential relations which are believed to be new.

2. The generalization indicated above may be stated in the following

* *Correspondance d' Hermite et de Stieltjes*, vol. 2, p. 73, letter 263.

† See for example, Hancock, *Elliptic Functions*, Chap. 5. Appell-Lacour, *Fonctions Elliptiques*, Chapters 11, 12.

THEOREM (A): *Let $\phi_1(x), \phi_2(x), \cdots, \phi_{2k}(x)$ be a set of $2k$ arbitrary functions for each of which the first n derivatives exist. Define*

$$(3)\quad F(\phi_j) \equiv \sum (-1)^{n_2+n_4+\cdots+n_{2k}} \frac{n!}{n_1!n_2!\cdots n_{2k}!} \phi_1^{(n_1)}(x)\phi_2^{(n_2)}(x)\cdots\phi_{2k}^{(n_{2k})}(x),$$
$$n = n_1 + n_2 + n_3 + \cdots + n_{2k},$$

the sum being extended over the indicated partitions of a given arbitrary integer n, and where

$$\phi_j^{(n_j)}(x) \equiv \frac{d^{n_j}}{dx^{n_j}}\phi_j.$$

Then, if $c_1, c_2, \cdots, c_{2k}$; $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{2k}$ are arbitrary constants,

$$(4)\qquad F(c_je^{\alpha_jx}\phi_j(x)) \equiv c_1c_2\cdots c_{2k}e^{(\alpha_1+\alpha_2+\cdots+\alpha_{2k})x}F(\phi_j),$$

provided

$$\sum_{i=1}^{k}(\alpha_{2i}-\alpha_{2i-1}) = 0.$$

COROLLARY: *If $\alpha_1=\alpha_2=\cdots=\alpha_{2k}=\alpha$, then*

$$(5)\qquad F(c_je^{\alpha x}\phi_j(x)) = c_1c_2\cdots c_{2k}e^{2k\alpha x}F(\phi_j).$$

From this identity, Hermite's result (2) is obtained at once on placing $k=1$.

The proof of the theorem will follow easily on performing the indicated substitution on $F(\phi)$ and recalling that

$$[c_je^{\alpha_jx}\phi_j(x)]^{(n_j)} = c_je^{\alpha_jx}(D_j+\alpha_j)^{n_j}\cdot\phi_j(x),$$

where $(D_j+\alpha_j)^{n_j}$ is the usual binomial differential operator, it being agreed that it is to affect $\phi_j(x)$ only, and finally noticing that $F(\phi_j)$ may be written in the symbolic form

$$F(\phi_j) = [D_1 - D_2 + D_3 - \cdots - D_{2k}]^n\cdot\phi_1(x)\phi_2(x)\cdots\phi_{2k}(x),$$

it being understood that each D_j^p will operate on $\phi_j(x)$ only.

A theorem which may be proved in the same manner as the preceding is the following:

THEOREM (B): *Let $\phi_1(x), \phi_2(x), \cdots, \phi_k(x)$ be a set of k arbitrary functions. Consider the function defined by*

$$(6)\qquad K(x) = \sum_{(n=n_1+n_2+\cdots n_k)} \frac{n!}{n_1!n_2!\cdots n_k!}\phi_1^{(n_1)}(x)\phi_2^{(n_2)}(x)\cdots\phi_k^{(n_k)}(x)$$
$$(n = n_1 + n_2 + \cdots + n_k).$$

If $\alpha_1, \alpha_2, \cdots, \alpha_k$ are arbitrary constants subject only to the condition $\alpha_1+\alpha_2+\cdots+\alpha_k=0$, then $K(x)$ is invariant under the substitution

$$\begin{pmatrix} \phi_1(x), & \phi_2(x), & \cdots, & \phi_k(x) \\ e^{\alpha_1 x}\phi_1(x), & e^{\alpha_2 x}\phi_2(x), & \cdots, & e^{\alpha_k x}\phi_k(x) \end{pmatrix}.$$

3. Suppose now, that the functions $\phi_j(x)$ belong to the class of doubly periodic functions of the third kind, so that they are meromorphic and satisfy periodicity relations of the form

$$(7) \qquad \begin{cases} \phi_j(x + 2\omega) = e^{\alpha_j x + \beta_j}\phi_j(x), \\ \phi_j(x + 2\omega') = e^{\gamma_j x + \delta_j}\phi_j(x), \end{cases}$$

$$\left(\alpha_j, \gamma_j \neq 0,\ 0 < \arg\frac{\omega'}{\omega} < \pi,\ \omega'\alpha_j - \omega\gamma_j = n_j\pi i,\ n_j = \text{arbitrary integer}\right),$$

a common period pair $(2\omega, 2\omega')$ being assumed for the entire set. With this additional hypothesis, theorems (A) and (B) give rise respectively to (C) and (D) below.

THEOREM (C): *If the constants α_j, γ_j in (7) satisfy the conditions*

$$\sum_{j=1}^{k} (\alpha_{2j} - \alpha_{2j-1}) = 0, \qquad \sum_{j=1}^{k} (\gamma_{2j} - \gamma_{2j-1}) = 0,$$

then the function $F(\phi_j(x))$ defined by equation (3) is, in general, a doubly periodic function of the third kind with periods $(2\omega, 2\omega')$, the corresponding multipliers being exp $(Ax+B)$, exp $(Cx+D)$, where

$$A = \sum_{j=1}^{2k} \alpha_j, \quad B = \sum_{j=1}^{2k} \beta_j, \quad C = \sum_{j=1}^{2k} \gamma_j, \quad D = \sum_{j=1}^{2k} \delta_j.$$

COROLLARY: *If* $\alpha_1+\alpha_3+\cdots+\alpha_{2k-1}=\alpha_2+\alpha_4+\cdots+\alpha_{2k}$ and $\gamma_1+\gamma_3+\cdots\ \gamma_{2k-1}=\gamma_2+\gamma_4+\cdots+\gamma_{2k}$, *then $F(\phi_j(x))$ will be an elliptic function or a doubly periodic function of the second kind according as $B=D=0$ or at least one of $B, D\neq 0$.*

THEOREM (D): *If the constants α_j, γ_j in (7) are such that $\sum_{j=1}^{k}\alpha_j=0$ and $\sum_{j=1}^{k}\gamma_j=0$, then the function $K(x)$ defined by (6) will be a doubly periodic function of the second kind with constant multipliers* exp B, exp D, *where $B=\sum_{j=1}^{k}\beta_j$ and $D=\sum_{j=1}^{k}\delta_j$.*

COROLLARY: *In case $B=D=0$, $K(x)$ will be an elliptic function with periods* $(2\omega, 2\omega')$.

4. It is known* that the properties of functions which satisfy (7) are deducible from those of a suitably defined function $G(x)$ which satisfies the simpler periodicity relations

$$(8) \qquad \begin{cases} G(x + \pi) = G(x) \\ G(x + \pi\tau) = e^{-2mix}G(x),\ m \neq 0,\ i = \sqrt{-1}\,, \end{cases}$$

* Appell-Lacour, loc. cit. pp. 388, 396.

where $0<\arg\tau<\pi$, and m is an integer which may be either positive or negative. It can be shown that m is the excess of the number of zeros over the number of poles of the function in a primitive period cell.

If it is supposed that the functions $\phi_j(x)$ are such that each satisfies (8), the entire set possessing a common period pair $(\pi, \pi\tau)$, but not necessarily a common excess m_j, then theorems (A) and (B) yield (E) and (F) respectively.

THEOREM (E): *If the functions* $\phi_j(x)$ *satisfying* (8), *be so chosen that* $\sum_{j=1}^{k}(m_{2j}-m_{2j-1})=0$, *then* $F(\phi_j(x))$, *defined by* (3), *will likewise satisfy* (8) *with* $m=m_1+m_2\cdots+m_{2k}$.

A concrete example of the class of functions from which the $\phi_j(x)$ in the preceding theorem may be selected is the following:

$$\chi_{m_j}(x,\alpha)\equiv\sum_{n=-\infty}^{\infty} q^{n(n-1)m_j}e^{2m_jnix}\operatorname{ctn}(\alpha-x-n\pi\tau), \tag{9}$$

where α is a parameter, $q=\exp\pi i\tau$ and m_j is a positive integer. This function is due to Appell and has played a fundamental role in some of his investigations on doubly periodic functions.*

COROLLARY: *If the* $\phi_j(x)$ *in* (E) *be so selected that, considered as a whole, they possess as many zeros as poles in a primitive period cell, and such that either* $m_1+m_3+\cdots+m_{2k-1}$ *or* $m_2+m_4+\cdots+m_{2k}$ *vanish, then* $F(\phi_j(x))$ *is an elliptic function.*

THEOREM: (F): *The function* $K(x)$ *defined by* (6) *is an elliptic function in case the* $\phi_j(x)$ *satisfy* (8) *and are so selected that, taken as a whole, they possess as many zeros as poles in a primitive period cell.*

5. To conclude we sketch briefly the derivation of some differential identities involving the elliptic theta functions;† these identities are a consequence of Hermite's formula (2). For example, if in this identity we let

$$U=\theta_4(x,q),\quad V=\theta_3(x,q),$$

then it follows from the properties of the theta functions that according as n is odd or even

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\theta_4^{(j)}(x,q)\theta_3^{(n-j)}(x,q)=\begin{cases}A\theta_1(2x,q^2), & (n\text{ odd})\\ B\theta_4(2x,q^2), & (n\text{ even})\end{cases} \tag{10}$$

where A, B, are independent of x. The determination of these constants may be made from a comparison of the coefficients of like trigonometric terms; thus,

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* See also an article by the present writer in Acta Mathematica, vol. 57, pp. 95–100.

† For the definition and properties of the theta functions we refer to Whittaker and Watson's *Modern Analysis*, Chap. XXI.

from a comparison of the coefficients of sin $2x$ (for n odd) and of the constant terms (for n even) it can be shown that

$$
(11)\qquad \begin{cases} A = (-1)^{(n+1)/2}2^n \sum\limits_{(\mu)} (-1)^{(\mu-1)/2}\mu^n q^{\mu^2/2}, \quad (n \text{ odd}) \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mu = \pm 1, \pm 3, \pm 5, \cdots) \\ B = (-1)^{n/2}4^n \sum\limits_{(\nu)} (-1)^\nu \nu^n q^{2\nu^2}, \quad (n \text{ even}) \\ \qquad\qquad\qquad\qquad\qquad\qquad (\nu = 0, \pm 1, \pm 2, \pm 3, \cdots). \end{cases}
$$

If $n=1$, there results, on applying the transformation of the second order (i.e., Landen's transformation) the well known relation:

$$
(12)\qquad \theta_4'(x)\theta_3(x) - \theta_4(x)\theta_3'(x) = \theta_2^2\theta_1(x)\theta_2(x),
$$

which is thus seen to be included as a special case.

There are, altogether, six identities similar to (10), of which the following is another:

$$
(13)\qquad \sum_{j=0}^{n}(-1)^j\binom{n}{j}\theta_4^{(j)}(x,q)\theta_1^{(n-j)}(x,q) = \begin{cases} C\theta_1(x, q^{1/2}), & (n \text{ even}) \\ D\theta_2(x, q^{1/2}), & (n \text{ odd}). \end{cases}
$$

where,

$$
(14)\qquad \begin{cases} C = \tfrac{1}{2}(-1)^{n/2} \sum\limits_{(\mu)} \mu^n q^{\mu^2/8}, \quad (n \text{ even}), \\ D = \tfrac{1}{2}(-1)^{(n-1)/2} \sum\limits_{(\mu)} (-1)^{(\mu-1)/2}\mu^n q^{\mu^2/8}, \quad (n \text{ odd}), \\ \qquad\qquad\qquad\qquad\qquad\qquad (\mu = \pm 1, \pm 3, \pm 5, \pm 7, \cdots) \end{cases}
$$

In the light of the principle of paraphrase,* all identities such as these imply certain arithmetical theorems. Thus, (13) can be shown to be equivalent to the following:

$$
(15)\qquad \begin{aligned} &\sum_{(x,y)} (-1)/(x+y)\left[(x+y)^n F(x-y) + (x-y)^n F(x+y)\right] \\ &\qquad = \begin{cases} \sum\limits_{(x,y)} (-1/x) y^n F(x), & (n \text{ even}) \\ \sum\limits_{(x,y)} (-1/x) y^n F(y), & (n \text{ odd}) \end{cases} \end{aligned}
$$

where $F(x)$ is an arbitrary odd or even function according as the arbitrary positive integer n is even or odd; $(-1/m) \equiv (-1)^{(m-1)/2}$; the sum on the left is extended over all solutions of $\alpha = x^2+y^2$, where α is an arbitrary integer of the form $4k+1$, x is any even integer, and y an odd integer. On the right, the sum ranges over all solutions of $2\alpha = x^2+y^2$ with x, y both odd and $\gtrless 0$.

* E. T. Bell, *Arithmetical Paraphrases*. Trans. A. M. S.. 1921.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

NOTE ON THE DIFFERENTIAL EQUATION $dy+P(x, y)dx=0$

By PETER FIELD, University of Michigan

I. Although the question of integrating the equation

$$dy + P(x, y)dx = 0 \tag{1}$$

has been studied by many writers, including Euler, I do not recall that it has been attempted to see under what circumstances a function of $P(x, y)$ is an integrating factor. As is to be expected it does not lead to new cases but simply to a different point of approach for cases that are known.

II. Assuming an integrating factor $\psi(P)$, equation (1) becomes

$$\psi(P)dy + P\psi(P)dx = 0. \tag{2}$$

The condition for exactness gives

$$\psi'(P)P_x = P\psi'(P)P_y + P_y\psi(P), \text{ or}$$

$$\frac{\psi'(P)}{\psi(P)} = \frac{P_y}{P_x - P P_y}.$$

Hence

$$\frac{\psi'(P)}{\psi(P)} dP = \frac{\psi'(P)}{\psi(P)} (P_x dx + P_y dy) = \frac{P_y}{P_x - P P_y} (P_x dx + P_y dy),$$

and

$$\psi(P) = e^{\int P_y (P_x dx + P_y dy)/(P_x - P P_y)}. \tag{3}$$

The integral in equation (3) is exact if

$$\frac{\partial}{\partial_y} \frac{P_x P_y}{P_x - P P_y} = \frac{\partial}{\partial_x} \frac{P_y^2}{P_x - P P_y} \text{ or}$$

$$P_x^2 P_{yy} - 2P_x P_y P_{xy} + P_y^2 P_{xx} = 0.$$

This is the condition* that the curves $P(x, y)=c$ have zero curvature. Consequently the solution applies only to the two well known cases:

(a) $P(x, y)=c$ are lines through a point,

(b) $P(x, y)=c$ are lines which envelop a curve. Stated as a Theorem: If in the equation $dy+P(x, y)dx=0$, the curves $P(x, y)=c$ are straight lines, then the equation can be made exact by the introduction of the integrating factor

* Scheffers, *Anwendung der Differential- und Integral-Rechnung auf Geometrie*, Bd. 1, s. 89.

$$e^{\int P_y dP/(P_x - PP_y)}.$$

III. As an illustration of (a), consider the equation

$$dy + \frac{x-y}{x+y}dx = 0.$$

Here

$$P(x, y) = \frac{x-y}{x+y}$$

and the integrating factor is

$$e^{\int(-2xydx+2x^2dy)/\{(x^2+y^2)(x+y)\}}.$$

The equation

$$\left(dy + \frac{x-y}{x+y}dx\right)e^{\int(-2xydx+2x^2dy)/\{(x^2+y^2)(x+y)\}} = 0$$

is exact.

As an illustration of (b) consider the equation

$$dy + (\sqrt{y+x^2} - x)dx = 0,$$

where the lines $\sqrt{y+x^2} - x = P(x, y) = c$ are tangents to the parabola $y + x^2 = 0$. The integrating factor is

$$e^{\int[dx/(3\sqrt{y+x^2})+dy/\{(6(x-\sqrt{y+x^2})\sqrt{y+x^2}\}]} = e^{\int dP/(-3P)} = (x - \sqrt{y+x^2})^{-1/3}.$$

After introducing the integrating factor the equation becomes

$$(x - \sqrt{y+x^2})^{-1/3}dy - (x - \sqrt{y+x^2})^{2/3}dx = 0,$$

which is an exact differential.

Of course, it must be admitted that so far as examples of type (b) are concerned, we have simply obtained a complicated method for solving a simple problem. The point of interest, however, is that both types of problem can be made exact by a general type of integrating factor.

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All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Einführung in die Theorie der kontinuierlichen Gruppen. By Dr. Gerhard Kowalewski. Volume IX of Mathematik und ihre Anwendungen. Leipzig, Akademische Verlagsgesellschaft, 1931. x+396 pages.

This is a stimulating presentation by one who was inspired by his close contact with Lie. In his preface the author indicates that the manner of presentation of some of the important topics was determined by his personal recollections and the notes he made at the time when he heard Lie lecture and had almost daily conferences with the great mathematician.

One would hesitate to recommend this book as a first approach to the subject. But to one familiar with the elements, the work will likely make a strong appeal because of the masterly manner of presentation. It is not easy reading, in places. The author does not hesitate to assume a knowledge of other fields of mathematics in order to bring out interesting applications of the Lie theory of continuous groups.

Although there are some 394 pages of text, there are only four chapters, thus indicating the thoroughness with which relatively few topics are treated. The headings of the chapters are:

I. Infinitesimal Transformations and One-Parameter Groups,
II. r-Parameter Groups and Their Infinitesimal Transformations,
III. The Lie Fundamental Theorems,
IV. Transformation Groups on the Straight Line and in the Plane.

We have here a book that seems to fulfill the author's hope, expressed in his preface, that those who have recognized the true significance of the Lie Theory and have the desire to study it further, will find this work a welcome aid.

A. Cohen

Review of Pre-College Mathematics. By C. J. Lapp, F. B. Knight, and H. L. Rietz. Chicago, Scott, Foresman and Company, 1934. 124 pages. $1.00.

This work-book—to quote the preface—provides "explanations of, and drill in, the fundamentals of arithmetic, algebra, geometry, and trigonometry, with the major emphasis on algebra." It is designed "as a remedy . . . for the mathematical weak spots of college freshmen, who come to their first mathematics or science courses . . . most of them . . . with insufficient degrees of competence." It is to be used either as "supplementary material for first year college courses in mathematics, preparation for college courses in physics or chemistry," or "general review for high school seniors." Its "content was determined by first-hand experience with college freshmen, actual analysis of the requirements of first courses in college, and research on frequency of errors among freshmen." "The book is the joint product of a professor of mathematics, a professor of physics, and a professor of educational psychology. Each has contributed his own point of view and his own experience to the construction of it."

The names of the authors are a sufficient guarantee that the book will adequately achieve its purpose. Examination of the text itself justifies this expectation. No attempt is made to raise the student to a higher viewpoint; that is the province of the college course. But the high school mathematics in which our students are frequently found to be weak is thoroughly reviewed. There are

some 24 pages of arithmetic and 60 pages of algebra, though the demarcation is not sharp. The geometry consists of a few pages of mensuration and simple problems of construction, and there is a brief summary of "some aspects of trigonometry," dealing only with the acute angle and right triangles. Logarithms are included, with a four-place table.

The book is, as described, a work-book. The page is large ($8\frac{1}{4}\times 11\frac{1}{4}$) and space is provided on the page for the solution of most of the problems. An interesting feature is a glossary of mathematical terms.

One finds little to criticize in the book. Perhaps the precise might not approve the bald and unqualified statement that $\pi=3.1416$; but is it not true that in one or more of the states of the Mississippi Valley this value has been ordained by act of the legislature? One looks in vain for any reference to the question of significant figures in arithmetical work; in the multiplication of decimals, all figures are kept in the answers. True, this question is never raised in the schools, but it would seem appropriate to introduce it here, especially in connection with problems of mensuration. It seems a little absurd to give the volume of a 13-inch sphere as 1150.349 cubic inches.

This is a matter, however, which each instructor will wish to present in his own way, and its omission is not a serious fault. The book will be welcomed by the college instructor of freshmen who wishes to give his classes additional drill on the basic high-school work which he needs to use in his college courses.

R. A. JOHNSON

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

LOCAL MATHEMATICS CLUBS

The Mathematics Club of Wayne University

The officers for the year 1933–1934 were: Margaret Dunford, President; Don D. Miller, Vice President; Ellen MacPetrie, Secretary.

The meetings and programs were as follows:

October 24, 1933: "Approximations and significant figures" by Mr. W. Borgman.
November 21, 1933: "Probability" by Beatrice Rosenfeld.
January 16, 1934: "Indeterminate equations" by Ellen MacPetrie.
February 20, 1934: "A method of summation" by W. Gordon Scott.
March 20, 1934: "Magic squares" by Melanya Pawlick.

April 10, 1934: "Line representations of hyperbolic functions" by C. A. Kedzierski.
May 15, 1934: "The Rhind Papyrus" by Associate Professor Worden.

ELLEN MACPETRIE, *Secretary*

The Mathematics Club of William and Mary

During the academic year of 1933–1934, our mathematics club, which is called the "Euclid" club, had seven meetings. The purpose of the club is to further the interest of mathematics on the campus. We had forty active members. The requirements for membership in the club are: A person must be either a major or minor in the field of mathematics and must have completed at least nine semester hours of work in this subject with an average grade of 85% and a grade of 91% or above in at least one course in mathematics.

The meetings and programs were as follows:

October 20, 1933: "Euclid's life and geometry" by Sarah Pope.
November 17, 1933: "Magic squares" by Marianne Norris.
December 15, 1933: "Mathematics in the building of a home" by Col. Earl C. Popp; "Projective geometry" by Alexander Haughout.
February 16, 1934: "Cubic equations as solved by Tartaglia" by Bruce M. Kent.
March 16, 1934: "Mathematics in harmony" by Edna Lemster.

In addition to the above meetings, we had the following social activities:

September 26, 1933: The club was entertained at a picnic at Jamestown by the members of the mathematics department.
April 26, 1934: The club held the annual Spring banquet.

BRUCE M. KENT, *Secretary*

The Mathematics Club of the University of Virginia

The Echols mathematics club has for its purpose, as the constitution states, "to promote better fellowship among its members and to foster a wider interest in mathematics at the University of Virginia."

The officers for the academic year 1933–1934 were: M. W. Aylor, President; H. W. Eves, Vice President; Geo C. Watson, Treasurer.

The meetings and programs were as follows:

October 5, 1933: "The foundations of the mathematical sciences" by Professor W. H. Echols.
October 26, 1933: "Hamilton's analogy between mechanics and optics" by Professor C. M. Sparrow.
November 9, 1933: "Some points related to the triangle" by A. D. Wallace.
November 24, 1933: "Some problems in finite differences" by Professor E. J. Oglesby.
January 25, 1934: "Hyperbolic functions and their representations by lines" by Professor B. Z. Linfield.
February 8, 1934: "Note on the curvatures of corresponding curves in conformal mapping" by Professor W. H. Echols.
March 1, 1934: "The soluble case of the problem of Newton" by F. V. Reno.
April 11, 1934: "A study of the conic defined by its line equation" by W. T. Puckett, Jr.
May 2, 1934: "The isologue of the centroid under the isogonal conjugate transformation" by Geo. C. Watson. At this meeting we elected our officers for the academic year 1934–1935.
May 17, 1934: "The Lie group of one parameter" by Professor J. J. Luck.

Refreshments were served following each meeting.

GEORGE C. WATSON, *Secretary*

The Mathematics Club of New Jersey State Teachers College

Sigma Phi Mu, the mathematics club of our institution, holds its meetings bi-monthly. All mathematics students who acquit themselves with distinction in their mathematics course are eligible for membership.

The officers for the academic year 1933–1934 were: Edna Hitchcock, President; Natalie Dalton, Vice President; Ida Krug, Treasurer; Gertrude Winchell, Secretary; Marge Jansen, Librarian; Professor V. S. Mallory, Faculty Adviser.

The meetings and programs were as follows:

September 23, 1933: "Preparation of mathematics text books" by Professor Stone.
October 7, 1933: Chicken chowder.
October 23, 1933: "Mathematical recreations" by Dorothy Schneider.
November 8, 1933: "History of verbal problems in algebra" by Miss Pratt.
December 13, 1933: Social meeting.
January 10, 1934: This meeting was given in honor of Professor Stone. The "Eternal Triangle" was presented.
January 29, 1934: "Magic squares" by Herbert Freed.
February 26, 1934: "Functional relationships with special reference to the use of models as visual aids in the presentation of these relationships" by Ferdinand Kertes.
March 14, 1934: "Philosophy of teaching" by Professor Mallory.
March 26, 1934: "Philosophy of mathematics" by Dorothy Schmitt.
April 11, 1934: "Linkages" by Mr. Leuder.
May 9, 1934: "Dramatization of Stephen Leacock's *A. B. C.*"
May 21, 1934: "Conic sections" by Dr. D. R. Davis.
June 11, 1934: Supper hike.

GERTRUDE WINCHELL, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr.,Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond these ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 154. *Proposed by V. Thébault, Le Mans, France.*

Find the smallest positive integer, not beginning with zero, such that if it is written down twice in succession so as to form a number of twice as many digits, that number will be a perfect square.

E 155. *Proposed by Maud Willey, Gulfport, Mississippi.*

Prove that

$$\sum_{i=0}^{n}\left[{}_nC_i \sum_{j=0}^{i} {}_iC_j \right] = 3^n.$$

E 156. *Proposed by J. A. Bullard, University of Vermont.*

If a parabola is inscribed tangent to two sides of an equilateral triangle with the third side forming the chord of contact, then the focus of the parabola lies at the centroid and the latus rectum is equal to the radius of the circumcircle. If three such parabolas are inscribed, what is the ratio of the area inside all three parabolas to the area of the triangle?

E 157. *Proposed by Raymond Garver, University of California at Los Angeles.* Prove that

$$2\sqrt{7}\cos\left[\tfrac{1}{3}\cos^{-1}(1/2\sqrt{7})\right] - 6\cos(2\pi/7) = 1.$$

E 158. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Several years ago today John Smith borrowed some money from his bank at a normal rate of simple interest and then vanished without paying anything on his debt. Today he suddenly reappeared at the bank and paid off his accumulated indebtedness, which amounted to precisely $204.13. How much did he borrow, at what rate did he borrow it, and how long did he keep it?

SOLUTIONS

E 125 [1934, 629] *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the vertex A and the points of contact of BC produced with each of the escribed circles corresponding to sides AC and AB respectively.

Solution by Wm. Douglas, Courtenay, B. C.

If D is the foot of the altitude from A on BC, and E and F are the given points of contact on BC, construct circles tangent to BC at E and at F, on the same side of BC as A, which are also tangent to DA produced. Their radii are ED and DF respectively. Let the line joining their centers, O_1 and O_2, intersect BC produced at G. Let GA produced meet EO_1 at P, and FO_2 at Q. Draw circles centered at P and Q, tangent to BC at E and F respectively. Their radii are proportional to ED and DF, and A lies on their line of centers. Therefore their common internal tangents intersect at A, and hence form the missing sides of the desired triangle, ABC.

Also solved by W. B. Clarke, W. M. Jackson, L. M. Kelly, Roy MacKay, Leon Recht, M. J. Turner, and Simon Vatriquant.

E 126 [1934, 629] *Proposed by M. C. Holmes, West Virginia University.*

A series of events, $e_1, e_2 \cdots e_n$, are given, with constant probabilities $p_1, p_2, \cdots p_n$ respectively of happening in any given trial. Show that in a series of repeated trials the chance that event e_i will happen before any of the others is given by

$$u_i = p_i/(p_1 + p_2 + \cdots + p_n)$$

if the events are mutually exclusive, and by

$$v_i = \frac{p_i(1-p_1)(1-p_2)\cdots(1-p_{i-1})(1-p_{i+1})\cdots(1-p_n)}{1-(1-p_1)(1-p_2)\cdots(1-p_n)}$$

E 128 [1934, 629]. *Proposed by J. A. van Groos, Oregon State College.*

If PQ is a chord of the parabola, $y^2=ax$, and the ordinates of P and Q are p and q respectively, with $p<q$, show that the area of the segment cut from the parabola by PQ is given by $(q-p)^3/6a$.

Solution by Roy MacKay, Eastern New Mexico Junior College.

The coordinates of P and Q are $(p^2/a, p)$ and $(q^2/a, q)$, and the equation of the chord PQ is $x_1=(p+q)y/a-pq/a$. The equation of the parabola may be written $x_2=y^2/a$. Then the area between the curve and the chord is

$$\int_p^q (x_1 - x_2)dy = \int_p^q [(p + q)y/a - pq/a - y^2/a]dy = (q - p)^3/6a.$$

Also solved by L. J. Adams, W. E. Buker, David Colbert, E. W. Franz, Abe Gelbart, Hansraj Gupta, J. A. Hurry, L. M. Kelly, F. L. Manning, H. L. Quarles, C. W. Trigg, Simon Vatriquant, Wm. Wernick, E. N. Yeager, and the proposer.

E 129 [1934, 629]. *Proposed by Leon Battig, University of Wisconsin.*

In the parallelogram, $ABCD$, points E and F are in sides AB and CD respectively. AF intersects ED in G. EC intersects FB in H. GH produced intersects AD in L and BC in M. Prove by high school geometry that $DL=BM$.

Solution by L. M. Kelly, Northeastern University, Boston, Massachusetts.

Through E and F draw two lines parallel to BC and meeting GH in J and K respectively. Then $EJ/LD=EG/GD=AE/DF$. Also $MC/EJ=HC/HE=FC/EB$. Therefore $MC/LD=(FC\cdot AE)/(EB\cdot DF)$.

Again, $KF/BM=FH/HB=FC/EB$, and $AL/KF=AG/GF=AE/DF$, so that $AL/BM=(AE\cdot FC)/(EB\cdot DF)$.

Finally, it follows that $MC/LD=AL/BM$, so $(BC-BM)/LD=(BC-LD)/BM$, and, subtracting 1 from both sides, $(BC-BM-LD)/LD=(BC-LD-BM)/BM$. Whence, cancelling $(BC-LD-BM)$ and inverting, we see that $LD=BM$. Note that in case the cancelled factor, $(BC-LD-BM)$, were zero, then LM is parallel to AB and CD and the proof is still more elementary.

Editorial Note. In his solution, W. E. Buker points out that this problem appeared in the problem department of *School Science and Mathematics*, with solutions published in November 1931 and October 1932.

Also solved by W. B. Clarke, David Colbert, J. W. Davis, Roy MacKay, Simon Vatriquant, and E. N. Yeager.

E 130 [1934, 630]. *Proposed by Wm. F. Cheney, Jr., Connecticut State College.*

In the following two sums, each different letter represents a different digit. Identify them and show that the solution is unique.

$$USA + FDR = NRA, \qquad USA + NRA = TAX.$$

Solution by H. L. Quarles, University of Mississippi.

It is immediately obvious that $R=0$, $5<A$, $S+1=A$, $S+D=10$ and $D<S$. Each letter must be less than ten. The possible values for A are then 6, 7, 8 and 9. We must exclude 6 as that would make $S=D=5$. We must exclude 7 as that would make $D=X=4$. We must exclude 9 as that makes $S=X=8$. Consequently $A=8$. From this it follows immediately that $S=7$, $D=3$ and $X=6$.

This leaves T, N, F and U to be determined, and 1, 2, 4, 5 and 9 as the only values possible for them to take. From the problem it appears that $U<N$, and that $F<N<T$. Hence T is the only letter that could be 9. Now if $T=9$, the second addition requires that $U=4$ and $N=5$, which will not fit in the first addition if $F=1$ or 2. Hence $T=5$, $N=4$, $U=1$ and $F=2$ is the unique solution. The problem then reads

$$178 + 230 = 408, \qquad 178 + 408 = 586.$$

Editorial Note. In his solution, Mr. C. W. Foard points out that from the given equations, USA is easily eliminated by subtraction, leaving the equation

$$NRA = \tfrac{1}{2}FDR + \tfrac{1}{2}TAX,$$

which may or may not have a bearing upon the problem confronting us.

Also solved by L. J. Adams, Annabel S. Boyce, W. E. Buker, David Colbert, M. L. Constable, Wm. Douglas, Churchill Eisenhart, Robert Elias, Daniel Finkel, Marvin Flake, E. W. Franz, Hansraj Gupta, E. L. Harp, E. D. Hartsook, Elizabeth Hayes, R. A. Johnson, Sidney Kaplan, H. R. Leifer, Morris Lieblich, Roy MacKay, F. L. Manning, Gertrude I. McCain, C. T. Oergel, W. R. Ransom, Herbert Spiro, Cornelia Strong, C. W. Trigg, M. J. Turner, Simon Vatriquant, Wm. Wernick, C. R. Worth, and B. C. Zimmerman.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3735. *Proposed by B. P. Gill, College of the City of New York.*

Solve in real, non-zero integers: $(x^2+y^2+z^2)(x+y+z)+xyz=0$. An equiv-

alent statement is: Can the equation $t^5+at+b=0$, $b\neq 0$, have three real integers t for roots?

3736. *Proposed by J. M. Feld, New York City.*

Prove that the two following sets of 12 points in three-dimensional projective space are projective with one another:

(a) The 8 vertices and the center of a cube and the three points at infinity in the direction of the edges;

(b) The 6 vertices of a regular (right) triangular prism, the three middle points of the lateral faces of the prism, and the three points at infinity in the directions of the edges of the bases.

Determine the number of different one-to-one projective correspondences that can be established between the two sets of 12 points.

3737. *Proposed by J. P. Ballantine, University of Washington.*

Derive the following formulas:

(a) $$\pi = \frac{10}{3} - 24\left\{\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{1}{5\cdot 6\cdot 7\cdot 8\cdot 9} - \cdots\right\},$$

(b) $$\pi = 3.15 - 360\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10} + \cdots\right\},$$

(c) $$\log 2 = \frac{17}{24} - 12\left\{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \frac{1}{4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots\right\},$$

(d) $$\pi = \frac{64}{21} + 96\left\{\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9} - \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \cdots\right\}.$$

3738. *Proposed by V. Thébault, Le Mans, France.*

Consider a convex quadrilateral $ABCD$ inscribed in a circle and the circles (α), (β), (γ), (δ) described on the sides AB, BC, CD, DA as diameters. Show that the diagonal AC is to the diagonal BD as the product of the common tangents to circles (α) and (β), (γ) and (δ) is to that of the tangents to circles (β) and (γ), (δ) and (α).

SOLUTIONS

3661 [1934, 50]. *Proposed by Maud Willey, Long Beach, Miss.*

What is the order of the group of movements into itself, in space of n dimensions, of the regular n-dimensional solid whose 2^n vertices have the coordinates $(\pm 1, \pm 1, \cdots, \pm 1)$?

Solution by Jeanette Fox, New Haven, Conn.

If we consider the coordinate axes $x_1, x_2, \cdots, x_n$ as being rigidly fixed in the

a continuous variation of the coefficients a_{ij} so that we have continuous motion from the initial to the final position. This suffices to make $D=1$; and it then follows that a_{ij} must equal its cofactor in D. We easily prove then that the summations in (1) with respect to j must be true. Thus the equations (1) define an infinite continuous group of motions which carries the configuration

$$\sum_{i=1}^{n} x_i^2 = r^2$$

into itself. Such a figure has therefore perfect symmetry and we may call it a sphere. Our group G is one of its finite subgroups. In the solution by the proposer, instead of the general conditions, a set of generating elements was used. Thus, for example, the equations

$$x_1' = x_1 \cos \phi - x_k \sin \phi,$$
$$x_k' = x_1 \sin \phi + x_k \cos \phi,$$
$$x_l' = x_l, \qquad l \neq 1,\ l \neq k,$$

are possible motions since $D=1$; and, if we set $\phi=\pi/2$ or $\phi=\pi$, we have motions which take the figure into itself. This was used by the proposer as a basis for counting the number of elements of G. It was also shown by the proposer that in space of $n+1$ dimensions the order of the group is $2^n n!$.

3662 [1934, 112]. *Proposed by W. B. Campbell, Rangoon, Burma.*

Within what region must the point P lie, in order that it be possible to draw four real normals from it to the ellipse $x=a \cos \phi$, $y=b \sin \phi$, $a>b$?

Solution by Dwight F. Gunder, Colorado Agricultural College.

The condition that the line joining the point $P(\alpha, \beta)$ to any point $(a \cos \phi, b \sin \phi)$ of the ellipse be a normal is,

$$\frac{\beta - b \sin \phi}{\alpha - a \cos \phi} = \frac{a \text{ s n } \phi}{b \cos \phi},$$

which gives at once the equation,

$$(1) \qquad (c^2 \sin \phi + b\beta) \cos \phi = a\alpha \sin \phi.$$

If this equation is expressed in terms of $\sin \phi$ only there results the quartic,

$$(2) \quad c^4 \sin^4 \phi + 2b\beta c^2 \sin^3 \phi + [a^2\alpha^2 + b^2\beta^2 - c^4] \sin^2 \phi - 2c^2 b\beta \sin \phi - b^2\beta^2 = 0,$$

where $c^2=a^2-b^2$. Since clearly there are always at least two real normals to an ellipse from P, [i.e., two real values of $\sin \phi$ satisfying (2)], the necessary and

sufficient condition that there be four real solutions of (2) is that its discriminant be positive. This condition is found at once to be,

$$(a^2\alpha^2 + b^2\beta^2 - c^4)^3 + 27a^2b^2\alpha^2\beta^2c^4 < 0. \tag{3}$$

If we replace the inequality sign by one of equality, the equation of the curve bounding the desired region is obtained. This equation is seen to be exactly the rationalized form of the equation of the evolute of the ellipse, i.e., of

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = c^{4/3}.$$

The area included by the evolute is thus seen to be the required one. On the evolute itself the normals are not all distinct but are all real; thus the solution is the closed region just defined.

Solved also by J. H. Butchart, L. Richardson, E. P. Starke, C. W. Trigg, and the proposer.

Editorial Note. A reference was given by Rothwell Stephens to a solution in Briot and Bouquet's *Elements of Analytical Geometry*, page 313, Boyd's translation. In this solution the condition (3) above, etc., was found from the reducing cubic corresponding to the biquadratic equation (2)

$$4a^2b^2\lambda^3 + (a^2\alpha^2 + b^2\beta^2 - c^4)\lambda + c^2\alpha\beta = 0.$$

This latter equation was obtained by a consideration of the three pairs of common chords of the given ellipse and the equilateral hyperbola

$$c^2xy + b^2\beta x - a^2\alpha y = 0,$$

corresponding to (1).

Morgan Ward furnished the information that the number and location of the normals is discussed by Bliss in his monograph *Calculus of Variations*, pp. 35–37.

The proposer reduced the problem to the consideration of the intersections of

$$y = \sin 2\phi, \qquad y = B \sin(\phi - A),$$
$$a\alpha \tan A = b\beta, \qquad c^2B = 2(a^2\alpha^2 + b^2\beta^2)^{1/2};$$

and, in the discussion of the results, found the equations of the evolute in terms of the parameter ϕ.

A method of discussion similar to that of the proposer may be obtained by setting $\sin\phi = y$, $\cos\phi = x$, in the above solution, and by examining the intersections of

$$x^2 + y^2 = 1, \qquad x(c^2y + b\beta) - a\alpha y = 0.$$

We consider the normal, or shortest distance, ON, from the origin O on one branch of the equilateral hyperbola to the other branch. A simple calculation gives

$$c^4(ON)^2 = [(a\alpha)^{2/3} + (b\beta)^{2/3}]^3.$$

The circle always cuts the branch through O in two real distinct points, and it cuts also the other branch in two real distinct points, if ON is less than unity; two coincident points, if ON is unity; two imaginary points, if ON is more than unity.

3664 [1934, 112]. *Proposed by Otto Dunkel, Washington University.*

The three independent variables u, v, w are the parameters of three systems of surfaces such that through each point P there passes one and only one surface of each system, and the three surfaces cut orthogonally at P. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be the unit vector tangents at P to the three curves u, v, w which are determined at that point, and such that these three vectors form a right handed rectangular system. Also let $ds = h_1 du$ be the differential of arc at P along the curve u, and let h_2 and h_3 have similar definitions. Prove the relations

$$\frac{\partial \boldsymbol{a}}{\partial u} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v}\boldsymbol{b} - \frac{1}{h_3}\frac{\partial h_1}{\partial w}\boldsymbol{c},$$

$$\frac{\partial \boldsymbol{a}}{\partial v} = \frac{1}{h_1}\frac{\partial h_2}{\partial u}\boldsymbol{b},$$

$$\frac{\partial \boldsymbol{a}}{\partial w} = \frac{1}{h_1}\frac{\partial h_3}{\partial u}\boldsymbol{c},$$

with two other sets obtained by cyclic interchanges; and show also that on each surface, say w = constant, the parametric curves u = constant and v = constant are lines of curvature.

See Weatherburn, *Advanced Vector Analysis*, p. 12, where equations are given which easily result from the above, but which are not so convenient for obtaining the curl of a vector in curvilinear coordinates, p. 16, formula (28).

Solution by A. S. Householder, Washburn College.

Let $\boldsymbol{r} = \boldsymbol{r}(u, v, w)$ be the vector to the point determined by u, v, w; then

$$\text{(1)} \qquad \boldsymbol{r}_u = h_1\boldsymbol{a}, \qquad \boldsymbol{r}_v = h_2\boldsymbol{b}, \qquad \boldsymbol{r}_w = h_3\boldsymbol{c};$$

$$\boldsymbol{r}_{uu} = \frac{\partial h_1}{\partial u}\boldsymbol{a} + h_1\frac{\partial \boldsymbol{a}}{\partial u},$$

$$\text{(2)} \qquad \boldsymbol{r}_{uv} = \frac{\partial h_1}{\partial v}\boldsymbol{a} + h_1\frac{\partial \boldsymbol{a}}{\partial v},$$

$$\boldsymbol{r}_{uw} = \frac{\partial h_1}{\partial w}\boldsymbol{a} + h_1\frac{\partial \boldsymbol{a}}{\partial w}.$$

We can express each second derivative of $\boldsymbol{r}$ as a linear combination of the three linearly independent first derivatives of $\boldsymbol{r}$;

$$
\begin{aligned}
\mathbf{r}_{uu} &= A_1\mathbf{r}_u + B_1\mathbf{r}_v + C_1\mathbf{r}_w,\\
\mathbf{r}_{uv} &= A_2\mathbf{r}_u + B_2\mathbf{r}_v + C_2\mathbf{r}_w,\\
\mathbf{r}_{uw} &= A_3\mathbf{r}_u + B_3\mathbf{r}_v + C_3\mathbf{r}_w,
\end{aligned}
\tag{3}
$$

Since the first derivative vectors on the right are mutually perpendicular, the coefficients are equal, except for factors h_i, each to a scalar product of a second derivative vector and a first derivative vector. These can be calculated by differentiating the identities

$$\mathbf{r}_u\cdot\mathbf{r}_u = h_1^2, \qquad \mathbf{r}_v\cdot\mathbf{r}_v = h_2^2, \qquad \mathbf{r}_w\cdot\mathbf{r}_w = h_3^2; \tag{4}$$

$$\mathbf{r}_v\cdot\mathbf{r}_w = \mathbf{r}_w\cdot\mathbf{r}_u = \mathbf{r}_u\cdot\mathbf{r}_v = 0. \tag{5}$$

From the last three we have

$$\mathbf{r}_{uv}\cdot\mathbf{r}_w + \mathbf{r}_v\cdot\mathbf{r}_{wu} = \mathbf{r}_{vw}\cdot\mathbf{r}_u + \mathbf{r}_w\cdot\mathbf{r}_{uv} = \mathbf{r}_{wu}\cdot\mathbf{r}_v + \mathbf{r}_u\cdot\mathbf{r}_{vw} = 0; \tag{6}$$

whence it follows that

$$0 = \mathbf{r}_{uv}\cdot\mathbf{r}_w = \mathbf{r}_{wu}\cdot\mathbf{r}_v = C_2 = B_3.$$

From the above results we find that

$$
\begin{aligned}
\mathbf{r}_u\cdot\mathbf{r}_{uu} &= A_1h_1^2 = h_1\frac{\partial h_1}{\partial u}, &\qquad \mathbf{r}_u\cdot\mathbf{r}_{uv} &= A_2h_1^2 = h_1\frac{\partial h_1}{\partial v},\\
\mathbf{r}_u\cdot\mathbf{r}_{uw} &= A_3h_1^2 = h_1\frac{\partial h_1}{\partial w}, &\qquad \mathbf{r}_v\cdot\mathbf{r}_{uv} &= B_2h_2^2 = h_2\frac{\partial h_2}{\partial u},\\
\mathbf{r}_w\cdot\mathbf{r}_{uw} &= C_3h_3^2 = h_3\frac{\partial h_3}{\partial u}.
\end{aligned}
$$

Then the identities $\mathbf{r}_{uu}\cdot\mathbf{r}_v+\mathbf{r}_u\cdot\mathbf{r}_{uv}=0$, $\mathbf{r}_{uu}\cdot\mathbf{r}_w+\mathbf{r}_u\cdot\mathbf{r}_{uw}=0$ give

$$B_1h_2^2 = -A_2h_1^2 = -h_1\frac{\partial h_1}{\partial v}, \qquad C_1h_3^2 = -A_3h_1^2 = -h_1\frac{\partial h_1}{\partial w}.$$

Substituting these values in (3) we obtain the second derivatives of $\mathbf{r}$ in terms of the first derivatives of $\mathbf{r}$; and then using (1) we have them expressed in terms of the unit tangential vectors:

$$
\begin{aligned}
\mathbf{r}_{uu} &= \frac{\partial h_1}{\partial u}\mathbf{a} - \frac{h_1}{h_2}\frac{\partial h_1}{\partial v}\mathbf{b} - \frac{h_1}{h_3}\frac{\partial h_1}{\partial w}\mathbf{c},\\
\mathbf{r}_{uv} &= \frac{\partial h_1}{\partial v}\mathbf{a} + \frac{\partial h_2}{\partial u}\mathbf{b},\\
\mathbf{r}_{uw} &= \frac{\partial h_1}{\partial w}\mathbf{a} + \frac{\partial h_3}{\partial u}\mathbf{c}.
\end{aligned}
\tag{7}
$$

The desired results follow now immediately from (2) and (7).

For the surface $w=$constant, the second coefficients of the two fundamental quadratic differential forms are equal to $\boldsymbol{r}_u\cdot\boldsymbol{r}_v$ and to $\boldsymbol{r}_{uv}\cdot\boldsymbol{c}$, respectively. Since both are zero the parametric curves are lines of curvature. This is a known theorem of Dupin.

Editorial Note. A proof with less computation is as follows: We may assume that the parameters u, v, w have been chosen so that $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, $\boldsymbol{r}_w$ form a right hand system of orthogonal vectors, and in this case the h's are all positive. We begin with $\boldsymbol{r}_v\cdot\boldsymbol{r}_w=\boldsymbol{r}_w\cdot\boldsymbol{r}_u=\boldsymbol{r}_u\cdot\boldsymbol{r}_v=0$, and we find, as in the above solution, that

$$(1)\qquad \boldsymbol{r}_{uv}\cdot\boldsymbol{r}_w = \boldsymbol{r}_{vw}\cdot\boldsymbol{r}_u = \boldsymbol{r}_{wu}\cdot\boldsymbol{r}_v = 0.$$

This shows at once that, for the surface $w=$constant, the consecutive normals along the curve $v=$constant ultimately intersect. Thus the parametric curves on this surface are lines of curvature. Then from $\boldsymbol{r}_u=h_1\boldsymbol{a}$, $\boldsymbol{r}_v=h_2\boldsymbol{b}$, we have

$$(2)\qquad \boldsymbol{r}_{uv} = \frac{\partial h_1}{\partial v}\boldsymbol{a} + h_1\frac{\partial \boldsymbol{a}}{\partial v} = \frac{\partial h_2}{\partial u}\boldsymbol{b} + h_2\frac{\partial \boldsymbol{b}}{\partial u},$$

$$(3)\qquad h_1\frac{\partial \boldsymbol{a}}{\partial v} - \frac{\partial h_2}{\partial u}\boldsymbol{b} = h_2\frac{\partial \boldsymbol{b}}{\partial u} - \frac{\partial h_1}{\partial v}\boldsymbol{a}.$$

If we denote by $\boldsymbol{d}$ the vector in (3) given in two forms, then $\boldsymbol{d}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. For, from $\boldsymbol{a}^2=\boldsymbol{b}^2=1$, we have $\boldsymbol{a}\cdot\partial\boldsymbol{a}/\partial v=\boldsymbol{b}\cdot\partial\boldsymbol{b}/\partial u=0$. Also from (1) we have $\boldsymbol{r}_{uv}\cdot\boldsymbol{c}=0$, and then (2) gives $\boldsymbol{c}\cdot\partial\boldsymbol{a}/\partial v=\boldsymbol{c}\cdot\partial\boldsymbol{b}/\partial u=0$. It now follows from (3) that $\boldsymbol{d}$ is also perpendicular to $\boldsymbol{c}$: hence $\boldsymbol{d}=0$. By a cyclic interchange of the letters we have the proof of the second and third equations of the problem.

Since $\boldsymbol{a}\cdot\partial\boldsymbol{a}/\partial u=0$, we may write

$$\frac{\partial \boldsymbol{a}}{\partial u} = \left(\boldsymbol{b}\cdot\frac{\partial \boldsymbol{a}}{\partial u}\right)\boldsymbol{b} + \left(\boldsymbol{c}\cdot\frac{\partial \boldsymbol{a}}{\partial u}\right)\boldsymbol{c}.$$

From $\boldsymbol{b}\cdot\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{a}=0$ we have

$$\boldsymbol{b}\cdot\frac{\partial \boldsymbol{a}}{\partial u} = -\boldsymbol{a}\cdot\frac{\partial \boldsymbol{b}}{\partial u} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v},$$

$$\boldsymbol{c}\cdot\frac{\partial \boldsymbol{a}}{\partial u} = -\boldsymbol{a}\cdot\frac{\partial \boldsymbol{c}}{\partial u} = -\frac{1}{h_3}\frac{\partial h_1}{\partial w},$$

where the second and third results of the problem, or similar results by cyclic interchange, have been used. Hence

$$\frac{\partial \boldsymbol{a}}{\partial u} = -\frac{1}{h_2}\frac{\partial h_1}{\partial v}\boldsymbol{b} - \frac{1}{h_3}\frac{\partial h_1}{\partial w}\boldsymbol{c},$$

and the proof is complete. The proof depends very largely on the assumption that the order of partial differentiation may be reversed, but this assumption is legitimate in such considerations.

F. D. Murnaghan writes that the formulae of the problem are a direct consequence of formulae (6.29) p. 33 and (6.18) p. 31 of the text *Theoretical Mechanics* by Ames and Murnaghan (Ginn and Company, 1929). Also that the parametric curves are lines of curvature is an immediate consequence of the fact that e.g., $\partial c/\partial u$ has the same direction as $\boldsymbol{a}$.

It has occurred to the proposer after the appearance of this problem in print that the desired formulae may be obtained from almost any text on differential geometry; and it appears just to refer to Weatherburn's *Differential Geometry*, Vol. I. These formulae are a direct consequence of Gauss's formulae, which are equivalent to formulae for the second derivatives of $\boldsymbol{r}$, in the more serviceable form (B) on page 91. However, the substitutions for reduction to the formulae of the problem may be more tedious than the above proof. Also on page 213 are given in (20) and (21) results which are equivalent to those in the solution by Householder leading to his (7). Finally, the results at the bottom of page 213 and at the top of 214 are the formulae of the problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

In March 1935, the Division of Applied Mechanics of the American Society of Mechanical Engineers began publication of a quarterly Journal of Applied Mechanics. The editorial board includes J. P. Den Hartog, R. Eksergian, J. Goff, J. C. Hunsaker, G. B. Karelitz, T. von Kármán, A. I. Lipetz, A. Nadia, J. Ormandrougd, R. E. Peterson, S. Timoshenko. The technical editor will be J. M. Lessells, of Swarthmore, Pa.

An informal meeting of persons interested in the advancement of logical studies, and particularly in the possibility of providing greater facilities for the publication of papers in symbolic or mathematical logic, was held on December 27th at New York University during the annual meeting of the Eastern Division of the American Philosophical Assocition. The sense of the meeting was that an Association should be organized for the advancement of logical studies through the establishment of a Journal, and possibly also by means of meetings occasionally held in conjunction with those of the AAAS or the American Philosophical Association; and that a committee of organization be appointed by the chairman. The committee consists of the following persons: C. A. Baylis, Philosophy, Brown; A. Church, Mathematics, Princeton; M. R. Cohen, Philosophy, C.C.N.Y.; H. B. Curry, Mathematics, Pennsylvania State College;

C. J. Ducasse, Philosophy, Brown; S. K. Langer, Philosophy, Radcliffe; J. B. Rosser, Mathematics, Princeton; H. B. Smith, Philosophy, University of Pennsylvania; P. Weiss, Philosophy, Bryn Mawr.

Since more definite plans for the proposed Journal must depend largely on the funds that would be available, all persons (whether or not themselves working in the field of logic) who would favor the launching of such a journal, and who would be willing to lend it their support by joining the proposed association (with dues of probably $3.00 a year) are asked to send their names to Professor C. J. Ducasse, Brown University.

The Commission on Examinations in Mathematics, appointed by the College Entrance Examination Board in April 1933, has just submitted its report. The membership of the Commission included the following: Professor Arnold Dresden, Swarthmore College, *Chairman*; Miss Grace S. Barker, Baldwin School, Bryn Mawr, Pa.; Professor Ralph Beatley, Harvard University; William Betz, East High School, Rochester, N. Y.; Professor William R. Longley, Yale University; Gordon R. Mirick, Lincoln School, New York, N. Y.; Rolland R. Smith, Classical High School, Springfield, Mass.; Professor Anna Pell Wheeler, Bryn Mawr College; Professor Norbert Wiener, Massachusetts Institute of Technology.

The report appears in full in the March number of the Mathematics Teacher. In order to expedite action by the College Entrance Examination Board and to permit the holding of the examinations proposed by the Commission at the earliest possible date, Professor Thomas S. Fiske, Secretary of the Board, desires to give all possible publicity to this report and will welcome suggestions and criticisms from those interested. Requests for copies of the report should be addressed to the Secretary of the College Entrance Examination Board, 431 West 117th Street, New York City. The charge for single copies is ten cents each. For fifty copies or more, ordered at the same time, the charge is five cents each.

A greatly enlarged program of teaching and research in mathematical statistics is being undertaken at Columbia University this year. Research under the auspices of the Carnegie Corporation is being conducted with Professor Harold Hotelling as director, with a view to clarifying the foundations of statistical methods and extending their scope, and particularly in the development of tests of significance and criteria of accurate estimation. For this work Dr. J. S. Doob has been appointed research associate, and Margaret H. Richards and W. C. Madow research assistants. Professor Felix Bernstein, founder and formerly head of the Göttingen Institute of Mathematical Statistics, is at Columbia University this year as visiting professor of mathematics. A course of training in mathematical statistics has been arranged by the coordination of courses in the departments of mathematics, economics, and astronomy; it includes *Probability*, by B. O. Koopman; *Statistical Inference* and *Mathematical Economics*, by Harold Hotelling; *Mathematics of Heredity and Evolution*, by Felix Bern-

stein; and training in the use of card tabulating and calculating machines, interpolation, and finite differences, by W. J. Eckert.

The following appointments to instructorships in mathematics are announced:

Brown University: Max Astrachan;

Vassar College: Frances E. Baker, during the sabbatical leave of Professor Cummings.

West Virginia University: Dr. R. H. Downing;

Sister Marie Cecelia Mangold, professor of mathematics at Trinity College, Washington, D. C., died February 9, 1934. She was a charter member of the Mathematical Association.

Professor E. B. Skinner, of the University of Wisconsin, died April 3, 1935 following a heart attack. He was technically professor emeritus since June 1934, but was still actively engaged in teaching. He had been a member of the Association for many years.

In the November number of this Monthly the erroneous statement was made that Professor Morgan Ward had been appointed to an assistant professorship at the Institute for Advanced Study. Professor Ward is on leave of absence from the California Institute of Technology during the year 1934–35, and is enrolled at the Institute with the title "worker."

Also in the December issue, the statement "Albany College, Portland, Oregon" should read "Albany College, Albany, Oregon."

The following courses in Mathematics are announced for the Summer of 1935.

University of California at Los Angeles. The following graduate courses will be offered: By Professor Garver: Probability and its applications to statistics. By Professor Whyburn: Differential equations of mathematical physics.

University of Chicago. First term, June 17 to July 4; second term, July 25 to August 30. In addition to Differential calculus, Elementary theory of equations, and Elementary differential equations, the following courses will be offered: By Professor Lunn: Fourier series and Bessel functions; Relativity. By Professor Logsdon: Projective analytic geometry; Higher plane curves. By Professor Albert: Introduction to higher algebra; Galois theory of equations. By Professor Bartky: Introduction to celestial mechanics I; Dynamics of rigid bodies. By Professor McShane: Theory of functions of a complex variable. By Dr. Hestenes: Topology. By Professors Bliss and Graves: Topics in the calculus of variations. The last named course will have the nature of a seminar in which Professor McShane, Dr. Hestenes, Dr. Reid, Dr. Coral and others who may be interested will participate.

University of Colorado. First term, June 13 to July 20; second term, July 22

to August 23. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor Light: Statistics (first term); History of mathematics (second term); Differential equations; Calculus of variations. By Assistant Professor Hazard; Teachers' course in mathematics (first term). By Associate Professor Kendall; Theory of equations (second term). By Professor Hutchinson; Functions of a real variable.

Columbia University. July 8 to August 16. In addition to courses in intermediate algebra, trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following courses are offered: By Professor Kasner: Fundamental concepts of mathematics; Geometric transformations and continuous groups. By Professor Fite: Introduction to higher algebra. By Dr. Berry: Differential equations. By Dr. Brown: Theory of functions of a real variable.

Cornell University. July 8 to August 16. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Hutchinson, Modern higher algebra. By Professor Hurwitz, Elementary differential equations. By Professor Gillespie, Advanced calculus. By Dr. Dye, Projective geometry. By Professor Carver, Advanced analytic geometry. Reading and research will be directed by Professors Hutchinson, Hurwitz, Gillespie, Carver; Assistant Professors Lawrence, Jones, Agnew, and Dr. Dye.

Duke University. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor Rankin; Teaching of mathematics; History of mathematics. By Professor Patterson: Theory of equations and determinants. By Professor Dale: Modern higher algebra; Infinite series. By Professor Elliot: Advanced calculus; Vector analysis. Professor Miles: Projective geometry; Advanced calculus. By Professor Roberts: Non-Euclidean geometry; Topology. By the staff: directed reading and thesis supervision.

University of Illinois. June 17 to August 10. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor Carmichael: The theory of numbers. By Associate Professor Lytle: History of mathematics; Teachers course. By Assistant Professor Brahana: Introduction to higher algebra; Algebra. By Assistant Professor Trjitzinsky: Analysis. By Dr. Ketchum: Differential geometry. By Dr. Steimley: Introduction to higher analysis.

University of Iowa. First term June 7 to July 18. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Miss Ruth Lane: Content and teaching of mathematics. By Assistant Professor Craig: Matrices and determinants; Statistics; Topics in analysis. By Assistant Professor Ward: Differential equations; Modern geometry; Introduction to differential geometry. By Associate Profes-

sor Wylie: Mathematics of finance; Astronomy; Meteors. By Professor Chittenden: Advanced calculus; General topology. By the staff: Reading and research.

Second term, July 20 to August 22. By Professor Chittenden: Theory of functions of a complex variable; Seminar in analysis. Professor Reilly: Potential theory; Advanced trigonometry; Seminar in interpolation. Associate Professor Woods: Elliptic integrals; Projective geometry. By Assistant Professor Conkwright: Differential equations; Theory of equations; Group theory. By the staff: reading and research.

Johns Hopkins University. June 24 to August 3. In addition to courses in college algebra and elementary calculus, one of the following courses is offered: By Professor F. D. Murnaghan: Modern geometry; Statistics; Elementary group theory with applications to physics and chemistry.

University of Kansas. June 12 to August 7. In addition to the usual elementary courses, the following advanced courses are offered: By Professor Jordan: Higher algebra (Bôcher). By Professor Smith: Modern synthetic geometry; Higher plane curves; Seminar. By Professor Wheeler: Mathematical theory of statistics.

University of Kentucky. First term. By Dean Boyd: Projective geometry. By Professor Downing: Analytic mechanics. By Professor LeStourgeon: Advanced calculus.

Second term. By Professor Latimer: Higher algebra.

University of Maine. July 1 to August 9. In addition to the usual elementary work, the following advanced courses are offered: By Associate Professor Bryan: Theory of numbers; Teachers seminar. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Differential equations, or other graduate courses by arrangement.

Massachusetts Institute of Technology. First period, June 11 to July 23. Calculus and differential equations, covering the work of the first two years; graduate courses in advanced calculus and functions of complex variable.

Second period, July 24 to September 4. Courses covering first two years repeated; course in advanced calculus continued; vector analysis. August 5 to September 7, courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in these subjects.

University of Michigan. June 24 to August 16. In addition to elementary courses in college algebra, trigonometry, plane and solid analytic geometry, calculus, differential equations, statistics, and finance, the following advanced courses will be offered: By Professor Anning: Graphical methods. By Professor Ayres: Advanced calculus; Infinite series, with special reference to Fourier series. By Professor Bradshaw: Advanced solid analytic geometry; Synthetic projective geometry. By Professor Coe: Analytic mechanics. By Professor Craig: Theory of probability; Advanced theory of statistics. By Doctor Elder:

Theory of numbers. By Professor Field: Applied mathematics—Engineering problems. By Professor Hildebrandt: Theory of functions of a real variable; Calculus of variations. By Professor Karpinski: Teaching of geometry; History of mathematics. By Professor Menge: Theory of equations and determinants; Mathematics of finance and insurance. By Professor Nyswander: Finite differences. By Professor Poor: Vector analysis. By Professor Rainich: Introduction to the foundations of mathematics; Groups of transformations. By Professor Hildebrandt and Professor Rainich: Seminar in pure mathematics.

University of Minnesota. First term June 17 to July 27. In addition to the usual elementary work, the following courses will be offered: By Professors Elizabeth Carlson and Gladys Gibbens: Reading in senior college mathematics. By Professor Gibbens: Differential equations. By Professor Hart: Advanced calculus; Algebraic introduction to statistics. By Professor Jackson: Fourier, Legendre, and Bessel series. By Professors Hart and Jackson: reading in advanced mathematics.

University of North Carolina. First term June 13 to July 24. In addition to the usual elementary work in algebra, trigonometry, analytic geometry and the calculus, the following courses will be offered. By Professor Henderson: Differential equations; Elementary relativity. By Professor Lasley: Analytic geometry of space; Differential geometry. By Professor Mackie: Advanced calculus. By Professor Winsor: College geometry; Theory of equations.

Second term, July 25 to August 31. By Professor Hobbs: Differential equations (continued); Theory of equations (continued). By Professor Browne: Analytic geometry of space (continued); Differential geometry (continued.)

Northwestern University. June 24 to August 17. In addition to the usual elementary work, the following advanced courses will be offered. By Professor Moulton: Probability. By Professor Wood: The teaching of mathematics in secondary schools. By Professor Wall: Higher algebra.

Ohio State University. June 17 to August 30. In addition to the usual elementary courses, the following advanced courses will be offered. By Professor Weaver: Differential equations; Projective geometry. By Professor Morris: Mathematical statistics; Probability. By Professor MacDuffee: Linear algebras; Finite groups. By Professor LaPaz: Calculus of variations; Tensor analysis. By the staff: reading and research.

University of Pittsburgh. July 1 to August 9. In addition to the usual courses in elementary subjects, the following more advanced courses will be offered: By Professor Foraker: Modern synthetic geometry; solid analytic geometry. By Professor Taylor: Advanced calculus; Introduction to the theory of relativity. By Professor Culver: Differential equations.

University of Southern California. First term, June 17 to July 26. The following advanced courses are offered: By Professor Gurney: The theory of equations

and determinants; Mathematical astronomy. By Associate Professor Steed: Vector analysis; Advanced analytical geometry.

Second term, July 27 to August 30. By Professor Ames: Theory of probability and statistics; The teaching of junior college mathematics; Seminar (subject to be announced during first term).

Syracuse University. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Decker: A choice between the History of mathematics and the Theory of invariants. By Professor Campbell: Either elementary solid analytical geometry or Affine geometry. By Professor Carroll: Methods in teaching mathematics.

Teachers College, Columbia University. July 8 to August 16. In addition to the usual courses in the teaching of arithmetic and the teaching of algebra given by Professor Upton, the following courses will be offered. By Mr. Smith: The teaching of geometry and a demonstration class in geometry. By Dr. Swenson: A demonstration class in eleventh year mathematics; A course in professionalized subject matter for the senior high school. By Mr. Bemis: Two demonstration classes, one on intuitive geometry and one on algebra both in the junior high school. By Mr. Shuster: A course in field work in mathematics. By Dr. Sanford: A demonstration class in social and economic arithmetic; A course on the history of mathematics. By Professor Hedrick: A course on teaching mathematics in junior colleges and in lower divisions of colleges and universities; A course on professionalized subject matter in algebra and geometry. By Professor Reeve: A mathematics field course in Germany. The group will sail from New York about June 20th and will leave Germany on August 10th.

University of Vermont. July 8, to August 16. In addition to the usual elementary work, the following advanced courses will be offered. By Professor Butterfield: Astronomy; Integral calculus; History of mathematics. By Professor Nicholson; Teaching algebra and plane geometry. By Professor Bullard: Analytical geometry; Differential calculus.

University of Wisconsin. Six weeks session July 1 to August 9. By Professor Evans; Analytic geometry; Differential equations; Topics in the theory of probability. By Professor Hart: College geometry; The teaching of mathematics. By Professor March: Advanced calculus; Topics in applied mathematics.

Special nine weeks session for graduates, July 1 to August 30. These courses may be taken for six weeks. By Professor Ingraham: Mathematics of educational statistics; Theory of numbers.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.
Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Southern California, Los Angeles, California, Saturday, March 2, 1935. Professor L. E. Wear presided.

The attendance was forty-two, including the following thirty-two members of the Association; O. W. Albert, E. E. Allen, L. D. Ames, Harry Bateman, Mable Lou Beckwith, Clifford Bell, Jessie R. Campbell, P. H. Daus, Iva B. Ernsberger, Raymond Garver, H. H. Gaver, Harriet E. Glazier, E. R. Hedrick, G. H. Hunt, C. G. Jaeger, G. R. Kaelin, G. R. Livingston, Ada McClellan, G. F. McEwen, D. B. Perry, Lena E. Reynolds, J. M. Robb, G. E. F. Sherwood, D. V. Steed, K. D. Swartzel, F. C. Touton, S. E. Urner, H. C. VanBuskirk, L. E. Wear, W. M. Whyburn, H. C. Willett, Euphemia R. Worthington.

The following officers were elected for the next year: Chairman, Prof. C. G. Jaeger, Pomona College; Vice-Chairman, Dr. S. E. Urner, Los Angeles Junior College; Program Committee, Miss Lena E. Reynolds, Fullerton Junior College, and Prof. A. D. Michal, California Institute of Technology. The next meeting was tentatively scheduled for March 7, 1936, at Fullerton Junior College.

The following six papers were read:

1. "Calendar reform" by Professor Emeritus F. P. Brackett, Pomona College, by invitation of the program committee.
2. "The generalization of functions defined as coefficients in an expansion" by Professor Harry Bateman, California Institute of Technology.
3. "On functions similar to Euler's phi function" by Mable Lou Beckwith, Brawley, California.
4. "Isotomic conjugates" by Professor P. H. Daus, University of California at Los Angeles.
5. "Note on equally spaced ordinates from averages of observed ordinates arbitrarily spaced" by Professor G. F. McEwen, Scripps Institution of Oceanography of the University of California.
6. "Vector proof of a theorem due to Beltrami" by Professor D. V. Steed, University of Southern California.

Abstracts of the papers follow, the numbers corresponding to those in the list of titles.

1. In a lecture illustrated with slides Professor Brackett discussed the history of calendar changes due to the fact that the length of the tropical year is 365.24220 mean solar days. Calendar reform will produce (1) a rational perpetual calendar, (2) simplification, (3) equal periods, especially quarters, (4) minimum changes, (5) stabilization of Easter. Several calendars including the so-called World Calendar were discussed.

2. In the case when the generating function is a product of binomial and exponential factors, a useful method of generalization is first to derive the functions of negative integral order with the aid of a recurrence relation, and then to complete the generalization with the aid of the cardinal function of interpolation or some related function. Professor Bateman gave special attention to the case in which the recurrence relation fails to determine the functions of negative integral order.

3. The number of integers $\phi_k(n)$, prime to n in the sequence $1\cdot 2\cdot 3 \cdots k$, $2\cdot 3\cdot 4 \cdots (k+1), \cdots, n(n+1) \cdots (n+k-1)$ was determined by Mable Lou Beckwith and shown to have a great similarity to Euler's phi function. A study of the number of integers prime to n in the sequence $1\cdot 2 \cdots k/k!$, $2\cdot 3 \cdots (k+1)/k!, \cdots, n(n+1) \cdots (n+k-1)/k!$ is in progress; the case $k=2$ has been determined and was discussed.

4. Professor Daus considered the quadratic Cremona transformation, defined by isotomic conjugates, by projective methods and by means of areal coordinates, pointing out the fixed elements and the fundamental conic which is the transform of the line at infinity.

5. In order to reveal any time relation obscured by high individual variability of ordinates observed at unequal intervals, a succession of group averages corresponding to irregular intervals is calculated. Smoothed or graduated ordinates at selected equal time intervals are needed in order to take advantage of uniform spacing in further studies of the series. Professor McEwen has developed a method for expressing the graduated ordinates in terms of divided differences of the mean ordinates for the irregular intervals used in the first computation.

6. Professor Steed presented a vector proof of the following theorem due to Beltrami: the tangential surface of a space curve intersects the osculating plane at a point P of the curve in a plane curve whose curvature at P is three-fourths of the curvature of the given curve at that point.

P. H. DAUS, *Secretary*

THE FALL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The third meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Grove City College, Grove City, Pennsylvania, on Saturday, October 13, 1934. Sessions were held at 10:30 and at 2:00, with a luncheon at 12:45. The morning session was opened with a welcoming address by President Weir C. Kettler of Grove City College. The program of the afternoon session was preceded by a business meeting at which the following officers were elected: Chairman, Professor C. S. Atchison; Secretary-Treasurer, Professor J. S. Taylor; Members of the Executive Committee, Professors W. E. Cleland and E. G. Olds. Professor C. S. Atchison presided at both sessions.

Sixty-one persons were in attendance, including the following twenty-eight

members of the Association: C. S. Atchison, B. R. Beisel, O. F. H. Bert, W. E. Cleland, Elizabeth B. Cowley, L. L. Dines, C. W. Foard, F. A. Foraker, Orrin Frink, Jr., W. O. Gordon, N. C. Grimes, E. E. Hess, H. C. Hicks, B. P. Hoover, A. V. Karpov, W. W. McCormick, T. W. Moore, David Moskovitz, E. G. Olds, F. W. Owens, J. B. Rosenbach, E. A. Saibel, J. C. Stayer, J. S. Taylor, R. W. Thomas, C. C. Wagner, E. D. Wells, and E. A. Whitman.

The following seven papers were presented:

1. "What can the small college do for the student who desires to specialize in mathematics?" by Professor W. E. Cleland, Geneva College.

2. "A problem in the mathematics of economics" by K. H. Stahl, Shaler High School, introduced by the Secretary.

3. "The vibrational motion of hydrocarbon chains" by Dr. L. S. Kassel, United States Bureau of Mines, introduced by Professor Atchison.

4. "Mathematics and character education" by Dr. Elizabeth B. Cowley, Allegheny High School.

5. "The representation as a single integral of the iterated integral over an area" by James Affleck, Gulf Research Laboratory, introduced by the Secretary.

6. "Multiple perspectivity" by Professor F. W. Owens, Pennsylvania State College.

7. "Distribution functions of characteristics of samples" by Professor E. G. Olds, Carnegie Institute of Technology.

The abstracts of these papers follow:

1. Professor Cleland believes that the limitations imposed on the mathematics departments of small colleges by the small teaching staff due to the comparatively small number of mathematics students may be overcome, at least in part, by judicious use of various devices: alternation of courses; close cooperation of the mathematics instructors with those in allied subjects; honors courses; individual assignments to able students. Successful use of these devices is not confined to small colleges, but all of them and especially the last, are peculiarly suited to conditions in the small college.

2. Technological improvements in an industry may effect changes in the selling price of the article produced. Assuming certain types of relation between selling price and demand, K. H. Stahl discussed the question of the selling price which will produce the maximum profit together with the effect on the number of men employed. While the discussion is entirely theoretical some interesting results were deduced with reference to the types of assumed relations connecting selling price and demand.

3. Dr. Kassel showed that the Hamiltonian function for a chain of four carbon atoms with a simple valence force system potential energy can be transformed by successive changes of variable to a form in which the wave equation is nearly separable into five harmonic oscillator equations and a sixth equation governing the internal rotation. The eigenfunctions of this rotation equation correspond to torsional oscillation about one or the other of the two possible plane structures for low energy, and to rotation about the central carbon-carbon

bond for higher energies. The non-separable terms in the equation were shown to be negligible by a perturbation theory treatment.

4. The need of character training as a part of educational work is apparent to everyone who is familiar with the statistics of crime. As writers on character education have largely ignored the fact that skillful teaching of mathematics can contribute to the development of character, Dr. Cowley studied the contribution that can thus be made.

5. While a double integral over an area is usually defined as the limit of a single summation it is ordinarily computed as an iterated integral. Under certain fairly general conditions James Affleck exhibited the fact that the double integral can be represented as a single integral of a function of area.

6. Professor Owens presented a general discussion of multiply perspective triangles in the plane together with a proof that there can not exist in a plane three real triangles each of which is in six-fold perspective with each of the others. Other extensions were discussed briefly.

7. Theoretical investigations of the frequency distributions of statistical parameters are of general interest because of the mathematical devices employed. After discussing the nature and use of distribution functions, Professor Olds called attention to derivations making use of n-dimensional geometry, complex variable theory, finite differences, and integral equations.

J. S. TAYLOR, *Secretary*

MATHEMATICAL PRINCIPLES IN THE THEORY OF SMALL SAMPLES*

By DUNHAM JACKSON, University of Minnesota

1. *Introduction.* So many students have occasion to seek an understanding of the theory of sampling that an exposition even of well known principles may serve a useful purpose in bringing together items that are scattered through a variety of books and journals, and in supplying explanations which in one account or another may have to be read between the lines. The purpose of this paper is to facilitate an appreciation of the fundamental memoirs in the field of small sample analysis on the basis of ordinary courses and textbooks in mathematics.

2. *Definitions and fundamental formulas.* The notion of frequency function is dominant throughout. The concept of frequency is taken as primary and es-

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* Presented before a joint session of the American Mathematical Society, the American Statistical Association, the Econometric Society, and Sections A and K of the American Association for the Advancement of Science, at Berkeley, June 20, 1934. In preparing the paper for publication the author has derived profit from remarks made by Professors Hotelling and Uspensky in the discussion following the oral presentation.

sentially undefined, the question of the objective significance of frequency functions being looked upon as comparable with that of the relation between the straight lines or circles of geometry and the outlines of material objects. A detailed study of this question is regarded as preliminary or supplementary to the logical development, rather than as constituting an organic part of it.

A variable x is said to have the frequency function $\phi(x)$ if the frequency of occurrence of x in the range $\alpha < x < \beta$ is measured by

$$\int_{\alpha}^{\beta} \phi(x)dx. \tag{1}$$

The frequency function will be thought of as defined (and non-negative) for all real values of the variable; if the actual occurrence of the variable is limited to a finite range, the frequency function is defined as identically zero outside that range. It is not required that $\phi(x)$ be continuous. If it is different from zero over an infinite interval, or if it has infinite discontinuities, questions of convergence naturally arise. In the most important cases these can be dealt with by routine methods, without special difficulty, and they will not be discussed in detail here.

Unless the contrary is indicated it will be understood that

$$\int_{-\infty}^{\infty} \phi(x)dx = 1. \tag{2}$$

The integral (1) then expresses the *probability* that x is between α and β. The concept of probability, however, is regarded as secondary to that of frequency, and will seldom be mentioned explicitly. In certain calculations a constant factor in the frequency function may be left undetermined in the intermediate stages of the work, and evaluated, if necessary, by a supplementary reckoning at the end.

Under the conditions implied by the preceding statements the frequency attached to any single value of the variable is zero (and so, incidentally, it is immaterial whether the interval (α, β) is regarded as open or closed). The important situations in which the variable takes on only a finite number or an enumerable infinity of different values call for a separate treatment, with sums in place of integrals, unless both types of distribution are comprehended under a single more general formulation by the use of Stieltjes integrals. For the present introductory account the representation by means of ordinary integrals will be regarded as sufficiently illustrative. Much of the theory, in any event, is concerned with certain specific forms of distribution of a continuous variable.

If x has the frequency function $\phi(x)$, with total frequency 1, in accordance with (2), the arithmetic mean of x is

$$m = \int_{-\infty}^{\infty} x\phi(x)dx.$$

Related by an obvious limiting process to the mean of a finite number of quantities, this formula has for present purposes the status of a definition. Similarly, if the mean is taken as the origin of measurement, so that

$$\int_{-\infty}^{\infty} x\phi(x)dx = 0,$$

the standard deviation of x is the (positive) square root of

$$\sigma^2 = \int_{-\infty}^{\infty} x^2\phi(x)dx.$$

The mean value of an arbitrary function $f(x)$, subject to conditions of integrability, is

$$\int_{-\infty}^{\infty} f(x)\phi(x)dx.$$

Corresponding definitions are to be laid down for the distribution of two or more variables. The variables (x, y) have the frequency function $\phi(x, y)$ if the double integral of $\phi(x, y)$ over a region of the (x, y)-plane measures the frequency of occurrence of pairs of values (x, y) belonging to that region. The definition will be thought of as extended over the entire plane, even when positive frequencies are restricted to a finite region, and it will ordinarily be supposed that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x, y)dy\, dx = 1.$$

This condition being fulfilled, the means of x and y are

$$m_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\phi(x, y)dy\, dx, \quad m_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\phi(x, y)dy\, dx,$$

and if the origin of measurement is chosen so that $m_x = m_y = 0$ the squares of the standard deviations are

$$\sigma_x^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\phi(x, y)dy\, dx, \quad \sigma_y^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2\phi(x, y)dy\, dx,$$

and the coefficient of correlation between x and y is

$$\rho_{xy} = \frac{1}{\sigma_x\sigma_y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\phi(x, y)dy\, dx.$$

The mean value of a function $f(x, y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\phi(x, y)dy\, dx.$$

In particular, the mean of $x+y$ is m_x+m_y.

Let x and y be independent variables with the frequency functions $\phi(x)$ and $\psi(y)$ respectively. Since the occurrence of x between α and β and the occurrence of y between γ and δ are independent events, the probability that x and y both fall within the ranges specified is the product of the probabilities for x and y separately, or

$$\int_\alpha^\beta \phi(x)dx \int_\gamma^\delta \psi(y)dy,$$

which is the same as the double integral

$$\int_\alpha^\beta \int_\gamma^\delta \phi(x)\psi(y)dy\, dx.$$

This gives at once a measure of frequency for a region made up of any number of rectangles with sides parallel to the coordinate axes, and so by a routine limiting process it appears that the frequency in a region of arbitrary shape is measured by the double integral of $\phi(x)$ $\psi(y)$ over that region. In other words, *the frequency function for x and y jointly is the product of the frequency functions describing their distributions separately*. If the notion of frequency function is maintained as fundamental throughout, and the notion of probability considered to be derived from it, this relation between the frequency functions can be regarded as a *definition* of independence of x and y. The independence of any number of variables is characterized similarly.

If x and y are independent variables with the frequency functions $\phi(x)$ and $\psi(y)$, each with total frequency 1, and if they are measured from their respective means as origin, so that $m_x = m_y = 0$, the square of the standard deviation of $x+y$, the mean of $(x+y)^2$, is given by

$$\sigma_{x+y}^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty (x+y)^2\phi(x)\psi(y)dy\, dx.$$

The term $2xy$ in the expansion of $(x+y)^2$ contributes nothing to the value of the integral, since

$$\int_{-\infty}^\infty \int_{-\infty}^\infty xy\phi(x)\psi(y)dy\, dx = \int_{-\infty}^\infty x\phi(x)dx \int_{-\infty}^\infty y\psi(y)dy = m_x m_y = 0,$$

and consequently

$$\sigma_{x+y}^2 = \int_{-\infty}^\infty x^2\phi(x)dx \int_{-\infty}^\infty \psi(y)dy + \int_{-\infty}^\infty \phi(x)dx \int_{-\infty}^\infty y^2\psi(y)dy = \sigma_x^2 + \sigma_y^2.$$

In the same way *the square of the standard deviation of the sum of any number of independent variables is the sum of the squares of their standard deviations.*

3. *Samples from an arbitrary population; elementary moments.* Let $x_1, x_2, \cdots,$

x_n be n independent variables, all having the same frequency function ϕ, so that the frequency function for their joint distribution is

$$\phi(x_1)\phi(x_2)\cdots\phi(x_n). \tag{3}$$

Under these circumstances a set of n values $x_1, x_2, \cdots, x_n$ will be said to constitute a *sample of n from a population with frequency function* $\phi(x)$, and the function (3) will be said to represent the *frequency distribution of samples*. The notion of frequency distribution of samples underlies all the succeeding work. Let the mean and the standard deviation corresponding to $\phi(x)$, the *mean of the population* and the *standard deviation of the population*, be 0 and σ. By the preceding paragraph the square of the standard deviation of $x_1+x_2+\cdots+x_n$ is $n\sigma^2$. Let $\bar{x}$ denote the *mean of a sample*,

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The square of the standard deviation of $\bar{x}$, the mean value of $\bar{x}^2$ as a function of $x_1, x_2, \cdots, x_n$, is represented by $(1/n^2)$ times the multiple integral which gives the mean value of $(x_1+x_2\cdots+x_n)^2$, and consequently this standard deviation σ_{mean} is given by

$$\sigma^2_{\text{mean}} = \sigma^2/n, \quad \sigma_{\text{mean}} = \sigma/n^{1/2}.$$

Let s denote* the standard deviation of a sample:

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n} = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \bar{x}^2.$$

The mean of x_k^2 is σ^2, for each value of k from 1 to n, the mean of x_k^2/n is σ^2/n, the mean of $(x_1^2+x_2^2+\cdots+x_n^2)/n$ is σ^2, the mean of $\bar{x}^2$, as just noted, is σ^2/n, and consequently *the mean value of* s^2 *is*

$$\sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2.$$

The mean of the sum of the squares of the deviations from the origin is σ^2; the sum of the squares of the deviations from the mean in the case of any sample is less than the sum of the squares of the deviations from the origin (unless the sample is one whose mean falls at the origin) by the least-square property of the arithmetic mean; and it is thus obviously to be anticipated that the mean of s^2 is less than σ^2. The relation just obtained measures the extent of this inequality.

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* See e.g. P. R. Rider, *A survey of the theory of small samples*, Annals of Mathematics, (2), vol. 31 (1930), pp. 577–628; p. 579. R. A. Fisher uses the letter s to denote a different quantity, namely the estimate of the value of σ obtained at the end of the next paragraph. While the whole purpose of the paragraph is to compare these quantities and distinguish between them, the use of a particular symbol to represent one or the other of them is of course a mere matter of notation.

Let the positive square root of the mean value of s^2 be denoted by $\bar{s}$, so that

$$\bar{s} = [(n-1)/n]^{1/2}\sigma, \qquad \sigma = [n/(n-1)]^{1/2}\bar{s}.$$

If the "parent population" is known only from a single sample, and if it is required to estimate the value of σ on the basis of the information given by the sample, the one observed value of s^2 may be regarded as the best available estimate of $\bar{s}^2$, and the corresponding estimate of σ is

$$\left(\frac{n}{n-1}\right)^{1/2} s = \left[\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n-1}\right]^{1/2}.$$

4. *Frequency distributions pertaining to an arbitrary population.* An important class of problems, with which the rest of this paper will be concerned, goes beyond the finding of a mean or of any finite number of moments, and calls for determination of the form of the frequency distribution for the values of a function when the distribution of the independent variable or variables* is given.

If $\phi(x, y)$ is the frequency function for the joint distribution of x and y, *the frequency function for the single variable x is*

$$\omega(x) = \int_{-\infty}^{\infty} \phi(x, y)dy.$$

For the frequency of occurrence of x in any interval (α, β) is the frequency of pairs (x, y) belonging to the strip of the (x, y)-plane for which $\alpha<x<\beta$, and this is

$$\int_{\alpha}^{\beta}\int_{-\infty}^{\infty} \phi(x, y)dy\,dx = \int_{\alpha}^{\beta} \omega(x)dx.$$

With the same meaning of $\phi(x, y)$ *the frequency function for the variable* $u=x+y$ is

$$\psi(u) = \int_{-\infty}^{\infty} \phi(x, u-x)dx.$$

For $\alpha<u<\beta$ when $\alpha-x<y<\beta-x$; these inequalities define a strip of the (x, y)-plane for which the corresponding frequency is

$$F(\alpha, \beta) = \int_{-\infty}^{\infty}\int_{\alpha-x}^{\beta-x} \phi(x, y)dy\,dx;$$

in the integration with respect to y, the substitution $u=x+y$, $y=u-x$, makes

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* It will occasionally be convenient, when no confusion is likely to result, to use the phrase "independent variables" in the ordinary sense of analysis to designate the variables on which a specified function depends, without any implication that these variables are independent of each other in the statistical sense. The two usages, though quite distinct for the purposes of a discussion such as this, are not fundamentally contradictory, since functional dependence can be regarded as a limiting case of statistical dependence.

$$\int_{\alpha-x}^{\beta-x} \phi(x, y)dy = \int_{\alpha}^{\beta} \phi(x, u - x)du;$$

and hence

$$F(\alpha, \beta) = \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} \phi(x, u - x)du\, dx = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} \phi(x, u - x)dx\, du = \int_{\alpha}^{\beta} \psi(u)du.$$

Let x have the frequency function $\phi(x)$, let $u = f(x)$ be a function of x having everywhere a positive derivative, and let $x = g(u)$ be the function inverse to $f(x)$. The interval $\alpha < u < \beta$ corresponds to the interval $g(\alpha) < x < g(\beta)$, and the frequency for this range is

$$\int_{g(\alpha)}^{g(\beta)} \phi(x)dx = \int_{\alpha}^{\beta} \phi[g(u)]g'(u)du.$$

The frequency function for u is

$$\psi(u) = \phi[g(u)]g'(u);$$

the functions $\psi(u)$ and $\phi(x)$ are connected by the relation

$$\psi(u)du = \phi(x)dx.$$

In the particularly important case that $u = f(x) = x^2$, the condition that $f'(x) > 0$ everywhere is not satisfied, and x is of course not a single-valued function of u. Furthermore, everything being real, negative values of u do not occur. By making the necessary changes in the argument of the preceding paragraph, with reference to the fact that the interval $\alpha < u < \beta$, if $\beta > \alpha \geqq 0$, corresponds to the two intervals $\alpha^{1/2} < x < \beta^{1/2}$, $-\beta^{1/2} < x < -\alpha^{1/2}$, it is seen that the frequency function $\psi(u)$ for u can be described by saying that

$$\begin{aligned} \psi(u) &= 0 && \text{for } u < 0, \\ \psi(u) &= \tfrac{1}{2}u^{-1/2}[\phi(u^{1/2}) + \phi(-u^{1/2})] && \text{for } u > 0. \end{aligned}$$

5. *Normal population; distribution of means and standard deviations of samples.* So far the assumed distributions have been of arbitrary form; it has not been supposed that they are normal or otherwise specialized. *From now on the discussion will be concerned with the special calculations which are possible when the underlying distribution of the independent variable or variables is normal.*

Let x and y be normally distributed about zero as mean with standard deviations σ_1 and σ_2 respectively, so that their frequency functions have the form

$$\phi_1(x) = C_1 e^{-ax^2}, \quad \phi_2(y) = C_2 e^{-by^2}, \quad a = 1/(2\sigma_1^2), \quad b = 1/(2\sigma_2^2);$$

the explicit values $C_1 = 1/[\sigma_1(2\pi)^{1/2}]$, $C_2 = 1/[\sigma_2(2\pi)^{1/2}]$, for total frequency 1, are not needed at the moment. Let it be supposed that x and y are independent. Then their joint distribution is given by

$$\phi(x, y) = C_1 C_2 e^{-ax^2 - by^2}.$$

The frequency function for $u=x+y$ is

$$\psi(u) = \int_{-\infty}^{\infty} \phi(x, u - x)dx = C_1C_2 \int_{-\infty}^{\infty} e^{-ax^2-b(u-x)^2}dx.$$

It is seen that

$$ax^2 + b(u - x)^2 = (a + b)\left(x - \frac{b}{a+b}\, u\right)^2 + \frac{ab}{a+b}\, u^2$$
$$= (a + b)v^2 + cu^2,$$

with the notation

$$v = x - \frac{b}{a+b}\, u, \qquad c = \frac{ab}{a+b} = \frac{1}{2(\sigma_1^2 + \sigma_2^2)}.$$

The value of u being regarded as constant for the integration with respect to x, so that incidentally $dv=dx$, the expression for $\psi(u)$ can be written in the form

$$\psi(u) = C_1C_2e^{-cu^2} \int_{-\infty}^{\infty} e^{-(a+b)v^2}dv = Ce^{-cu^2},$$

where C depends neither on u nor on x. The last formula describes a normal frequency distribution with standard deviation $\sigma_u=(\sigma_1^2+\sigma_2^2)^{1/2}$. *If two independent variables are normally distributed with standard deviations σ_1 and σ_2, their sum is normally distributed with standard deviation $(\sigma_1^2+\sigma_2^2)^{1/2}$.* The essential point here is that the distribution of u is normal; its standard deviation is necessarily that given by a general proposition already established. Furthermore, without carrying through the intermediate steps in the calculation of the constants, it is clear that for total frequency 1 the value of C must be $1/[\sigma_u(2\pi)^{1/2}] = 1/[2\pi(\sigma_1^2+\sigma_2^2)]^{1/2}$.

If x, y, and z are independent and normally distributed, the quantity $x+y+z$ can be regarded as the sum of the two independent* normally distributed variables $x+y$ and z, and so is itself normally distributed. The conclusion can be carried over by induction, without further calculation, to the sum of any number of variables. *If $x_1, x_2, \cdots, x_n$ are independent and normally distributed with standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_n$, their sum is normally distributed with standard deviation $(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)^{1/2}$.*

In particular, if $x_1, x_2, \cdots, x_n$ constitute a sample of n from a normally distributed population with standard deviation σ, the frequency function for the population being of the form

$$Ce^{-x^2/(2\sigma^2)},$$

then $x_1+x_2+\cdots+x_n$ is normally distributed with standard deviation $n^{1/2}\sigma$,

* It is tacitly assumed as "obvious" that if x, y, and z are independent and if $u=x+y$ then u and z are independent. It can be verified without difficulty that this is in fact a consequence of the definition of independence.

and *the means of samples*, the values of the quantity $\bar{x}=(x_1+x_2+\cdots+x_n)/n$, *are normally distributed with standard deviation* $\sigma/n^{1/2}$. Again the essential feature of the conclusion is the normal form of the distribution; the magnitude of the standard deviation is known from an earlier passage.

If x is normally distributed, with frequency function $\phi(x)=Ce^{-x^2/(2\sigma^2)}$, the distribution found in a previous paragraph for $u=x^2$ is simplified by the fact that $\phi(x)=\phi(-x)$; the function $\psi(u)$, identically zero for $u<0$, has the form

$$\psi(u) = Cu^{-1/2}e^{-u/(2\sigma^2)}$$

for $u>0$.

This is a special case of an important type of frequency function in which the factor $u^{-1/2}$ is replaced by u^p, with an arbitrary value of $p>-1$. The more general type is essentially the χ^2-distribution of the theory of "goodness of fit." To avoid any appearance of dependence on that theory, however, and the bringing up of questions associated with it, the distribution will be referred to here as a Γ-distribution, in view of the fact that the function describing it is at the same time essentially the integrand in the integral representation of the Gamma function. Specifically, a variable u will be said to have a *Γ-distribution with exponent p* if its frequency function is 0 for $u<0$, and of the form $Cu^pe^{-u/(2\sigma^2)}$ for $u>0$.

A fundamental theorem with regard to such frequency functions is that *if u and v are independent variables having Γ-distributions with exponents p and q respectively and the same value of σ, their sum has a Γ-distribution with this value of σ and with exponent** $p+q+1$.

The distributions of u and v are given by

$$\psi_1(u) = 0 \text{ for } u < 0, \quad \psi_1(u) = C_1u^pe^{-u/(2\sigma^2)} \text{ for } u > 0,$$
$$\psi_2(v) = 0 \text{ for } v < 0, \quad \psi_2(v) = C_2v^qe^{-v/(2\sigma^2)} \text{ for } v > 0.$$

The frequency function for their joint distribution is $\psi(u, v)=\psi_1(u)\,\psi_2(v)$, and that for $w=u+v$ is

$$\int_{-\infty}^{\infty}\psi(u, w-u)du = \int_{-\infty}^{\infty}\psi_1(u)\psi_2(w-u)du.$$

As the integrand vanishes identically, however, for $u<0$ and for $w-u<0$, $u>w$, the last integral is for $w>0$ the same as

$$\int_0^w \psi_1(u)\psi_2(w-u)du = C_1C_2\int_0^w u^p(w-u)^qe^{-w/(2\sigma^2)}du$$
$$= C_1C_2e^{-w/(2\sigma^2)}\int_0^w u^p(w-u)^qdu,$$

* The relation between the exponents is more strikingly expressed in terms of the corresponding arguments of the Gamma function, which are $p+1$, $q+1$, and $p+q+2$, the third being the sum of the first two.

which by the substitution $u = wt$, $du = w\,dt$, becomes

$$C_1C_2w^{p+q+1}e^{-w/(2\sigma^2)}\int_0^1 t^p(1-t)^q dt,$$

and this is of the form $Cw^{p+q+1}\,e^{-w/(2\sigma^2)}$, the coefficient C being independent of w.

If x and y are independent and normally distributed with the same standard deviation σ (the mean being zero in each case), the variables $u = x^2$ and $v = y^2$ have each a Γ-distribution with exponent $-1/2$, and x^2+y^2 consequently is in a Γ-distribution with exponent 0. If x, y, and z are independent and normally distributed with standard deviation σ and mean 0, the quantity $x^2+y^2+z^2$ can be regarded as the sum of the two independent variables x^2+y^2 and z^2, which are in Γ-distributions with exponents 0 and $-1/2$, and so has a Γ-distribution with exponent $1/2$. Hence, by induction, *if $x_1, x_2, \cdots, x_n$ are independent and normally distributed, with mean* 0 *and standard deviation σ in each case* (and so if $x_1, x_2, \cdots, x_n$ form a sample of n from a normal population having the characteristics indicated), *the variable $w = x_1^2 + x_2^2 + \cdots + x_n^2$ has a Γ-distribution with exponent $(n-2)/2$, its frequency function for $w > 0$ being of the form*

$$Cw^{(n-2)/2}e^{-w/(2\sigma^2)}.$$

With regard to the important problem of the *distribution of the squares of the standard deviations of samples of n from a normal population*, i.e., the distribution of the quantity

$$s^2 = \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2}{n},$$

where $\bar{x} = (x_1+x_2+\cdots+x_n)/n$, the essential facts are strikingly brought out by means of a representation in terms of n-dimensional geometry.* This geometrical treatment will not be repeated here. Instead it will be shown how the discussion can be put in completely analytical form, by use of the concept of *orthogonal linear transformation* of n variables.†

An orthogonal linear transformation (more fully characterized with the further adjective *unitary*) may be described by a set of equations

$$\begin{aligned} u_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n, \\ u_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_n &= c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n, \end{aligned}$$

in which the coefficients satisfy the conditions

* See e.g. R. A. Fisher, *Applications of "Student's" distribution*, Metron, vol. 5, no. 3 (1925), pp. 90–104, pp. 91–93; Rider, loc. cit., pp. 579–583.

† See Fisher, loc. cit., pp. 97–99; Bôcher, *Introduction to Higher Algebra*, p. 154.

$$\sum_{k=1}^{n} c_{ik}c_{jk} = 0, \quad i \neq j; \qquad \sum_{k=1}^{n} c_{ik}^2 = 1, \quad i = 1, 2, \cdots, n.$$

Corresponding equations hold if the sums of squares and product sums are taken by columns instead of by rows. A fundamental property is that the sum of the squares of the variables is invariant:

$$u_1^2 + u_2^2 + \cdots + u_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

The inverse transformation, giving the x's in terms of the u's, is expressed by means of the same coefficients with rows and columns interchanged: $x_j = \sum_i c_{ij} u_i$. The determinant of the coefficients is necessarily ± 1. It will be sufficient here to consider transformations with determinant $+1$.

If the x's are independent and normally distributed, each with standard deviation σ, the u's are independent and normally distributed with the same standard deviation. The frequency function for the joint distribution of the x's is

$$Ce^{-\Phi}, \quad \Phi = (x_1^2 + x_2^2 + \cdots + x_n^2)/(2\sigma^2).$$

The frequency of occurrence of $(u_1, u_2, \cdots, u_n)$ in any specified region is the frequency of occurrence of $(x_1, x_2, \cdots, x_n)$ in the corresponding region; the latter is measured by the integral of $Ce^{-\Phi}$ over the region with respect to the x's; but since the Jacobian determinant of the transformation, being the same as the determinant of the coefficients, is equal to 1,

$$\int\int \cdots \int e^{-\Phi} dx_1 dx_2 \cdots dx_n = \int\int \cdots \int e^{-\Phi} du_1 du_2 \cdots du_n,$$

the integrals being extended over corresponding ranges; consequently the frequency of occurrence of $(u_1, u_2, \cdots, u_n)$ in any region is measured by the integral of $Ce^{-\Phi}$ over that region with respect to the u's; the function Φ when expressed in terms of the u's has the form

$$(u_1^2 + u_2^2 + \cdots + u_n^2)/(2\sigma^2),$$

since $u_1^2 + u_2^2 + \cdots + u_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2$; and so the frequency function for the joint distribution of the u's is that for n independent normally distributed variables each with standard deviation σ.

In particular, let $c_{11} = c_{12} = \cdots = c_{1n} = 1/n^{1/2}$. The requirement that $\sum_k c_{1k}^2 = 1$ is obviously fulfilled. It is possible in an infinite variety of ways to choose a set of coefficients c_{2k} to satisfy the condition $\sum_k c_{1k}c_{2k} = 0$, then to choose a set c_{3k} so that $\sum_k c_{1k}c_{3k} = \sum_k c_{2k}c_{3k} = 0$, and so on, the determination at each stage calling for the solution of a system of homogeneous equations with fewer equations than unknowns.* It requires merely the adjustment of a constant factor in each row to provide that $\sum_k c_{ik}^2 = 1$, $i = 2, 3, \cdots, n$, and that the de-

* See Fisher, loc. cit., pp. 97–98.

terminant of the whole set of n^2 coefficients shall be $+1$. The definition of $c_{11}, \cdots, c_{1n}$ makes $u_1 = n^{1/2}\bar{x}$. Also, since

$$ns^2 = \sum_k x_k^2 - n\bar{x}^2 = \sum_k x_k^2 - u_1^2,$$

while on the other hand $\sum_k x_k^2 = \sum_k u_k^2$, it follows that

$$ns^2 = u_2^2 + u_3^2 + \cdots + u_n^2.$$

Thus it appears that ns^2 *is expressible as the sum of the squares of* $n-1$ *independent normally distributed variables with standard deviation* σ; it has a Γ-distribution with exponent $(n-3)/2$. The integral expressing the frequency of occurrence of s in a specified (positive) range is of the form

$$C\int (s^2)^{(n-3)/2}e^{-ns^2/(2\sigma^2)}d(s^2) = C'\int s^{n-2}e^{-ns^2/(2\sigma^2)}ds.$$

At the same time it is seen again that $\bar{x}$ is normally distributed with standard deviation $\sigma/n^{1/2}$, since u_1 is normally distributed with standard deviation σ.

6. *Student's distribution.* Inasmuch as u_1 is independent of $u_2, \cdots, u_n$, the variables $\bar{x}$ and s are distributed independently of each other. The frequency function for their joint distribution is (with a new value of C) of the form

$$Cs^{n-2}e^{-P}, \quad P = n(\bar{x}^2 + s^2)/(2\sigma^2),$$

for $s > 0$, being identically zero for $s < 0$, since s is naturally to be taken as the positive square root of the expression defining s^2. From this can be deduced* the distribution of the variable known as "Student's z." Let

$$z = \bar{x}/s, \quad \bar{x} = zs.$$

The joint frequency function for z and s is

$$Cs^{n-1}e^{-n(z^2+1)s^2/(2\sigma\ )}$$

for $s > 0$; for in calculating the frequency of occurrence of $(\bar{x}, s)$ in a specified region, or of (z, s) in the corresponding region, if the integration with respect to $\bar{x}$ is performed first, $d\bar{x} = s\, dz$ for this integration.† The frequency function for the single variable z is

$$C\int_0^\infty s^{n-1}e^{-n(z^2+1)s^2/(2\sigma^2)}ds.$$

By the substitution $t = (z^2+1)^{1/2}s$, $ds = (z^2+1)^{-1/2}dt$, this becomes

* See Fisher, loc. cit., pp. 91–93; Rider, loc. cit., pp. 579–583.

† Formally expressed in terms of infinitesimal frequencies, with $-P$ as an abbreviation for the long exponent, the relation is simply that

$$s^{n-2}e^{-P}d\bar{x}\, ds = s^{n-1}e^{-P}dz\, ds.$$

$$C(z^2 + 1)^{-n/2} \int_0^\infty t^{n-1} e^{-nt^2/(2\sigma^2)} dt = C'(z^2 + 1)^{-n/2},$$

where C' is independent of z; *the frequency function for z is a constant multiple of*

$$\frac{1}{(z^2 + 1)^{n/2}}.$$

Inasmuch as the variable z is introduced for the purpose of dealing with situations in which the value of σ is unknown, an important feature of the z-distribution is that it is independent of σ.

More generally, if w is any variable which has a Γ-distribution with exponent p, the frequency function being

$$Cw^p e^{-w/(2\sigma^2)},$$

if $v = w^{1/2}$, and if u is independent of w and normally distributed with standard deviation σ, then the frequency function for $z = u/v$ is a constant multiple of

$$\frac{1}{(z^2 + 1)^{(2p+3)/2}}.$$

For, by the relation

$$w^p e^{-w/(2\sigma^2)} dw = 2v^{2p+1} e^{-v^2/(2\sigma^2)} dv,$$

the frequency function for $v(>0)$ is $2Cv^{2p+1} e^{-v^2/(2\sigma^2)}$; hence that for the joint distribution of u and v is of the form

$$C'v^{2p+1} e^{-(u^2+v^2)/(2\sigma^2)};$$

this gives for z and v the frequency function

$$C'v^{2p+2} e^{-(z^2+1)v^2/(2\sigma^2)},$$

and for the single variable z the function

$$C' \int_0^\infty v^{2p+2} e^{-(z^2+1)v^2/(2\sigma^2)} dv = C'(z^2 + 1)^{-(2p+3)/2} \int_0^\infty t^{2p+2} e^{-t^2/(2\sigma^2)} dt$$
$$= C''(z^2 + 1)^{-(2p+3)/2}.$$

The result thus obtained is applicable in a variety of problems. An important case is that leading to a test of significance of a *difference between two means*.* Let $x_1, x_2, \cdots, x_{n_1}$ be a sample of n_1 from a normal population with standard deviation σ, and let $y_1, y_2, \cdots, y_{n_2}$ be a sample of n_2 (where n_2 may or may not be the same as n_1), drawn independently of the first sample, from the same population. Let

$$\bar{x} = (x_1 + x_2 + \cdots + x_{n_1})/n_1, \quad \bar{y} = (y_1 + y_2 + \cdots + y_{n_2})/n_2.$$

* See Fisher, loc. cit., pp. 94–96.

Then $\bar{x}$ is normally distributed with standard deviation $\sigma/n_1^{1/2}$, y is normally distributed with standard deviation $\sigma/n_2^{1/2}$, $\bar{x}-\bar{y}$ is normally distributed* with standard deviation $[(n_1+n_2)/(n_1n_2)]^{1/2}\sigma$, and

$$u = \left(\frac{n_1n_2}{n_1 + n_2}\right)^{1/2}(\bar{x} - \bar{y})$$

is normally distributed with standard deviation σ. If s_1 and s_2 are the standard deviations of the first and second samples respectively,

$$n_1s_1^2 = x_1^2 + \cdots + x_{n_1}^2 - n_1\bar{x}^2, \quad n_2s_2^2 = y_1^2 + \cdots + y_{n_2}^2 - n_2\bar{y}^2.$$

Let

$$u_1 = (x_1 + \cdots + x_{n_1})/n_1^{1/2} = n_1^{1/2}\bar{x}, \qquad u_2 = (y_1 + \cdots + y_{n_2})/n_2^{1/2} = n_2^{1/2}\bar{y},$$

and let $u_3, \cdots, u_{n_1+n_2}$ be defined so that the equations expressing $u_1, \cdots, u_{n_1+n_2}$ in terms of the n_1+n_2 variables $x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}$ are those of a unitary orthogonal transformation. Then

$$u_1^2 + u_2^2 + \cdots + u_{n_1+n_2}^2 = x_1^2 + \cdots + x_{n_1}^2 + y_1^2 + \cdots + y_{n_2}^2,$$

and consequently

$$n_1s_1^2 + n_2s_2^2 = u_3^2 + \cdots + u_{n_1+n_2}^2;$$

the quantity $w = n_1s_1^2 + n_2s_2^2$ is expressible as the sum of the squares of n_1+n_2-2 independent normally distributed variables with standard deviation σ. So w has a Γ-distribution with exponent $p = (n_1+n_2-4)/2$. The value of $(2p+3)/2$ being $(n_1+n_2-1)/2$, *the variable*

$$z = \frac{u}{w^{1/2}} = \left(\frac{n_1n_2}{n_1 + n_2}\right)^{1/2} \cdot \frac{\bar{x} - \bar{y}}{(n_1s_1^2 + n_2s_2^2)^{1/2}}$$

has for its frequency function a constant multiple of

$$\frac{1}{(z^2 + 1)^{(n_1+n_2-1)/2}}.$$

It is to be noted that the requisite count of "degrees of freedom" is automatically obtained as a result of the analysis, not something that has to be determined before the analysis can begin.

For further applications to the problem of estimating the significance of regression coefficients reference may be made to Fisher's original paper.†

7. *Distribution of correlation coefficients.* A problem leading to considerations

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* The difference $\bar{x}-\bar{y}$ can be regarded as the *sum* of the independent variables $\bar{x}$ and $-\bar{y}$.

† Fisher, loc. cit., pp. 96–102; also R. A. Fisher, *Statistical Methods for Research Workers* (fourth edition, Edinburgh and London, 1932), Chapter 5.

of less elementary character is that of the distribution of correlation coefficients.* Here the use of geometrical representation in space of many dimensions is particularly effective. In fact, in the simplest and most important case, that in which the variables are independent in the parent population, the result can be predicted on the basis of geometrical analogy (or correspondingly proved, if the requisite logical foundation has been laid for the geometrical reasoning) with almost no calculation at all—though this is far from being true more generally, when the variables in the underlying distribution are correlated.

Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be a sample of n pairs drawn independently from a normal (x, y)-distribution in which x and y are themselves independent, a distribution having a frequency function of the form

$$Ce^{-P}, \quad P = \frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2}.$$

The frequency function for the $2n$ variables $x_1, \cdots, y_n$ has the form

$$C'e^{-Q}, \quad Q = (\sum x_k^2)/(2\sigma_1^2) + (\sum y_k^2)/(2\sigma_2^2).$$

(The sign $\sum$ here and subsequently indicates in every case summation over values of the index from 1 to n.) Let

$$\bar{x} = (\sum x_k)/n, \qquad \bar{y} = (\sum y_k)/n,$$

$$s_1 = [(1/n)\sum(x_k - \bar{x})^2]^{1/2}, \qquad s_2 = [(1/n)\sum(y_k - \bar{y})^2]^{1/2}.$$

The correlation coefficient of the sample is

$$r = \frac{\sum(x_k - \bar{x})(y_k - \bar{y})}{ns_1s_2}.$$

The problem is that of finding the distribution of r, considered as a function of $x_1, \cdots, y_n$, when these variables have the distribution already indicated.

Geometrically, r is the cosine of the angle† in n-dimensional space between the vector with the components $(x_1 - \bar{x}, \cdots, x_n - \bar{x})$ and the vector having the components $(y_1 - \bar{y}, \cdots, y_n - \bar{y})$. If A is the point with coordinates $(x_1 - \bar{x}, \cdots, x_n - \bar{x})$ and B the point with coordinates $(y_1 - \bar{y}, \cdots, y_n - \bar{y})$, O being the origin, r is the cosine of the angle AOB. Since $\sum(x_k - \bar{x}) = \sum(y_k - \bar{y}) = 0$, both OA and OB lie in the $(n-1)$-dimensional plane space characterized by the condition that the sum of the coordinates is zero. Within this $(n-1)$-dimensional space all directions are possible for OA, and all directions are possible for OB; since

* See R. A. Fisher, *Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population*, Biometrika, vol. 10 (1914–15), pp. 507–521; also *Statistical Methods*, op. cit., Chapter 6.

† See e.g. Fisher, Biometrika, loc. cit.; D. Jackson, *The trigonometry of correlation*, American Mathematical Monthly, vol. 31 (1924), pp. 275–280; D. Jackson, *The elementary geometry of function space*, American Mathematical Monthly, vol. 31 (1924), pp. 461–471.

the frequency function for $(x_1, \cdots, x_n)$ and that for $(y_1, \cdots, y_n)$ depend only on distance from the origin, all directions for OA are equally likely,* and similarly for OB; and since the y's are independent of the x's, all directions OB (in the $(n-1)$-dimensional space) are equally likely to be associated with any specified direction OA. So the distribution of r is that of $\cos\theta$, if θ is the angle between a specified radius of a sphere in $(n-1)$-dimensional space and a radius whose terminal point ranges uniformly over the surface of the sphere. The frequency for an infinitesimal range from r to $r+dr$ is measured by the area of the corresponding zone on the sphere. If the radius of the sphere is taken as unity, this area is except for an infinitesimal of higher order the product of $|d\theta|$ by the circumference of an $(n-3)$-dimensional small circle of radius $\sin\theta$ on the $(n-2)$- dimensional surface of the sphere. The circumference being a constant multiple of $\sin^{n-3}\theta$, while $|d\theta| = dr/\sin\theta$ when $r=\cos\theta$ and $dr>0$, the frequency is proportional to

$$\sin^{n-3}\theta\,|\,d\theta\,| = \sin^{n-4}\theta dr = (1 - r^2)^{(n-4)/2}dr.$$

The problem of the distribution of r is of an altogether different order of difficulty if x and y in the population from which the samples are drawn, instead of being independent, have a coefficient of correlation ρ. It is not the purpose of this article to repeat the calculation† by which the frequency function is obtained in terms of r and ρ. It is intended in the remaining lines merely to show how certain preliminary geometric considerations leading to the formulas with which the detailed calculation starts can be translated into analytic language, so that the argument becomes entirely analytic in character.‡ The work naturally applies in particular when $\rho=0$, and makes it possible to replace the geometric discussion given above by a completely analytic formulation. It is noteworthy that in all that has been done in the preceding pages, although the notion of

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* If K is the point with coordinates $(x_1, x_2, \cdots, x_n)$ and M the point with coordinates $(\bar{x}, \bar{x}, \cdots, \bar{x})$, OA is parallel to MK; for any specified M, the point K may be any point in the $(n-1)$-dimensional hyperplane through M perpendicular to OM; within this hyperplane all points equidistant from O, and so all points equidistant from M, are equally likely.

If the notion of sample is modified so that the x's are not required to be statistically independent among themselves, or the y's among themselves, it is not essential that the distribution be normal; it is enough that the frequency functions for the x's and for the y's depend only on distance from the origin and that the two groups be independent of each other, the frequency function for the $2n$ variables being of the form $\phi(\sum x_k^2)\psi(\sum y_k)$.

† Fisher, Biometrika, loc. cit.

‡ See also H. Hotelling, *The distribution of correlation ratios calculated from random data*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 657–662; H. Hotelling, *The generalization of Student's ratio*, Annals of Mathematical Statistics, vol. 2 (1931), pp. 360–378. For an analytic treatment proceeding along quite different lines see J. Wishart and M. S. Bartlett, *The distribution of second order moment statistics in a normal system*, Proceedings of the Cambridge Philosophical Society, vol. 28 (1932), pp. 455–459; J. Wishart and M. S. Bartlett, *The generalised product moment distribution in a normal system*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 260–270

multiple integral is fundamental, questions of the transformation of multiple integrals have been of the simplest sort, the transformation being accomplished in each case either by explicit change of one variable at a time or by means of constant coefficients with unit determinant. It will be necessary presently to lay stress on certain Jacobian determinants of less simple form.

The pairs $(x_1, y_1), \cdots, (x_n, y_n)$ are to be drawn from an (x, y)-population whose frequency function can be written as

$$Ce^{-P}, \quad P = \frac{1}{1-\rho^2}\left[\frac{x^2}{2\sigma_1^2} - \frac{2\rho xy}{2\sigma_1\sigma_2} + \frac{y^2}{2\sigma_2^2}\right].$$

The frequency function for $x_1, \cdots, y_n$ is of the form

$$C'e^{-Q}, \quad Q = \frac{1}{1-\rho^2}\left[\frac{\sum x_k^2}{2\sigma_1^2} - \frac{2\rho\sum x_k y_k}{2\sigma_1\sigma_2} + \frac{\sum y_k^2}{2\sigma_2^2}\right].$$

The symbols $\bar{x}$, $\bar{y}$, s_1, s_2, and r have the same meanings as before. Since

$$ns_1^2 = \sum x_k^2 - n\bar{x}^2, \qquad ns_2^2 = \sum y_k^2 - n\bar{y}^2,$$
$$nrs_1s_2 = \sum(x_k - \bar{x})(y_k - \bar{y}) = \sum x_k y_k - n\bar{x}\bar{y},$$

it follows that

$$\sum x_k^2 = n(\bar{x}^2 + s_1^2), \quad \sum y_k^2 = n(\bar{y}^2 + s_2^2), \quad \sum x_k y_k = n(\bar{x}\bar{y} + rs_1s_2).$$

The quantities σ_1, σ_2, and ρ being given constants, *the exponent Q is a function of the five variables $\bar{x}$, $\bar{y}$, s_1, s_2, and r, and the same is true of the frequency function for $x_1, \cdots, y_n$.*

To facilitate the explicit carrying out of the requisite integrations a transformation is to be defined, in successive stages, from the $2n$ variables $x_1, \cdots, y_n$ to a new set of $2n$ variables, five of which shall be those just specified.

First, let

$$\begin{aligned} u_1 &= (x_1 + x_2 + \cdots + x_n)/n^{1/2}, & v_1 &= (y_1 + y_2 + \cdots + y_n)/n^{1/2},\\ u_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n, & v_2 &= c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_n,\\ &\cdots\cdots\cdots & &\cdots\cdots\cdots\\ u_n &= c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n, & v_n &= c_{n1}y_1 + c_{n2}y_2 + \cdots + c_{nn}y_n, \end{aligned}$$

the coefficients being those of a unitary orthogonal transformation, and the same coefficients being used for the x's and for the y s. If these equations are regarded as defining a transformation from the $2n$ variables $x_1, \cdots, y_n$ to the $2n$ variables $u_1, \cdots, v_n$, in which the u's are independent of the y's and the v's independent of the x's, the determinant of the coefficients, which is at the same time the Jacobian determinant of the transformation, is 1, and integration with respect to the x's and y's is equivalent to integration with respect to the u's and v's over the corresponding domain. With the identities $\sum x_k^2 = \sum u_k^2$,

$\sum y_k^2 = \sum v_k^2$, is associated now the identity $\sum x_k y_k = \sum u_k v_k$. Let the symbol $\sum'$ be used to indicate summation from $k=2$ to $k=n$, the index 1 being omitted. Then, as $u_1 = n^{1/2}\bar{x}$, $v_1 = n^{1/2}\bar{y}$,

$$ns_1^2 = \sum x_k^2 - n\bar{x}^2 = \sum x_k^2 - u_1^2 = \sum' u_k^2, \quad ns_2^2 = \sum' v_k^2,$$

$$nrs_1s_2 = \sum x_k y_k - n\bar{x}\bar{y} = \sum x_k y_k - u_1 v_1 = \sum' u_k v_k,$$

$$r = \sum' u_k v_k/(ns_1s_2) = \sum' u_k v_k/\left[\sum' u_k^2 \sum' v_k^2\right]^{1/2}.$$

Now let $S = [\sum' u_k^2]^{1/2}$, and let a set of variables $\xi_1, \cdots, \eta_n$ be defined in terms of $u_1, \cdots, v_n$ by the equations

$$\begin{aligned}
\xi_1 &= u_1, \quad & \xi_2 &= u_2, \cdots, \xi_n = u_n,\\
\eta_1 &= v_1, \quad & \eta_2 &= (u_2/S)v_2 + (u_3/S)v_3 + \cdots + (u_n/S)v_n,\\
& & \eta_3 &= \gamma_{32}v_2 + \gamma_{33}v_3 + \cdots + \gamma_{3n}v_n,\\
& & & \cdots\cdots\cdots\cdots\cdots\cdots\\
& & \eta_n &= \gamma_{n2}v_2 + \gamma_{n3}v_3 + \cdots + \gamma_{nn}v_n,
\end{aligned}$$

where the γ's are functions of the u's (independent of the v's) chosen so as to give a unitary orthogonal transformation of the last $n-1$ variables. If in particular it is specified that $\gamma_{ij}=0$ for $j>i\geqq 3$, and if a convention is made as to algebraic signs, the conditions imposed on the transformation determine the γ's as continuous functions of $u_2, \cdots, u_n$ for $u_2 \neq 0$, the specialization of the variable u_2 (or, alternatively, u_3) resulting from the designated choice of the γ's which are to vanish.* The Jacobian of the $2n$ variables ξ, η with respect to the $2n$ variables u, v reduces to the determinant of the coefficients in the last $n-1$ rows and columns (in spite of the dependence of these coefficients on the u's), and this determinant is equal to 1.

The solution of the last $n-1$ equations for the v's in terms of the η's takes the form

$$\begin{aligned}
v_2 &= (u_2/S)\eta_2 + \gamma_{32}\eta_3 + \cdots + \gamma_{n2}\eta_n,\\
v_3 &= (u_3/S)\eta_2 + \gamma_{33}\eta_3 + \cdots + \gamma_{n3}\eta_n,\\
& \cdots\cdots\cdots\cdots\cdots\cdots\\
v_n &= (u_n/S)\eta_2 + \gamma_{3n}\eta_3 + \cdots + \gamma_{nn}\eta_n.
\end{aligned}$$

By the conditions of orthogonality, as the coefficients in the first column on the right are proportional to the u's, $\sum' u_k \gamma_{jk} = 0$ for $j=3, \cdots, n$; hence $\sum' u_k v_k = (\sum' u_k^2)\eta_2/S = \eta_2[\sum' u_k^2]^{1/2}$. Furthermore, $\sum' v_k^2 = \sum' \eta_k^2$. So

* From the geometric analogy by which the formulas are suggested it is in the Laplacian sense "easy to see" that the γ's can be otherwise chosen so as to be continuous except for $u_2 = u_3 = \cdots = u_n = 0$; but an explicit proof does not appear to lend itself to simple formulation, and the assignment of an exceptional status to the entire locus $u_2 = 0$ does not give rise to any essential difficulty in the further reasoning.

$$r = \frac{\sum' u_k v_k}{[\sum' u_k^2 \sum' v_k^2]^{1/2}} = \frac{\eta_2}{[\sum' v_k^2]^{1/2}} = \frac{\eta_2}{[\sum' \eta_k^2]^{1/2}}.$$

In place of the $n-1$ variables $\eta_2, \cdots, \eta_n$ let a new set of $n-1$ variables R, $\theta, \theta_1, \theta_2, \cdots, \theta_{n-3}$ be introduced by means of the equations

$$\begin{aligned}
\eta_2 &= R \cos \theta, \\
\eta_3 &= R \sin \theta \cos \theta_1, \\
\eta_4 &= R \sin \theta \sin \theta_1 \cos \theta_2, \\
&\cdots\cdots\cdots\cdots \\
\eta_k &= R \sin \theta \sin \theta_1 \cdots \sin \theta_{k-3} \cos \theta_{k-2}, \qquad k = 4, 5, \cdots, n-1, \\
&\cdots\cdots\cdots\cdots \\
\eta_{n-1} &= R \sin \theta \sin \theta_1 \cdots \sin \theta_{n-4} \cos \theta_{n-3}, \\
\eta_n &= R \sin \theta \sin \theta_1 \cdots \sin \theta_{n-4} \sin \theta_{n-3}.
\end{aligned}$$

By forming successively the quantities $\eta_n^2 + \eta_{n-1}^2$, $\eta_n^2 + \eta_{n-1}^2 + \eta_{n-2}^2, \cdots$, it is seen that $\sum' \eta_k^2 = R^2$. All real values of $\eta_2, \cdots, \eta_n$ are obtained by allowing R to range from 0 to ∞, each of the variables $\theta, \theta_1, \cdots, \theta_{n-4}$ from 0 to π, and θ_{n-3} from $-\pi$ to π, the equations defining the θ's in terms of the η's being

$$\cos \theta = \frac{\eta_2}{R} = \frac{\eta_2}{[\sum' \eta_k^2]^{1/2}}, \qquad \cos \theta_1 = \frac{\eta_3}{[\eta_3^2 + \cdots + \eta_n^2]^{1/2}},$$

$$\cos \theta_2 = \frac{\eta_4}{[\eta_4^2 + \cdots + \eta_n^2]^{1/2}}, \cdots,$$

$$\cos \theta_{n-3} = \frac{\eta_{n-1}}{[\eta_{n-1}^2 + \eta_n^2]^{1/2}}, \qquad \sin \theta_{n-3} = \frac{\eta_n}{[\eta_{n-1}^2 + \eta_n^2]^{1/2}}.$$

It is to be noticed as an essential feature of the transformation that $\cos \theta = r$.

The Jacobian of $\eta_2, \cdots, \eta_n$ with respect to the new variables is of the form

$$\begin{vmatrix}
\cos \theta & -R \sin \theta & 0 & \cdots & 0 \\
T_{31} \sin \theta & T_{31} R \cos \theta & T_{33} R \sin \theta & \cdots & 0 \\
T_{41} \sin \theta & T_{41} R \cos \theta & T_{43} R \sin \theta & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
T_{n1} \sin \theta & T_{n1} R \cos \theta & T_{n3} R \sin \theta & \cdots & T_{n,n-1} R \sin \theta
\end{vmatrix},$$

where the T's are functions of $\theta_1, \cdots, \theta_{n-3}$, *independent of R and θ*, and the T's in the second column are the same as those in the first column. In this $(n-1)$-rowed determinant, R is a common factor of all the elements in the last $n-2$ columns, and $\sin \theta$ is a common factor of all the elements in the last $n-3$ columns. If a factor $\sin \theta$ is transferred from one of the later columns to the

first column, and the remaining common factor $R^{n-2}\sin^{n-4}\theta$ taken out before the whole, the expression becomes

$$R^{n-2}\sin^{n-4}\theta\begin{vmatrix}\sin\theta\cos\theta & -\sin\theta & 0\cdots & 0\\ T_{31}\sin^2\theta & T_{31}\cos\theta & T_{33}\cdots & 0\\ T_{41}\sin^2\theta & T_{41}\cos\theta & T_{43}\cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ T_{n1}\sin^2\theta & T_{n1}\cos\theta & T_{n3}\cdots & T_{n,n-1}\end{vmatrix}.$$

Now let the second column be multiplied by $\cos\theta$ and added to the first column; the elements of the first column are replaced by $0, T_{31}, T_{41}, \cdots, T_{n1}$. In the determinant thus obtained the only element different from zero in the first row is $-\sin\theta$ at the head of the second column, and the cofactor of this element is made up entirely of zeros and T's. So the value of the Jacobian is expressible in the form

$$T(\theta_1, \cdots, \theta_{n-3})R^{n-2}\sin^{n-3}\theta,$$

where T is independent of R and θ.

Similarly, $\xi_2, \cdots, \xi_n$ or $u_2, \cdots, u_n$ can be expressed in terms of the previously defined S and a set of new variables $\phi, \phi_1, \cdots, \phi_{n-3}$. In this case it is not necessary to specify even the manner of dependence of the Jacobian on ϕ; it is enough to say that it is of the form

$$F(\phi, \phi_1, \cdots, \phi_{n-3})S^{n-2},$$

where F is independent of S.

By combination of the various transformations that have been defined it appears that $x_1, \cdots, x_n, y_1, \cdots, y_n$ can be expressed in terms of $u_1 = n^{1/2}\bar{x}$, $v_1 = n^{1/2}\bar{y}$, $S = n^{1/2}s_1$, $R = n^{1/2}s_2$, $\theta = \text{arc}\cos r$, and $2n-5$ other variables, by a transformation with Jacobian of the form

$$GS^{n-2}R^{n-2}\sin^{n-3}\theta,$$

where G is independent of the five variables specially mentioned. Replacement of these five by $\bar{x}$, $\bar{y}$, s_1, s_2, and r as independent variables affects the absolute value of the Jacobian merely to the extent of the introduction of constant factors and a further factor $|d\theta/dr| = 1/\sin\theta$, whereby the Jacobian becomes

$$G_1 s_1^{n-2}s_2^{n-2}\sin^{n-4}\theta = G_1 s_1^{n-1}s_2^{n-2}(1 - r^2)^{(n-4)/2},$$

with G_1 independent of $\bar{x}$, $\bar{y}$, s_1, s_2, and r.

This, together with the fact that the frequency function for $x_1, \cdots, y_n$ is a function of the five variables, say $\Phi(\bar{x}, \bar{y}, s_1, s_2, r)$, means that the frequency of occurrence of

$$(\bar{x}, \bar{y}, s_1, s_2, r, \phi, \phi_1, \cdots, \phi_{n-3}, \theta_1, \cdots, \theta_{n-3})$$

in a specified domain, being the same as the frequency of $x_1, \cdots, y_n$ in the corresponding domain, is measured by the integral of

$$\Phi(\bar{x}, \bar{y}, s_1, s_2, r)G_1 s_1^{n-2} s_2^{n-2}(1 - r^2)^{(n-4)/2} d\bar{x} \cdots dr\, d\phi \cdots d\theta_{n-3}$$

over the specified domain for these variables. In other words, the frequency function for the $2n$ variables $\bar{x}, \cdots, \theta_{n-3}$ is

$$\Phi G_1 s_1^{n-2} s_2^{n-2}(1 - r^2)^{(n-4)/2}.$$

The frequency function for $(\bar{x}, \bar{y}, s_1, s_2, r)$ is obtained by integrating with respect to the remaining $2n-5$ variables over the entire range of the latter, and so *is a constant multiple of*

$$\Phi s_1^{n-2} s_2^{n-2}(1 - r^2)^{(n-4)/2}.$$

This agrees with the result obtained by geometric argument, and the above discussion is in fact merely a detailed statement of the analytic facts which the geometric language concisely describes.

If $\rho=0$ the function Φ is independent of r, and in this case the frequency function for r alone, found by integrating with respect to $\bar{x}$, $\bar{y}$, s_1, and s_2 over the entire range of these variables, is a constant multiple of $(1-r^2)^{(n-4)/2}$, as already stated. The complexity of the more general problem is due to the dependence of Φ on r.

PRODUCTIVE SCHOLARSHIP IN THE UNDERGRADUATE COLLEGE*

By R. L. Jeffery, Acadia University

In this address we shall have especially in mind the College which is not a part of a larger institution featuring a Graduate School. It is not now easy to see why I chose this subject. The undertaking would have been more pleasant and profitable had I selected a topic which was strictly mathematical. But in all probability there was back of my decision the feeling that those of us who are engaged in College work do not take the interest in Productive Scholarship that teaching at its highest level demands.

It is not possible for me to speak first hand in every case. I have, nevertheless, an intimate knowledge of conditions in several Colleges, and over a period of years I have picked up what information I could in regard to others. It seems to me that there is no exaggeration in saying that for the most part it is not even recognized that productive work has a definite place in the activities of the undergraduate teacher. At any rate, if this recognition does exist, it is merely formal. It is by no means sufficiently virile to color the general atmosphere. Under conditions such as these the teacher who is interested in progressive scholarly work does not find it congenial. That there should be even one institution of higher learning in which a teacher finds it unpleasant because of an interest in scholarly work, is unfortunate. It seemed worth while to open this question up for discussion; first to show that a certain amount of productive

* Address delivered by invitation before the Association at Pittsburgh, Jan. 1, 1935.

work is essential to first class teaching; then to show that in every College active participation in scholarly work is practicable.

Let me say at the start that it is not my intention to try to make out a case for Research for its own sake. There can be no possible question about the value of Research. But it is doubtful if this can be used effectively as an argument for a greater interest in Research in our Colleges. It is always met with the statement that theplace for Research is in the Universities or the Research Institutes where there are adequate facilities, and for this statement there is much justification. There are reasons, however, why the undergraduate teacher should have an active interest in Research apart from the possible value of any results which he obtains, and it is this point of view that I shall try to set forth. I shall not use the term Research. It has become distasteful to me for the reason that it is applied to no end of activities which are not related in any way to anything scholarly. This feeling is shared by many. For our purpose the term Productive Scholarship seems more to be desired. As Flexner points out, even the job of determining the rate of growth of bacteria in cotton undershirts has been called Research. I do not think that such work as this ever has been, or ever will be, classified as Productive Scholarship.

In the type of institution which we have in mind it is not feasible to consider the problem of productive work apart from its relation to all the other activities in which teachers are called upon to engage. But no matter what phase of these activities we consider, we find that it imposes a definite obligation to an active participitation in creative effort. In the first place, a large part of our time is given up to elementary instruction. And in every course we teach we find parts of the work which for some reason do not seem to go over. On these points class after class stalls for an unreasonably long time. There is a tendency for us to accept this situation as inevitable; to consider that there is something inherently difficult in the part of the work in question; or to let it go with the statement that the students we are getting nowadays are not up to what they should be. Of late, however, I have come to see this problem in a different way. Perhaps my own experience has led me to attribute undue emphasis to it. At any rate I have become convinced that when the situation we are describing arises the trouble is much more likely to be with our methods than with the work or the students. And once this fact is faced the problem does resolve itself into one of creative effort. We have to think the whole thing through from every possible angle, and keep working at it until we devise a method of presentation by which the work goes over easily and at once. To accomplish this one's ingenuity is sometimes pretty heavily taxed. But the longer I teach the more I am convinced that there is room for much of this sort of creative effort. It is a real source of satisfaction to clear away obstacles from places at which we have come to expect our classes to be held up, to say nothing of the time and energy that is saved. This sort of work frequently leads to worthwhile publication. The most striking instance of this that has come to my notice is a paper entitled "Marginal Notes" by Hildebrandt in the April 1929 number of the MONTHLY. There could well be

more of this sort of publication. Work at this level is valuable for the reason that we can secure the full cooperation of our undergraduate students. It is our first opportunity to initiate them to the method and the spirit of original investigation. And I find that interest is tremendously heightened if they can go to the literature to see what others are thinking on the same subjects. I would like to see these problems which arise out of elementary teaching much more freely discussed in our official journal. If College teaching is to be kept at a uniformly high level, the interchange of ideas that results from this type of Productive Scholarship is necessary.

Let us now look at this problem of Productive Scholarship from a different angle. As I said at the start, there is by no means unanimity of opinion concerning the value of an interest in creative work. But on one point there is universal agreement: the undergraduate teacher should know his subject matter. He should know its origin and its significance; its relation to other fields; the point of view, methods, and results of contemporary workers in his field. He should be so thoroughly aware of and awake to his subject that his enthusiasm carries over to those whom he is trying to teach. I am only repeating a small part of what has been said many times and in many ways, and in all this there is a large measure of justification. But at the same time I do not believe it is possible for one to fully appreciate any field of intellectual endeavour unless one is trying to do original work in some phase of that field. And even if it were possible, I do not believe that one can come anywhere near to the degree of excellence demanded, without at least being in a position to make some contribution. The background of experience and training of any one person is not the same as that of any other. And once a person arrives at a full knowledge of any field of work, he must necessarily see some phases of it from a new angle. He is in a position to suggest new lines of investigation, new methods of working, to simplify old methods of working, or at least to correlate the whole field in the light of his own background of training and experience. And unless he is willing to derive his satisfaction wholly from the hard work of others without himself trying to make any contribution to the general progress, he is under obligation to do one, or all, of these things to the best of his ability. On the other hand, if he does all, or even one, of these things, he has broken through to a fairly high level of Productive Scholarship. If we fail to reach this level, what can we say for ourselves? We can be charitable, and say that we have not troubled ourselves to master our subject to the degree of perfection that our work demands. But then we are admitting that we are not fit for our positions. If, on the other hand, we maintain that we are fitted for our posts, we must acknowledge that our minds are closed to original ideas, or that we are too indifferent and indolent to make known such worthwhile ideas as we do have. Gladly would I retreat from this exacting stand were it possible to do so. But for long I have been convinced that one or the other of these two positions was the only alternative to an active interest in productive work. And it seems to me that neither of them is very desirable.

Again take the Honors work that is being featured by many Colleges. If this is in a healthy state the work of the students is under the direction of teachers who are themselves students. Furthermore, there is nothing that arouses the enthusiasm of Honor students so quickly or so intensely, as the realization that they are creating something new, even if it is not of great significance. And with proper direction, these students can work effectively on significent problems. No teacher is in a position to assign such problems, or to encourage students to work hard at them, unless he himself is working at a higher level. And there is no way to be properly keyed to this situation other than by an active interest in Productive Scholarship. Honors work carried on under any other conditions lacks the zest that makes it worth while. The teacher who is assuming the responsibility of developing any phase of the field in which he is interested, unconsciously brings his students to a realization that what they are doing is part and parcel of a worth while project. Such an atmosphere provides an interest, and a stimulus to hard work, which can be achieved in no other way. The teacher who has Honors work under his direction cannot side-step his obligation to be actively interested in Productive Scholarship.

It is true that there are fields of influence open to the College teacher which are not directly related to instruction in his subject. There is the opportunity that College work offers for close association between teacher and student, with all that this implies. It provides a contact that can be used to arouse intellectual curiosity and ambition; to establish a hatred for prejudice, intolerance, and oppression; and to work towards setting up a keen sense of discrimination. Again, I am only repeating a part of what has often been said before. It is in such ways as these that a teacher renders his most important service. But whatever he accomplishes in this respect must be done incidentally. His influence is directly proportional to his own personal qualities of character and intellectual insight. And it is difficult to see how a teacher can keep his own intellectual tools at a keen edge, unless he continues to submit himself to the hard discipline that creative work imposes.

Come at this problem from any angle we please, we find that creative scholarly work is one of the obligations, perhaps the main obligation, of the undergraduate teacher. I do not see why there should ever be any question about this, or why College administrators should fail to see it in this light. But fail they often do, as one can easily convince himself by reading the literature on the subject, of which there is no end. Colleges are supported to provide an opportunity for young people to work hard and seriously towards their own intellectual development. But how can the students be expected to settle down to serious study, when the teachers, who are supposed to be their leaders, have long since ceased from any substantial intellectual effort? If there were no reason at all for our participation in creative work, other than the example of hard study that we set our students, this would be sufficient.

So far we have been dealing somewhat in generalities. There still remains the practical question as to whether or not conditions in our Colleges are such

that an active interest in productive work is at all possible. The answer to this is yes. It is true that one cannot always do the sort of work that one would like to do, nor so much of it. Teaching schedules are heavy; the demands on one's time in a social way are more exacting than in the larger centers, and it is neither wise nor desirable to side-step these demands. A very real difficulty is inadequate library facilities. But if such resources as exist are concentrated on one or two fields, a working nucleus is soon accumulated. It is frequently possible to get reprints of important papers. Authors are very generous in this respect. It might even be advisable to change ones interest to fit in with such library equipment as there is. This possibility was brought to my attention in an embarrassing way. In the hearing of the President I dropped a cryptic remark about the meagerness of our own mathematical collection. We were in the Library at the time. He made no reply, but casually picked up the three volumes of Whitehead and Russell's *Principia* and asked me if I had read them. There is never a sufficient reason for failing to keep up an interest in progressive scholarly work. There is always an opportunity for a beginning. If we avail ourselves of such opportunities as present themselves, more will develop as the need arises. Persistent effort over a period of years inevitably brings its reward of substantial achievement.

We can easily call to mind departments in small colleges in which worth while scholarly work is being accomplished. But these are the exception. There is no good reason why they should be the exception. If we believe the half of what we hear and read about our Colleges, there is not one of them that can in any sense be called a seat of learning. Yet observations over a period of years have convinced me that any College can be a seat of learning in a very real sense. It is not so much a matter of size and resources, as it is of attitude and emphasis. And these should be in the direction of creative work and Productive Scholarship, at any rate so far as the faculty is concerned. At present such things as organization, curriculum making, and class room procedure, hold the stage. These are important, and should be given adequate attention, but they should by no means be the main show.

While none of us, I hope, take either ourselves or our work with undue seriousness, yet there are occasional times when we wonder where we are going. Is our work of significance in the march of progress, or is it merely a lock-step in a racket? If one were to put in a sentence the function of a Liberal Arts College, he could probably do no better than this: *The Purpose of the Liberal Arts College is to provide a well rounded cultural background.* The trouble with such a statement is that it means something different to each of us. The things that it implies are largely the imponderables, and these are difficult to discuss in concrete terms. I am not at all sure that it is wise to try to discuss them. But for the sake of summing up, and perhaps emphasizing, what I have already said, I shall state what at any rate I think is my own attitude towards this question: *I believe that in the Liberal Arts College the faculty and students should work together, with a real interest centered in one way or another in creative work in the*

Sciences, in the Humanities, and in the Fine Arts. I believe that the result of such a group working together in this manner is that a way of working and living comes to be achieved which is in the highest sense cultural. I do not believe that there is any interest other than that in creative work which can serve as a unifying background and central purpose to accomplish this end.

IRREDUCIBILITY OF POLYNOMIALS

By H. L. DORWART, Williams College

Introduction. 1. A polynomial is said to be *reducible* in a given field if it is expressible as a product of polynomials with coefficients in this field, otherwise, it is said to be *irreducible.*

This paper will be concerned only with polynomials whose coefficients are rational integers and with reducibility in the rational field. In this field a well known theorem of Gauss shows, that if the highest coefficient is taken to be unity, then all factors may be supposed to have integral coefficients.

Kronecker* has given a general solution of the problem of completely factoring or decomposing a rational integral polynomial into the product of irreducible polynomials in a finite number of steps. In outline, his method is as follows.

Let $f(x)$ be a polynomial of nth degree whose coefficients are rational integers. If $f(x)$ is reducible, it will have a factor of degree $\leqq n/2$. Let s be the greatest integer $\leqq n/2$. We then seek a factor $g(x)$ of $f(x)$ of degree $\leqq s$.

Form the function values $f(b_0), f(b_1), \cdots, f(b_s)$ for $s+1$ arbitrary integral arguments $b_0, b_1, \cdots, b_s$. If $f(x)$ is divisible by $g(x)$, then $f(b_0)$ is divisible by $g(b_0)$; $f(b_1)$ is divisible by $g(b_1)$ etc. But $f(b_i)$ has a finite number of factors, hence there are only a finite number of possibilities for each $g(b_i)$. To every possible combination of the values $g(b_0), g(b_1), \cdots, g(b_s)$ there corresponds one and only one polynomial $g(x)$, which can be found by the aid of Lagrange's or Newton's interpolation formulas. Hence there will be a finite number of polynomials $g(x)$ which are possible factors of $f(x)$. By actual division, it can be determined whether or not any of these actually are factors.

It is evident that these calculations will usually be prohibitive in length. Furthermore, the knowledge of polynomials that is frequently desired is not the actual decomposition of the reducible ones, but an answer to the question whether a particular polynomial is reducible or irreducible. Consequently a simple test or criterion which would give this information is desirable. Unfortunately, no such criterion which will apply to all classes of polynomials has yet been devised; but a number of these tests, or *irreducibility criteria* as they are called, have been found which give valuable information for particular

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* Kronecker, Journ. f. Math. vol. 92 (1882), p. 11. [See also Runge, Journ. f. Math. vol. 99 (1886), p. 89; Molk, Acta math. vol. 6 (1885), p. 10; Mandl, Journ. f. Math. vol. 113 (1894), p. 252; Hancock, Ann. de l'éc. norm. (3) vol. 17 (1900), p. 89; Weber, *Algebra* vol. 2, p. 563; Koenig, *Algebraische Grössen*, p. 127.]

classes of polynomials. The remainder of this paper will be devoted to a summary of the results of a number of writers on this subject. It is not claimed that this summary is exhaustive, but it aims to give a general survey of the field.

The irreducibility criteria under discussion will be divided into three types: I) those depending on the divisibility properties of the coefficients, II) those depending on the comparative size of the coefficients, III) those depending on arithmetical properties of the values of the polynomial for integral arguments.

It should be noted that all of the criteria that will be mentioned form sufficient but not necessary conditions for irreducibility, i.e., a positive result from a test indicates an irreducible polynomial, but a negative result does not necessarily indicate a reducible polynomial.

I. *Criteria depending on the divisibility properties of the coefficients of the polynomial*

2. The earliest and probably best known irreducibility criterion is the Schoenemann*-Eisenstein† theorem:

If, in the integral polynomial

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

all the coefficients except a_0 are divisible by a prime p, but a_n is not divisible by p^2, then the polynomial is irreducible.

Simple proofs of this theorem are contained in such books as Dehn's *Algebraic Equations*, Hancock's *Algebraic Numbers*, etc., and will not be reproduced here.

An important application of this theorem is the proof of the irreducibility of the so-called "cyclotomic polynomial"

$$f(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + 1,$$

where p is a prime.

If, instead of $f(x)$, we consider $f(x+1)$, where

$$f(x+1) = \frac{(x+1)^p - 1}{(x+1) - 1} = x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + p,$$

the theorem is seen to apply directly, and the irreducibility of $f(x+1)$ implies the irreducibility of $f(x)$.

The Schoenemann-Eisenstein theorem has been generalized by Netto‡ and by Koenigsberger,§ and Koenigsberger's theorem has been generalized by

* Schoenemann, Journ. f. Math. vol. 32 (1846), p. 100.

† Eisenstein, Journ. f. Math. vol. 39 (1850), p. 166.

‡ Netto, Math. Ann. vol. 48 (1897), p. 81.

§ Koenigsberger, Journ. f. Math. vol. 115 (1895), p. 53.

Bauer* and Perron.† These results in most part are contained in the general investigations of Dumas.‡

The contribution of Dumas to this subject can best be understood in terms of the Newton polygons which Dumas associates with polynomials. Let the coefficients a_i of the polynomial

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n, \quad a_n \neq 0,$$

in general be divisble by p^{α_i}, and let the number pairs

$$(0, 0), (1, \alpha_1), \cdots, (n, \alpha_n)$$

be formed. Then suppose that the points represented by these number pairs are plotted in a rectangular coordinate system, omitting points for which $\alpha_i = 0$. For the points so plotted, a Newton polygon S, beginning at the origin and ending at the point (n, α_n), can be constructed such that all of the points will lie above or on S.

Let

$$S_1, S_2, \cdots, S_k$$

be the sides of S, and let

$$q_1, q_2, \cdots, q_k \quad \text{and} \quad r_1, r_2, \cdots, r_k$$

be the projections of these sides on the x and y axes, where

$$q_1 + q_2 + \cdots + q_k = n, \text{ and } r_1 + r_2 + \cdots + r_k = \alpha_n.$$

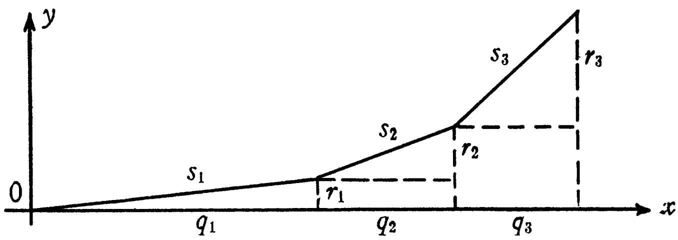

FIG. 1

Since the numbers q_i and r_i are rational integers, there is a greatest common factor e_i such that

$$q_i = e_i\lambda_i, \quad r_i = e_i\rho_i,$$

where λ_i is relatively prime to ρ_i. If the first side coincides with the x axis, $r_1 = \rho_1 = 0$, and in this case λ_1 is taken to be 1.

For these polygons, Dumas proves the theorem:

* Bauer, Journ. f. Math. vol. 128 (1905), p. 298.

† Perron, Math. Ann. vol. 60 (1905), p. 448.

‡ Dumas, Journ. de math. (6) vol. 2 (1906), p. 191.

The polygon of a product is composed of the polygons of the factors, where the sides are ordered according to increasing slopes.

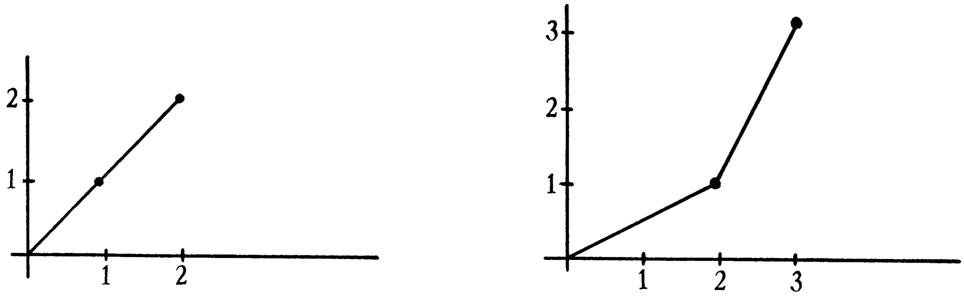

FIG. 2

For example, the factors x^2+6x+4, x^3-2x+8 have the polygons shown in Fig. 2 for the prime 2, which are combined in order of slopes to form the polygon shown in Fig. 3 for the product $x^5+6x^4+2x^3-4x^2+40x+32$.

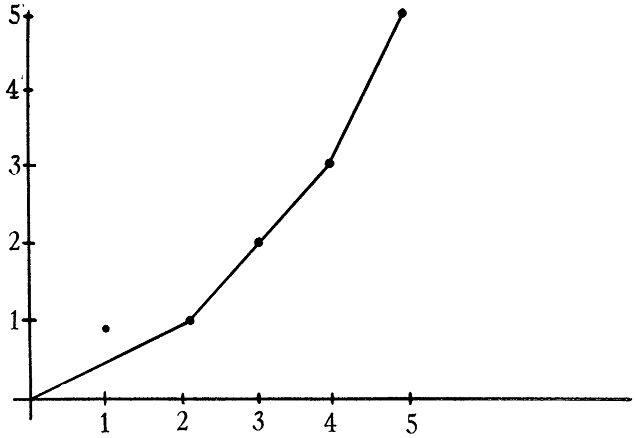

FIG. 3

From this theorem follows his irreducibility theorem:

The polynomial $f(x)$ can have only those factors of degree m for which m can be represented in the form

$$m = \sum_{i=1}^{k} t_i \lambda_i$$

where t_i denotes one of the numbers $0, 1, \cdots, e_i$.

For the Schoenemann-Eisenstein polynomials, the Newton polygons become the lines joining the origin to the point $(n, 1)$, and hence irreducibility follows at once.

In later papers, Bauer* has obtained generalizations of the Schoenemann-Eisenstein theorem in a different manner, and finally, Ore† has been able to expand Bauer's results and to show how all of these irreducibility theorems can be uniformly developed.

Recently Hopper and Ore,‡ in place of the open polygons of Dumas, have introduced a closed convex polygon, depending both on the size and divisibility properties of the coefficients. For this polygon, an approximate multiplication theorem holds, and this may be used to deduce irreducibility criteria dedepending on the size of the coefficients.

3. In a Mémoire presented to l'Académie des Sciences in 1865 and later incorporated in his *Algèbre supérieure*, Serret has developed a number of results concerning polynomials irreducible relative to a prime modulus.

If $f(x)=g(x)\cdot h(x)+p\cdot k(x)$, or $f(x)\equiv g(x)\cdot h(x)$, (mod p), where $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are integral polynomials in x and p is a rational prime, then $f(x)$ is said to be divisible by $g(x)$, (or by $h(x)$), relative to the modulus p. If, on the other hand, $f(x)$ is divisible by no integral polynomial other than itself or a constant (mod p), $f(x)$ is said to be irreducible (mod p). Irreducibility (mod p) of course implies ordinary irreducibility.

Among other theorems, Serret proves that

$$x^p - x + a, \quad a \not\equiv 0 \pmod{p}$$

is irreducible, and that

$$x^n - g$$

is irreducible for certain n and g. (A complete account of these results can be found in Serret's *Algèbre* or in Dickson's *Linear Groups*.)

A table of irreducible polynomials for the first four prime moduli§ has been compiled by Church.

4. Recently, Schur¶ has obtained the following results which have important applications.

Every polynomial of the form

$$f(x) = 1 + g_1\frac{x}{1!} + g_2\frac{x^2}{2!} + \cdots + g_{n-1}\frac{x^{n-1}}{(n-1)!} \pm \frac{x^n}{n!}$$

where the g_i are rational integers, is irreducible.

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* Bauer, Journ. f. Math. vol. 128 (1905), p. 87, vol. 132 (1907), p. 21, vol. 134 (1908), p. 15.

† Ore, Math. Zeitschrift vol. 18 (1923), p. 278, vol. 20 (1924), p. 267.

‡ Hopper and Ore, Bulletin of the Amer. Math. Soc., vol. 40 (1934), p. 216.

§ Church, Annals of Math., vol. 36 (1935), p. 198.

¶ Schur, Berl. Sitzungsber. (1929), p. 125 and p. 370

The class of polynomials to which this theorem applies includes the polynomials which are obtained by taking a finite number of terms of the expansions of cos x and of e^x, as well as the polynomials of Laguerre which are defined by the relations

$$\frac{e^x}{n!}\,\frac{d^n(x^n e^{-x})}{dx^n} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{x^i}{i!}$$

Schur proves the above theorem by generalizing a theorem of Tchebychef concerning prime numbers.

Other theorems obtained by Schur are:

For $n>1$, every polynomial of the form

$$f(x) = 1 + g_1\frac{x^2}{u_2} + g_2\frac{x^4}{u_4} + \cdots + g_{n-1}\frac{x^{2n-2}}{u_{2n-2}} \pm \frac{x^{2n}}{u_{2n}},$$

where the g_i are rational integers and where

$$u_{2j} = 1\cdot 3\cdot 5\cdots(2j-1),$$

is irreducible.

Likewise, in general, polynomials of the form

$$g(x) = 1 + g_1\frac{x^2}{u_4} + g_2\frac{x^4}{u_6} + \cdots + g_{n-1}\frac{x^{2n-2}}{u_{2n}} \pm \frac{x^{2n}}{u_{2n+2}}$$

are irreducible. Exceptions occur only if $2n$ is of the form 3^r-1 $(r\geqq 2)$, and in this case the only factor that $g(x)$ may have is $x^2\pm 3$.

In particular, these two theorems show that the polynomials of Hermite,

$$H_m(x) = (-\ 1)^m e^{x^2/2}\,\frac{d^m(e^{-x^2/2})}{dx^m}$$

for even degree $m>2$ are irreducible, and for odd degree, after removal of the factor x, are irreducible.

In general, every polynomial of the form

$$h(x) = 1 + g_1\frac{x}{2!} + g_2\frac{x^2}{3!} + \cdots + g_{n-1}\frac{x^{n-1}}{n!} \pm \frac{x^n}{(n+1)!}$$

is irreducible. The exceptions are

1) *If n is of the form 2^r-1 $(r\geqq 2)$, $h(x)$ may have the factors $x\pm 2$,*

2) *If $n=8$, $h(x)$ may be the product of two irreducible polynomials of degrees 2 and 6 respectively.*

From this last theorem, it follows that a finite number of terms of either of the series

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots, \quad \text{or} \quad \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots.$$

in general, form irreducible polynomials.

II. *Criteria which depend on the comparative size of the coefficients*

5. Perron* has found several interesting criteria of this type. He first notes that the algebraic equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

where the a_i are rational integers and $a_n \neq 0$ is irreducible if the absolute values of $n-1$ of its roots are less than 1. For if $f(x)$ can be decomposed, one of its factors must have roots whose absolute values are less than 1 while their product, which is equal to the constant term of the factor, is an integer and is therefore greater than or equal to 1. This contradiction can be removed only by conceding the irreducibility of $f(x)$.

From this simple principle, Perron is able to prove, with the aid of an auxiliary theorem, the following criterion.

If the coefficients of $f(x)$ satisfy the relation

$$|a_1| > 1 + |a_2| + |a_3| + \cdots + |a_n|$$

then $f(x)$ is irreducible.

Likewise from the observation that $f(x)$ is irreducible if $f(x)$ has a pair of conjugate complex roots and the $n-2$ other roots are in absolute values less than 1, he obtains the theorem:

If the coefficients of $f(x)$ satisfy the relation

$$\sqrt{a_2} \geqq 4^{n-1}(|a_1| + |a_3| + \cdots + |a_n|)$$

then $f(x)$ is irreducible.

The criteria of this type of Hopper and Ore have already been mentioned.

III. *Criteria which depend on arithmetical properties of the values of the polynomial for integral arguments*

6. An interesting paper by Stäckel† contains the following theorems.

For integral values of x, a reducible integral polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

* Perron, Journ. f. Math. vol. 132 (1907), p. 288. (Some of Perron's results are contained in his *Algebra*, vol. 2, p. 26.)

† Stäckel, Journal f. Math. vol. 148 (1918), p. 101.

of degree n can represent at most $2n$ prime numbers, and, as soon as the absolute value of the integer k exceeds a certain limit, $f(k)$ will represent only composite numbers.

If $f(x)$ is reducible,

$$f(x) = g(x) \cdot h(x),$$

$f(k)$ in general is a composite number, since $g(k) \cdot h(k)$ represents a prime number if and only if one of the factors is a prime and the other one is ± 1. But there are only a finite number of integral values of x for which one of these factors can be ± 1, namely the integral roots of the equations

$$g(x) = \pm 1, \quad h(x) = \pm 1.$$

In fact, the number of integral roots of these equations cannot exceed $2n$, since the sum of the degrees of $f(x)$ and $g(x)$ is n. As to the second part of the theorem, it is evident that as soon as the absolute value of k exceeds the absolute value of the greatest integral root of the equations

$$g(x) = \pm 1 \text{ and } h(x) = \pm 1,$$

$f(k)$ can no longer be a prime number but must be composite.

Among the integral numbers which can be represented by an integral polynomial $f(x)$, there are always infinitely many composite numbers, whether or not the polynomial is reducible.

There is no integral polynomial $f(x)$ which for integral x represents only prime numbers.

There are, however, irreducible polynomials which represent only composite numbers.

If an integral polynomial $f(x)$ represents infinitely many prime numbers it is irreducible.

For every integral polynomial $f(x)$ there is a positive integer S such that $f(x)$ is irreducible if there exists an integer k greater in absolute value than S for which $f(k)$ represents a prime number.

To see the truth of this theorem it is necessary only to observe, first, that there are only a finite number of integral divisors $g(x)$ of $f(x)$, and second, that the absolute value of the maximum integral root of all the equations

$$g(x) = \pm 1$$

furnishes a value for S.

Without the use of the fundamental theorem of algebra, Stäckel was able to obtain an expression for S depending only on the degree n and the absolute value of the maximum coefficient. Using the fundamental theorem, it is possible to obtain a much better value for S.

For example, Pólya and Szegö* prove that if the three conditions;

1) the zeros of $f(x)$ lie in the half-plane $Rx < k - 1/2$
2) $f(k-1) \neq 0$
3) $f(k)$ is a prime number

are satisfied, then $f(x)$ is irreducible. The following neat application is credited to A. Cohn.† Let

$$p = d_0 d_1 \cdots d_i \cdots d_n, \quad 0 \leqq d_i \leqq 9, \quad i = 0, 1, \cdots, n, \quad d_0 \geqq 1$$

be a prime in the decimal system. Then the polynomial

$$d_0 x^n + d_1 x^{n-1} + \cdots + d_{n-1} x + d_n$$

is irreducible.

This follows from the fact that for a polynomial of this type, all the zeros lie in the half-plane $Rx < 4$. Hence $k = 10$ satisfies the conditions above.

Possibly the chief interest in this application lies in the ease with which it enables one to construct examples of irreducible polynomials, e.g., since 2879 is a prime, $2x^3 + 8x^2 + 7x + 9$ is an irreducible polynomial.

Weisner‡ has proved the following theorem for the determination of irreducibility by leading and final coefficients.

Let L and M be lower and upper bounds, respectively, of the absolute values of the roots of a reducible polynomial

$$A(x) = \sum_{\nu=0}^{n} a_\nu x^{n-\nu} \qquad (n \geqq 2,\ a_0 \neq 0)$$

with integral coefficients, and suppose that

$$|a_n| = kp^m \qquad (k \geqq 1,\ m \geqq 1)$$

where p is a prime which does not divide a_{n-1} if $m > 1$. A. *If $L \geqq 1$, then $k \geqq L$.* B. *If $M \geqq 1$, then $p^m \leqq |a_0| M^{n-1}$.*

Polynomials which satisfy the hypothesis of this theorem but violate the conclusion are therefore irreducible.

A modification of this theorem for polynomials without linear factors is given in Weisner's paper and is illustrated by showing that the polynomials

$$x^n + x \pm p^m$$

are irreducible if p^m is any prime-power > 2.

Linear transformations are then made on the polynomials, and criteria are deduced whose sense is that, subject to certain conditions, a polynomial is irreducible if it is represented by the integer $\pm kp^m$, where k is relatively small. These criteria are well adapted to determining the irreducibility of numerical

* Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis.* II, page 350, theorem 127.

† Pólya and Szegö, l.c., page 351, theorem 128.

‡ Weisner, Bulletin of the Am. Math. Soc., vol. 40 (1934), p. 864.

polynomials and lead to the construction of large classes of irreducible polynomials which are discussed in other theorems.

7. The following general criterion has been found by Pólya.*

If for n integral values of x, the integral polynomial f(x) of degree n has values which are different from zero, and, without regard to sign, less than

$$\frac{\left(n - \left[\frac{n}{2}\right]\right)!}{2^{n-[n/2]}}$$

then f(x) is irreducible.

Applied to polynomials which take the values ± 1 n times, this theorem says that such polynomials are irreducible for $n>7$.

By a different method, Pólya proves that a polynomial $f(x)$ of odd degree n for which $|f(x)| = p$ for n integral values of x is irreducible if $n \geqq 17$, and that such a polynomial of even degree $n \geqq 17$ may have only two irreducible factors of equal degree.

A number of other writers have obtained results of this type. References to their papers are contained in the introduction to a paper by Dorwart and Ore,† In this paper, new methods are introduced whereby not only most of the results of the earlier writers are obtained in an extremely simple and natural manner, but new results are found, and former results in most cases extended or generalized.

Dorwart and Ore first determine all polynomials which take the values ± 1 more times than their degrees. They find the theorem:

A polynomial f(x) of n-th degree which takes the values ± 1 for more than n integral values of x has one of the following forms

			x	0	1	2	3
(1)	$\pm P_1(x)$	$= x(x-1)(x-3)+1$	values:	1,	1,	-1,	1
	$\pm P_2(x)$	$= (x-1)(x-2)-1$	"	1,	-1,	-1,	1
	$\pm P_3(x)$	$= 2x(x-2)+1$	"	1,	-1,	1	
	$\pm P_4(x)$	$= 2x-1$	"	-1,	1		
	$\pm P_5(x)$	$= x-1$	"	-1,		1	

or f(x) is equivalent to one of these polynomials, i.e., it can be derived from them by a substitution $x' = \pm x + \alpha$.

From this theorem, the following two theorems are quickly obtained.

A polynomial f(x) of n-th degree taking the values ± 1 for the m integral arguments $a_1, a_2, \cdots, a_m$, where $4 < m \leqq n$, can have factors only of the form

* Pólya, Jahresber. Deutsche Math. Ver., vol. 28 (1919), p. 31.

† Dorwart and Ore, Annals of Math. vol. 34 (1933), p. 81, vol. 35 (1934), p. 195.

$$g(x) = (x - a_1) \cdots (x - a_m)h(x) \pm 1.$$

The degree of a factor is therefore never less than m, and when $m > n/2$, $f(x)$ *is irreducible.*

The polynomials

$$f(x) = a(x - a_1) \cdots (x - a_n) \pm 1$$

are always irreducible except when $f(x)$ *is equivalent to one of the following three polynomials*

$$x(x-1)(x-2)(x-3) + 1 = [x(x-3)+1]^2,$$

$$x(x-2) + 1 = (x-1)^2, \quad 4x(x-1) + 1 = (2x-1)^2.$$

Generalizations of these results are made, and then similar methods are applied to polynomials taking prime values. The following theorems result.

Let p be a prime and let $f(x)$ *be a polynomial taking the value* $+p$ $(-p)$ *for m different integral values of x. For* $m > 4$, *the polynomial* $f(x)$ *cannot take the value* $-p$ $(+p)$ *for any other integral x.*

Let $f(x)$ *be a polynomial taking the values* $\pm p$ *for the m integral arguments* $a_1, \cdots, a_m$. *For* $m > 5$, $f(x)$ *must take the same value* $+p$ *or* $-p$ *for all* a_i *and consequently have the form*

$$f(x) = (x - a_1) \cdots (x - a_m)h(x) \pm p. \tag{2}$$

A polynomial of the form (2) *for* $m > 10$, *can have factors only of degrees* $\geqq m/2$.

A polynomial of the form

$$f(x) = a(x - a_1) \cdots (x - a_n) \pm p, \tag{3}$$

for $n > 6$, *is irreducible if n is odd, and if n is even, it may have only two factors of the degree* $n/2$.

The problem of determining actual cases in which a polynomial (3) with an even n may be reducible has been considered by Brauer.*

Dorwart† has shown that the determination of such reducible polynomials of even degree is equivalent to the solution of a special case of the problem of Equal Sums of Like Powers of diophantine analysis, that is, it involves finding the distinct, integral solutions of the system

$$\sum_{i=1}^{n/2} a_i^k = \sum_{j=n/2+1}^{n} a_j^k, \quad k = 1, 2, \cdots, n/2 - 1.$$

Partial solutions for this problem exist and these in turn give decompositions for the polynomials. Also, this new formulation seems to indicate that these decompositions exist for arbitrarily high n. The following examples of degrees 8, 10 and 12 involve the smallest primes that have been found for which polynomials of these degrees are reducible.

* Brauer, Jahresber. Deutsche Math. Ver., vol. 43 (1933), pp. 124–129.

† Dorwart, Duke Math. Journal, vol. 1 (1935), p. 70.

$$x(x-1)(x-2)(x-4)(x-7)(x-9)(x-10)(x-11)+179$$
$$= [x(x-4)(x-7)(x-11)+179][(x-1)(x-2)(x-9)(x-10)-179].$$
$$(x^2-1)(x^2-5^2)(x^2-7^2)(x^2-8^2)(x^2-9^2)+5039$$
$$= [(x-1)(x-5)(x+7)(x+8)(x-9)$$
$$+ 5039][(x+1)(x+5)(x-7)(x-8)(x+9)-5039].$$
$$(x^2-1)(x^2-5^2)(x^2-6^2)(x^2-9^2)(x^2-10^2)(x^2-11^2)+100799$$
$$= [(x^2-1)(x^2-9^2)(x^2-10^2)$$
$$- 100799][(x^2-5^2)(x^2-6^2)(x^2-11^2)+100799].$$

Ore* has generalized the above results for polynomials taking a fixed prime value a certain number of times to those taking any prime values and also has determined the exact number of prime values which a reducible polynomial may take. He introduces the term *regular* for polynomials which do not have factors of the forms (1). For regular polynomials, Ore finds the theorem:

A regular polynomial $f(x)$ of n-th degree which takes the values ± 1 s times and prime values t times can be decomposed at most in r factors, where

$$r \leqq \frac{n+t}{s+t}.$$

If $2s+t>n$, $f(x)$ is irreducible.

As a special case, he finds the theorem:

A regular polynomial of n-th degree which takes n prime values including ± 1, can at most be decomposed in two factors, and if it takes $n+1$ prime values it is irreducible.

Removing the restriction of regularity, he finds the following theorems.

Any polynomial $f(x)$ which takes s values ± 1 and t prime values is irreducible if

$$2s+t>n+4.$$

This theorem says that a polynomial with $n+5$ prime values is irreducible. In general, this is a much better value than the one, $2n+1$, given by Stäckel. However, further investigations give still better results.

A polynomial taking $n+3$ prime values is always irreducible, unless it is of 4th or 5th degree and of the form

$$f(x) = P_2(\pm\, x+\alpha)P_i(\pm\, x+\beta), \quad i = 1, 2.$$

A polynomial taking $n+4$ prime values is irreducible unless it is of 4th degree and of the form

$$f(x) = P_2(\pm\, x+\alpha)P_2(\pm\, x+\beta).$$

A polynomial taking $n+5$ prime values is irreducible.

* Ore, Jahresber. Deutsche Math. Ver., vol. 44 (1934), p. 147.

Existence of these exceptional polynomials is shown by examples. The reducible polynomial of 5th degree

$$f(x) = [(x-1)(x-2)-1][(x-7)(x-5)(x-4)+1]$$

takes only prime values for the eight values $x=0, 1, \cdots, 7$, and the reducible polynomial of 4th degree

$$f(x) = [(x-1)(x-2)-1][(x-5)(x-6)-1]$$

has the same property.

For the determination of the irreducibility of polynomials from the taking of a single value, Ore has the theorem:

Let $f(x)$ be an integral polynomial, k an integer, and $t>1$ a real number such that no zeros of $f(x)$ lie in the circle $|x-k| \leqq t$. If then $f(k)=rp$, where p is a prime and $|r|<t$, then $f(x)$ is irreducible.

Various applications and extensions are given or indicated which are useful for the establishment of irreducibility through numerical calculations.

And finally, Weisner* has considered polynomials of degree n which assume the same value k, where k is *any* integer $\neq 0$, for n distinct integral values of x. Such polynomials are evidently equivalent to

$$f(x)=ax(x-t_1)\cdots(x-t_{n-1})\pm k,$$

and for them, Weisner proves the following theorem.

The polynomial $f(x)$ is irreducible if at least one of the n inequalities

$$a>2^nk^2+1, \qquad t_i>(3+\lambda)k \qquad (i=1,\cdots,n-1)$$

is satisfied, where $\lambda=\lambda(n)$ is defined by

$$\lambda(2)=1,\quad \lambda(3)=4,\quad \lambda(4)=6,\quad \lambda(5)=3,\quad \lambda(6)=1,\quad \lambda(n)=0 \textit{ if } n\geqq 7.$$

Weisner concludes his paper with the observation that the inequalities of this theorem can undoubtedly be weakened by further analysis, without affecting the irreducibility of $f(x)$, but that they suffice to establish the following general theorems.

Only a finite number of non-equivalent reducible polynomials of degree n exist which assume a given integral value $\neq 0$ for n different integral values of the variable.

If k is a fixed integer $\neq 0$, and n is sufficiently large, every polynomial of degree n which assumes the value k for n distinct integral values of its argument is irreducible.

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* Weisner, Bulletin of the Am. Math. Soc., vol. 41 (1935), p. 248.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

GAUSS AND THE FRENCH ACADEMY OF SCIENCE

By ARNOLD EMCH, University of Illinois

1. In the following note I shall present some facts concerning the scientific activity of Gauss and its recognition by the French Academy which throw a peculiar light upon a statement in Ball's (*A Short Account of the History of Mathematics*), 5th edition (1912), p. 448, and which reads as follows:

"His *Disquisitiones Arithmeticae* appeared in 1801. A large part of this had been submitted as a memoir to the French Academy in the preceding year, and had been rejected in a most regrettable manner; Gauss was deeply hurt, and his reluctance to publish his investigations may be partly attributable to that unfortunate incident."*

This is probably the source upon which Professor G. M. Watson relied in his address to the Mathematical Association in 1933, as published in The Mathematical Gazette, Vol. xvii, 1933, p. 16:

"It is well known that the reluctance of Gauss to publish his discoveries was due to the rejection of his *Disquisitiones Arithmeticae* by the French Academy, the rejection being accompanied by a sneer which as Rouse Ball has said, would have been unjustifiable even if the work had been as worthless as the referees believed. It is the irony of fate that, but for this sneer, the (*Traité des fonctions elliptiques*), the work of a Frenchman, might have assumed a different and vastly more valuable form, and Legendre might have been spared the pain of realizing that many years of his life had been practically wasted, had the method of inversion come to be published when Legendre's age was fifty instead of seventy-six."†

2. These are very unusual and strong statements so that the writer was naturally curious to learn the source of Watson's information. A letter of inquiry brought no reply. A careful examination of Gauss's collected works and existing biographical material showed no evidence whatever of such an event; and Professor Martin Brendel of the University of Freiburg, who is at present in charge of the Gauss "Nachlass" (unpublished papers and letters) and working on a much needed competent biography of Gauss, writes to me that he knows of no such occurrence in Gauss's life, or evidence in the Gauss Archive. Likewise, a scrutiny of the publications of the French Academy before and after 1800 in the

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* In this connection see the question published in Scripta Mathematica, vol. 3 (1935), p. 98. EDITOR.

† The opinion that Watson might have gotten his information from Ball's history was expressed to me by Professor Archibald of Brown University.

University of Illinois library, showed absolutely no evidence for Ball's dubious assertion. To make sure that I had not missed anything from the French sources I wrote to Professor Émile Picard, perpetual (permanent) secretary to the French Academy of Science, who was kind enough to accompany his answer by an official transcript from the records of the Academy pertaining to its relation with Gauss.

The only reference to the scientific activity of Gauss at that time was made by Legendre in the session of Jan. 26, 1802, where he reported on a geometrical discovery by Gauss published in the *Disquisitiones Arithmeticae*. No other comment whatsoever is made.

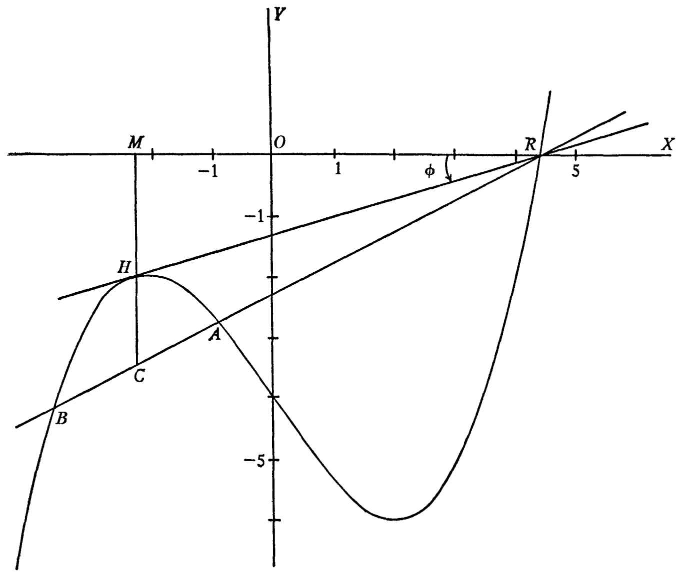
Instead of being rejected by the French Academy, Gauss, as Picard writes, at a relatively early age received full recognition for his investigations. The election of Gauss as a corresponding member took place on Jan. 30, 1804, when he was 27 years of age. Five years later, in 1809 he received the Lalande prize for the theory of planets and, finally in 1820, Gauss, when 43 years of age, was elected as a foreign associate, the highest honor possible for a non-Frenchman.

In conclusion Professor Picard rejects any insinuation that the great Gauss had at any time been unfairly treated by the French Academy of Science. The record is clear and all in favor of Gauss.

The Graphical Interpretation of the Complex Roots of Cubic Equations

By Garcia Henriquez, Santo Domingo, Rep. Dominicana

The following method for interpreting graphically the complex roots of cubic equations, while possibly not new, may be of interest.



The figure represents a cubic with one real root. (The curve is drawn for the equation $8y=x^3-12x-32$.) From the intersection, R, of the curve with the X-axis, a tangent RH is drawn to the cubic. The abscissa, OM, of H is the real part of the complex roots. The absolute value of the imaginary parts is the square root of tan $\phi=\tan ORH$. The tangent required may be easily constructed by drawing any secant, RAB, through R, intersecting the curve in A and B. An ordinate through C, the midpoint of AB, cuts the curve in H, the point of tangency.

Proof. The cubic with roots r, $a\pm ib$, where r, a and b are real, has for its equation

$$f(x) = (x - r)(x^2 - 2ax + a^2 + b^2) = 0.$$

Any secant through R is

$$y = m(x - r),$$

which intersects the curve in two additional points given by

$$x^2 - 2ax + a^2 + b^2 - m = 0,$$

i.e., in the points whose abscissas are

$$a \pm \sqrt{m - b^2}.$$

The secant will be a tangent if $m=b^2$. In this case the abscissa of H is a, as required. The slope of RH is b^2, which is the square of the imaginary coefficient of the complex roots.

Editorial Note. As this issue is about to go to press, Professor C. F. Barr of the University of Wyoming calls our attention to the fact that this same construction was given in this MONTHLY, vol. 25 (1918), p. 268, and in the Annals of Mathematics, vol. 19 (1917), p. 157. Professor Barr points out the further interesting fact that if the slope of the line RAB is twice that of the line RH, then the distance from A (or B) to the line MC will be b.

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Calculus of Variations in the Large. By Marston Morse. Colloquium Publications of the American Mathematical Society, volume XVIII. New York, 1934. x+368 pages. \$4.50.

Fourier Transforms in the Complex Domain. By the late R. A. C. Paley and Norbert Wiener. Colloquium Publications of the American Mathematical Society, volume XIX. New York, 1934. viii+184 pages. \$3.00.

Differential Geometry. By W. C. Graustein. New York, The Macmillan Company, 1935. xii+230 pages. $3.00.

Dynamic Economics. Theoretical and Statistical Studies of Demand, Production, and Prices. (Monographs of the Cowles Commission for Research in Economics, No. 1.) By C. F. Roos. Bloomington, Indiana, The Principia Press, 1934. xvi+275 pages. $3.50.

Erreurs des Mathématiciens des Origines à Nos Jours. By Maurice Lecat. Bruxelles, Ancienne Librairie Castaigne, 1935. xii+166 pages.

Esquisse du Progrès de la Pensée Mathématique. Des Primitifs au IX[e] Congrés international des Mathématiciens. By J. Pelseneer. Paris, Hermann et Cie. 1935. 160 pages.

Metodi Matematici. Essenza, tecnica, applicazioni. By Gino Loria. Milan, Ulrico Hoepli, 1935. xii+375 pages. 20 lire.

Business Mathematics. By I. L. Miller. New York, D. Van Nostrand Company, 1935. xii+376 pages. $3.50.

College Algebra. By W. B. Ford. Third Edition. New York, The Macmillan Company, 1935. viii+304 pages. $1.90.

(According to the preface of this new edition of a well-known text, reviewed in this MONTHLY by J. H. Weaver, 1928, page 22, the changes are chiefly in the entire reconstruction of the exercise lists.)

Brief Analytic Geometry. By T. E. Mason and C. T. Hazard. Boston, Ginn and Company, 1935. xii+196 pages. $2.00.

(This is an abridgment of the *Analytic Geometry* of Carmichael, Mason and Hazard, reviewed in this MONTHLY by A. D. Campbell in 1928, page 195. The present edition is adapted to a schedule of fifty to sixty lessons.)

REVIEWS

Methodische Einführung in die Höhere Mathematik. By K. Reinhardt. Leipzig, B. G. Teubner, 1934. iv+270 pages. Rm. 14.

The author believes that a youth can comprehend and understand the idea of finding the area between a curve and the X-axis much easier than he can assimilate the notion of the slope of a tangent to the curve. For the German schoolboy has made determinations of area—such as the quadrature of the circle—in his secondary school training long before he has taken up the measurement of the slope of tangent lines. On account of this belief Professor Reinhardt has presented us with a book in which integral calculus is developed first, completely independent of differential calculus; in fact the latter appears simply as the inverse operation of the former.

The text has been planned for use by the German youth when he first arrives at the University and to form a connecting link with his previous mathematical preparation. Although it represents an introduction to the calculus, it cannot be compared with our ordinary calculus texts; the American youth would

find difficulty in studying it before his junior or senior year as it contains many topics which appear here in courses on the theory of functions of a real variable. The choice of material includes not only the integration of the ordinary simple functions, but contains also the derivation of the fundamental rules for manipulating integrals, such as the methods for introducing new variables and for integrating a product of two functions. Lack of space confines the author to limit his discussion to the study of the functions of a single variable; nevertheless among its pages we find a proof of the irrationality of π and a chapter on the fundamental law of algebra.

The method of procedure is illustrated in the opening chapter where the are a between the parabola $y=cx^2$ and the X-axis, between the limits $x=0$ and $x=a$, is found by evaluating the sum of a series and passing to its limit. The area under one arch of the sine curve is derived in a similar manner. Thus the geometric approach to the subject is utilized; the text is illustrated with 131 figures. Succeeding chapters deal with series, the conditions for their convergence, the limits of the sum, difference, product and quotient of series, rational and irrational numbers. In chapters IX and XI we find a discussion of continuous functions of a real variable including the notions of a maximum and a minimum. In alternating chapters the author introduces all the necessary curves and, by a treatment similar to the one in chapter I, determines the area under them so that he has available for later use material which gives him the integral of all the ordinary simple functions.

The connection between an integral and the limit of a sum is made in chapter XII. Then follow polar coordinates, definite and improper integrals, rules for changing limits, for introducing other variables, etc. The derivative makes its appearance in chapter XIX as the inverse of integration and its geometric interpretation is stated. The derivatives of the ordinary functions are listed, followed by Rolle's Theorem and the Mean Value Theorem. The indefinite integral and the fundamental theorem of integral calculus appear in chapter XXI. After a discussion of Maclaurin's Theorem and power series comes the subject of infinite continued fractions and a development of tan x as a continued fraction. The final chapters deal with Fourier Series, Taylor's Theorem and certain topics of algebra already mentioned.

The exercises, 123 in number, placed at the end of the chapters, are not for the purpose of drilling the student in the use of the developed theory, but serve to amplify and supplement that theory.

If at times the aesthetic beauty of the book seems slightly marred by long, algebraic manipulations, as in chapter XXIII, the author has succeeded in making integral calculus a separate entity. It is consistent and sufficiently rigorous in treatment; it is a worthwhile experiment. Perhaps the reviewer has lived too long in the middle west where the rank and file of American youths who appear in our classrooms are products of modern educational methods and know little mathematics, but in reading this book shortly after hearing Professor Cairns at Williamstown one realizes only too sadly how much we lag behind the foreign

trained student. While our students cannot read this book, is it too much to hope that our teachers of calculus will do so?

J. R. MUSSELMAN

Einführung in die Theorie der Zähen Flüssigkeiten. (No. 10 of the Mathematik in Monographien und Lehrbüchern.) By Wilhelm Müller. Leipzig, Akademische Verlagsgesellschaft, 1932. x+367 pages.

This book is an important addition to the literature of Hydrodynamics. It brings together in one volume a detailed discussion of those cases of the flow of a viscous fluid which have so far received solution and which can be classified according to the Reynolds law of similitude. The author presupposes a familiarity with differential equations, including certain special equations, such as Bessel's, some knowledge of complex variables and mapping, and an acquaintance with the theory of ideal fluids. The work is characterized by the generous use of intuitive methods. As evidenced by the large number of pictures and graphs, the author has been at great pains to show how the mathematical developments are connected with physical notions and the results of technical research.

The first chapter is devoted to the fundamental equations and important theorems for a viscous fluid. This is followed by several shorter chapters which lead the reader by gradual stages from the simplest applications to those of a more complicated character. Chapters VII and VIII discuss viscous flow around obstacles for small and large Reynolds number respectively The contents of these chapters are derived mainly from the researches of Oseen and Prandtl respectively. The author then devotes two chapters to the rotation of bodies in viscous fluids. This is followed by an exposition of the Oseen asymptotic theory. This chapter is necessarily condensed and hard to read. One does better to consult Oseen's Hydrodynamics (No. 1 of the same series). The final chapter gives a survey of the phenomena associated with turbulence.

The misprints I have noted are too numerous to be listed here. While most of them are evident, there are some which can scarcely fail to cause confusion. The Gibbs vector notation is a fruitful source of error. For example, equation (9) on page 26 reads

$$\mathbf{r} \times \nabla \cdot \mathbf{v}\mathbf{v} = \nabla \cdot \mathbf{r} \times \mathbf{v} \cdot \mathbf{v}.$$

The left side is ambiguous. Actually $\mathbf{r}\times(\Delta \cdot \mathbf{v}\mathbf{v})$ is intended. The right side is meaningless. It should be $\nabla \cdot (\mathbf{v}\mathbf{r}\times\mathbf{v})$. We are convinced that most of the first chapter would gain in clearness as well as in freedom from error if the methods of modern vector analysis were used. Such detailed criticism cannot, however, detract from the essential value of the book. For anyone seriously interested in Hydrodynamics the book is indispensable.

C. A. SHOOK

Statics. By A. S. Ramsey. Cambridge, The University Press, 1934. xii+296 pages. $3.00.

This is written as a companion volume to the author's book on Dynamics.

Starting with the idea of force as an intuitive conception, the composition of forces, moments, couples, and conditions of equilibrium are treated in the usual way. This is followed by discussions of bending moments, graphical statics, friction, centers of gravity, work, equilibrium of cables, and the deflection of beams. Most of the analysis is algebraic but integration is used when necessary, and in the discussion of cables and beams simple differential equations appear. Although the conditions of equilibrium are given in general form at the beginning, most of the applications are two-dimensional. The book contains nearly 500 examples. Vector ideas appear in almost every section, but vector analysis is not used. In the earlier chapters some confusion might result from the use of the same notation for scalar and vector. Thus the components of a vector are defined as vectors, yet the components of a vector P are given as $P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$, the letter P being in each case in bold faced type.

H. B. PHILLIPS

Aristotle, Galileo, and the Tower of Pisa. By Lane Cooper. Cornell University Press, 1935. 102 pages. $1.50.

This interesting contribution to the history of mathematics and science is unique in several respects, notably in its excellent scholarship. The author is not a mathematician nor an historian, but a professor of the English language and literature at Cornell University. The book demands our further attention inasmuch as it is the first volume to be formally accepted for publication by the Council of the Cornell University Press. But its chief merit lies, as aforesaid, in the profound erudition and painstaking scholarship of its author.

In brief, Professor Cooper explodes the myth that Galileo, having ascended the leaning tower at Pisa, by a single dramatic experiment refuted an assertion of Aristotle that had not been challenged since the days of ancient Greece, and which is generally, although incorrectly, given as follows: If two different weights of the same material were let fall from the same height, the heavier would reach the ground sooner than the lighter in the proportion of their weights. Out of a confusion of data, and with great care, the author methodically assembles the evidence showing how the now famous story came into being and upon what a slender basis it rests. Among other unexpected revelations, it would appear that, contrary to general belief, Galileo did not experiment with falling bodies from the Pisan Campanile; that, moreover, Galileo himself was inconsistent in his writings; that Stevin and other earlier European scientists did their experimenting and criticized Aristotle before Galileo had broken with the Aristotelian tradition on the alleged basis of a single "epoch-making" experiment; and that many modern writers have unwittingly or inexcusably contributed to the further development of the legend, including such well-known authorities as Dampier-Whetham, Fahie, Ivor Hart, F. L. Darrow, and others.

Several additional illuminating factors bearing on the history of the myth are also brought to light, to wit: the possible vitiating effect of the Latin influence on the traditional views regarding "free fall" as discussed during the

Renaissance; a number of vague and conflicting passages from Galileo's writings alluding to some experiments which he "repeatedly made"; some evidence indicating that Aristotle, too, appears to have made experiments; and finally, the strange assertion made by Galileo in his early work *De Motu* that "in free fall wood starts off more quickly than lead." The last forty pages or so are devoted to a collection of original passages in Latin, Greek, Italian, German and English, all bearing on the problem in hand, and coming from the pens of such illustrious historical figures as Lucretius, Leonardo da Vinci, Jerome Cardan, Simon Stevin, Vincenzio Viviani, and of course, a number of relevant passages from Aristotle and Galileo. All in all, the book is well annotated, its style dignified, its scholarship irreproachable, and its content unusual.

W. L. SCHAAF

Introduction to Theoretical Physics. (International Series in Physics, F. K. Richtmyer, editor.) By J. C. Slater and N. H. Frank. New York, McGraw-Hill Book Company, 1933. xx+576 pages. $5.00.

This text ranges over almost the entire field of mathematical physics, as is indicated by the following list of important concepts which are treated: simple, damped and forced oscillators, scalar and vector potentials, Lagrange's and Hamilton's Equations of Motion, phase space, vibrating string and vibrating membrane, elasticity, fluid mechanics, flow of heat, electricity and magnetism, electrons and wave mechanics. The aim of the authors was to give a unified course in the introduction to the study of the methods of mathematical physics, combined with a detailed treatment of the structure of matter from the modern point of view.

The authors have, on the whole, succeeded in writing a very interesting and useful text. To college teachers of mathematics who are interested in students who are going on into work in theoretical physics, it is of interest to note that the authors presume that the student is familiar with the calculus and the ordinary elementary methods of solving differential equations. The starting point in the present text is that of power series representation of functions, and their determination from differential equations whose coefficients are analytic except for poles. In particular, the linear equation with constant coefficients is thus treated and the usual exponential and trigonometric solutions are obtained as series and then identified by means of their Taylor's expansions. In connection with the use of series to solve differential equations, a brief discussion of convergence of power series is given.

While much of the physics is from the ultramodernistic point of view, (with the exception that all mention of relativity is omitted) it is to be regretted that more physicists do not talk the language of modern mathematics. For example, it is quite apparent that the use of the term function is confined entirely to that of elementary function or at least in the sense of a single simple formula. Modern tensor analysis should be given more prominence, and there is every

reason to be advanced for giving more attention to the importance of the minimum principle with some attention to the beginnings of the calculus of variation. The reviewer is committed to the advantages of using experimental methods in mathematics, but that should go deeper than the older heuristic and intuitional methods which belong to the nineteenth century.

Finally, it should be stated that no small part of the usefulness of this book as an excellent text is due to the splendid set of problems at the end of each chapter. These, numbering almost uniformly ten to the chapter, are of the type which are calculated to broaden and deepen the student's knowledge of the subject and are not mere numerical substitutions into formulas developed in the text.

H. J. ETTLINGER

Correlation and Sampling. By W. D. Baten. Ann Arbor, Edwards Brothers, 1934. 57 pages. $1.00.

This book, a photo-lithograph of the author's manuscript, is designed for a second semester's course in Statistics. The student, to be equipped for such a course, needs a knowledge of the mean, standard deviation and skewness of a frequency distribution, and of how to use a table of areas for the normal curve. No calculus is used and there are very few occasions where its use would be convenient.

Chapters 1, 2 and 5 contain the material on correlation. The first two chapters are excellent. The subject is developed logically and concisely. The normal equations for a set of observational equations are proved to give a least square solution for the set. The predicting equation—the best fitting equation in the least square sense—is found for any frequency distribution. The standard error of prediction, or standard error of estimate, is defined and then used to define the coefficient of correlation. Chapter 2 ends with a generalization of correlation for n variables and the general formula in determinant form. This method of treating r emphasizes most effectually its meaning as a measure of the accuracy with which one measurement of an individual may be predicted from other of its measurements. Partial correlation and the correlation ratio, η, are defined by further generalizing the definition of r. Only one minor criticism is possible for the first two chapters:—the method of calculating r for a two dimensional table does not have the neatness of other methods where all results are tabulated in the margins instead of the body of the table, but the calculations are neither more laborious nor slower. Chapter 5 on non-linear regression is too brief to be satisfactory. No method is given by which η may be calculated for a two dimensional table. Indeed the student may suppose that η cannot be calculated unless the relation between the variables is known. There is no discussion of the value or lack of value of r, in cases of non-linear regression.

Chapter 4 discusses two problems of sampling: given a finite or infinite population for which the mean, standard deviation and skewness are known, to find the probability that the mean of a sample will lie within certain limits, given several samples from an unknown population to determine if the samples are

consistent. The first problem is solved satisfactorily if the curve of means is a normal curve. The second problem is solved by using the standard deviation of each sample as the mean of the standard deviations of all possible samples and from this assumption deriving a formula expressing the standard deviation of the curve of means in terms of the standard deviations of each sample. The test of consistency is the comparison of the standard deviations of the curve of means so derived. There is no discussion of the theory of probability, probability in all the problems in which it is used being determined by reference to a table of normal areas. Chapter 5 contains discussions of the probable error of a sum and the significance of the difference between the means of two samples from the same population.

The author presents most of his material so well that one wishes more had been included, but there seems to be sufficient for a semester's work. The supply of problems will not be exhausted by the most exorbitant demands—they make up, in fact, about one third of the entire contents of the book.

L. T. MOORE

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

1933–1934

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and in the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of Washington University

Greetings and best wishes to all the chapters: The Missouri Beta Chapter is happy to report a very successful year under the following officers, elected on May 13, 1933: W. O. Pennell, Director; S-Marie Vaughn, Vice Director; Jessica Young Stephens, Secretary; Charles A. Huff, Assistant Secretary; William E. Stephens, Treasurer; Fern M. Oestereich, Librarian; Annie Meroe Burnet, Cecilia Lehmann, Viola Muench, and William Roa, Student Members of the Executive Committee.

There were forty-one active members during the year 1933–1934. Twenty new members were initiated on April 14, 1934 who were distributed as follows: from the College of Liberal Arts—four; from the Schools of Engineering and Architecture—ten; from the School of Graduate Studies—four; and two Honorary members.

Nine meetings were held during the year as follows:

October 11, 1933: "Linkages" by Dr. J. J. Quinn: "Systems of differential equations" by C. M. Fixman.

November 7, 1933: "Television" by Mr. Dubel; "Numerology" by S. Littman.

Decmeber 4, 1933: "Number machines" by Dr. H. P. Lawther; "Getting together"—parts that compose a telephone and how they go together—a movie; "The family album"—showing several new inventions which are related to the telephone; "What country please?"—showing the extent to which overseas telephone communication has developed.

January 10, 1934: "Transfinite numbers" by J. A. Joseph.

February 6, 1934: "Determination of Avagadro's numbers" by A. H. Baum.

February 20, 1934: "Office methods for calculating life insurance reserves" by Harry Sarason.

March 5, 1934: "The tonoscope" by A. M. Razovsky.

March 15, 1934: Election of new members.

April 5, 1934: "Some seismological problems and their solutions" by Dr. J. B. Macelwane.

April 14, 1934: "Eighty thousand miles on a sailing ship" by Professor H. R. Grummann; Initiation and banquet.

May 12, 1934: Annual business meeting and social gathering. Report of the Stephens prize committee that no prize is to be awarded for the academic year 1933–1934; Treasurer's report and the election of officers for the academic year 1934–1935.

HILDA KOHM, *Secretary*

Honorary Science Club of Washington and Jefferson College

The meetings and programs of Phi Chi Mu, our honorary science club, were as follows:

December 13, 1933: "Historical attempts at the quadrature of the circle" by W. F. Sayenga.

January 14, 1934: "Philosophy of mathematics" by M. Korol.

February 18, 1934: "Gaussian logarithms" by R. A. Wylie.

March 13, 1934: "Illustrations of functions" by J. P. Knestrick.

April 10, 1934: "Geometric methods of finding roots" by J. L. White.

April 24, 1934: "History of complex numbers" by W. F. Pringle.

May 18, 1934: "Brief history of geometry" by J. N. Montgomery; "Geometric expression of arithmetic and algebra" by W. A. Schan.

WILLIAM F. SAYENGA, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 159. *Proposed by F. A. Lewis, University of Alabama.*

If N is a positive integer composed of n successive 9's, and M is any positive integer less than $N+2$, then the sum of the digits in the product MN is $9n$.

E 160. *Proposed by Daniel Finkel, Brooklyn, New York.*

If v, w, x, y and z are different digits, and $P=vwx$, $Q=wxv$, $R=xyz$, $S=zxy$

[note that these are not products], and if further, $4R=P$, $3R=Q$, $3S=P$; determine the values of the letters and show that the solution is unique.

E 161. *Proposed by W. B. Clarke, San Jose, California.*

Construct a triangle whose circumcenter lies on its orthic triangle.

E 162. *Proposed by J. M. Feld, New York City.*

Factor as far as possible:

$$(kx-y+z)(x+ky-z)(x-y-kz)-(kx+y-z)(x-ky-z)(x-y+kz).$$

E 163. *Proposed by W. A. Carver, Lakewood, Ohio.*

A man purchased at a post-office some one-cent stamps, three-fourths as many two's as one's, three-fourths as many five's as two's, and five eight-cent stamps. He paid for them all with a single bill, and there was no change. How many stamps of each kind did he buy?

E 164. *Proposed by Wm. Fitch Cheney, Jr., Connecticut State College.*

A problem in the extraction of square root was worked out in the customary fashion and then a different letter was assigned to each different digit and substituted for the digit wherever it occurred. The result was:

$$\begin{array}{cccccc|cccc}
w & o & o & d & e & n & b & e & d & \\
w & s & & & & & & & & \\
\cline{1-3}
 & d & o & d & & & & & & \\
 & d & e & s & & & a & w & e & \\
\cline{2-6}
 & & s & n & e & n & & & & \\
 & & s & n & e & n & a & w & s & d \\
\cline{3-6}
\end{array}$$

Determine the numerical value of each letter and show that the solution is unique.

SOLUTIONS

E 131 [1935, 44]. *Proposed by W. B. Clarke, San Jose, California.*

In a general plane triangle a line is drawn from each vertex to the point which is half way around the perimeter from that vertex. These lines are concurrent in the point V. Then a line is drawn from the midpoint of each side to the point which is half way around the perimeter from that midpoint. These three lines are concurrent in the point M. If G is the centroid of the triangle and I the incenter, prove that V, M, G and I are collinear, and that the segments VM, MG and GI are in the ratio $3:1:2$.

Solution by H. W. Smith, Oklahoma Agricultural and Mechanical College.

Let the three sides of the triangle be a, b and c, with $c\leqq b\leqq a$. Take C as the origin of oblique coordinates, with CB as the x-axis and CA as the y-axis. Then

the vertices of the triangle are $A(b, 0)$, $B(a, 0)$ and $C(0, 0)$. The midpoints of the sides are $(a/2, 0)$, $(0, b/2)$ and $(a/2, b/2)$. The centroid is $G(a/3, b/3)$ and the incenter is found after some reduction to be $I(ab/2s, ab/2s)$. $[2s=a+b+c]$.

The point half way around the perimeter from A is $P(s-b, 0)$, and the point half way around from B is $Q(0, s-a)$. The lines AP and BQ intersect at the point $V(a-ab/s, b-ab/s)$.

If the midpoints of a and b are J and K respectively, then the points half way around the perimeter from J and K are $E(0, s-a/2)$ and $F(s-b/2, 0)$ respectively. The lines EJ and FK then intersect at $M(a/2-ab/4s, b/2-ab/4s)$.

Any two of the four points G, I, V and M may then be used to determine the equation of a line, and since the coordinates of the remaining two points each satisfy this equation, the four points are collinear.

The lengths of the segments are then found to be $VM=L/4s$, $MG=L/12s$ and $GI=L/6s$, where

$$L^2 = a^2(2s - 3b)^2 + b^2(2s - 3a)^2 + 2ab(2s - 3b)(2s - 3a) \cos C.$$

Hence the ratio of the three segments is 3:1:2.

Also solved by Roy MacKay, Simon Vatriquant and the proposer.

E 132 [1935, 44]. *Proposed by H. T. R. Aude, Colgate University.*

Show that if $2a$ is the harmonic mean of the two rational numbers b and c, then the sum of the squares of the three numbers, a, b and c, is the square of a rational number.

Solution by W. R. Ransom, Tufts College, Massachusetts.

Since $2a=2bc/(b+c)$, we have $2ab+2ac-2bc=0$, from which it is immediately apparent that $(b+c-a)^2=a^2+b^2+c^2$.

Also solved by L. J. Adams, E. F. Allen, J. A. Benner, W. E. Buker, Daniel Finkel, D. W. Hall, E. H. Johnson, Sidney Kaplan, Roy MacKay, F. L. Manning, W. N. Mebane, Jr., Robert Rosenbaum, H. W. Smith, E. P. Starke, C. W. Trigg, Simon Vatriquant and J. A. Ward.

E 133 [1935, 44]. *Proposed by L. S. Johnston, University of Detroit.*

The center and one vertex of a conic are given, as well as the focus nearer to the given vertex. The only available instrument is a draftsman's ordinary right isosceles triangle. It is required to construct the center of curvature of the conic at the given vertex. (Is it possible, under these same conditions, to construct the ends of the minor axis?)

Solution by the proposer.

With the available instrument we can draw parallel and perpendicular lines, and can lay off on the sides of a right angle equal distances from the vertices of the angle. There are several solutions possible, but the simplest one seems to be as follows:

Through O and A respectively draw on the same side of OA the lines m and

n perpendicular to OA. On m lay off $OB=OA$, and on n lay off $AG=AF$. Draw BF. Through G draw the parallel to OA, intersecting BF at E. Draw EC perpendicular to OA at C, which is the required center of curvature for the vertex A.

Proof. Setting $OA=a$, $OF=c$, we have $CE/OB=CF/OF$, or $AG/a=CF/c$. But $AG=a-c$, whence $CF=c(a-c)/a$. Since $OC=c-CF$, we have $OC=c^2/a$, and, since $CA=a-OC$, we have $CA=(a^2-c^2)/a=b^2/a$, which we know to be the radius of curvature of the ellipse at the vertex. Hence C is the required center of curvature.

Some interesting results of this construction are (1) the perpendicular to OA at F intersects the line BG at P, the extremity of the latus rectum of the ellipse; and the line BG is tangent to the ellipse at P; (2) the segments OC, OF and OA are in geometric progression; the converse of this statement is also true, i.e., if four ordered collinear points O, C, F and A, define three segments OC, OF and OA in geometric progression, then C is the center of curvature for the vertex A of the ellipse with center, focus and vertex respectively at O, F and A.

The construction is easily modified for the center of curvature of the hyperbola at the vertex, if the center, vertex and focus are given in appropriate position.

Editorial Note: The proposer included an ungraduated straight edge as a second available instrument in the statement of the problem, to simplify the draughtsman's construction of parallels. The department editor omitted it, however, as theoretically unnecessary to the solution.

Also solved by L. M. Kelly, E. P. Starke and Simon Vatriquant.

E 134 [1935, 44]. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given the circumcenter, the point of contact of side BC with the escribed circle corresponding to side AC, and the intersection of BC produced with the bisector of the exterior angle at A.

Solution by Roy MacKay, Eastern New Mexico Junior College.

Construction. Denote the given circumcenter by O, the given point of contact by X and the intersection of the external bisector of A with BC by Z. With center O and radius OX cut XZ again at Y, and reletter X and Y if necessary, so that Y is between Z and X. Drop OM perpendicular to XY at M and construct AD perpendicular to XY at D, where D divides XY internally in the same ratio that Z divides XY externally. Let the circle on diameter OZ meet the perpendicular bisector of MD at S. Draw DS to meet the perpendiculars to XY at Y, D, M and X, in Q, A, R and P respectively. The common internal tangents of the circles (P, PX) and (Q, QY) intersect at A and meet XY at B and C to form the required triangle.

Proof. The common internal tangents of circles (P, PX), (Q, QY) divide PQ internally in the ratio $PX/QY=ZX/ZY$. Hence these tangents meet PQ on AD. Since the line of centers bisects the angle between two common internal

tangents, AZ is the bisector of the exterior angle A of the triangle ABC, and AZ meets the diameter of the circumcircle perpendicular to BC at one extremity. Since tangents from an external point to a circle are equal, it is easily proved that $XB=CY$, so OM is the perpendicular bisector of BC and therefore R is on the circumcircle of the triangle ABC. From the construction, OS is the perpendicular bisector of the chord AR, and hence O is the circumcenter of the triangle ABC.

Discussion. No solution is possible unless OZ is greater than OX. There may be two solutions arising from the two points S determined by the circle on OZ as diameter, and the perpendicular bisector of MD. If O is on ZX, these two solutions are congruent right triangles; otherwise one solution is acute angled and the other is obtuse angled at A.

Also solved by L. M. Kelly.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department, In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3739. *Proposed by Paul Erdös, The University, Manchester, England.*

Given $n+1$ integers, $a_1, a_2, \cdots, a_{n+1}$, each less than or equal to $2n$, prove that at least one of them is divisible by some other of the set.

3740. *Proposed by Paul Erdös, The University, Manchester, England.*

From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).$$

3741. *Proposed by H. D. Ruderman, James Madison High School, Brooklyn, N. Y.*

Find the value of the sum

$$\sum_{i=1}^{n} \tan^2 \frac{i\pi}{2n+1},$$

where n is a positive integer.

3742. *Proposed by Maud Willey, Long Beach, Miss.*

Prove that the group of movements into itself, in space of n dimensions, of a regular solid with $n+1$ vertices is the alternating group of degree $n+1$; and

that, in space of $n+1$ dimensions, the group for the same solid is the symmetric group of degree $n+1$.

3743. *Proposed by Norman Anning, University of Michigan.*

Two congruent coplanar parabolas have the same line as axis and open in the same direction. Tangents are drawn to the inner from any point of the outer. Prove that the area bounded by the tangents and the arc joining their points of contact is invariant.

SOLUTIONS

3640 [1933, 561]. *Proposed by V. Thébault, Le Mans, France.*

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the pedal triangles of the two isogonally conjugate points P, P' with respect to the triangle ABC. The parallels through P, P' to the sides of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$ meet the sides BC, CA, AB, in two sets of three collinear points. These two lines are respectively perpendicular to the lines joining P', P to the orthocenters of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$.

Solution by the Proposer.

The straight lines drawn from P parallel to $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$, are perpendicular, respectively, to PA, PB, PC. The intersections of these parallels with BC, CA, AB, respectively, are known to lie on a straight line Δ. In a similar manner the parallels from P' to the sides of $\alpha\beta\gamma$ determine a straight line Δ'.

It is easy to prove that Δ, for example, is perpendicular to the straight line PH_1, where H_1 is the orthocenter of triangle $\alpha_1\beta_1\gamma_1$ formed by the other intersections of $P\alpha$, $P\beta$, $P\gamma$ with the pedal circle (ω) of P and P'. For, the vertices $\alpha_1, \beta_1, \gamma_1$ are the inverses of α, β, γ in the inversion (P, $P\alpha \cdot P\alpha_1$). But the triangle $\alpha_1\beta_1\gamma_1$ is symmetric to $\alpha'\beta'\gamma'$ with respect to ω, the center of the pedal circle. Since P' is also the symmetric of P with respect to ω, the straight line Δ, which is perpendicular to PH_1, is also perpendicular to its parallel $P'H'$, where H' is the orthocenter of $\alpha'\beta'\gamma'$.

The same reasoning applies to Δ' and the triangle $\alpha\beta\gamma$.

Editorial Note. The above solution was sent by the proposer along with the problem merely as a brief outline of his method of proof. More details of the proof will be given. Let AP cut BC in A_a; and let the perpendicular at P to AP cut BC, CA, AB in A_a', A_b', A_c'. It will be shown that the pencils $P(A_a, A_a'$, B_b', $C_c')$ and $A(A_a, A_a'$, B_b', $C_c')$, where the notation is similar to the above, are projective. Then, since there is a self-corresponding ray through A_a, the points A_a', B_b', C_c' must lie on a straight line Δ. On BC we have $(A_a, A_a', B_a'$, $C_a') \barwedge (A_a', A_a, B, C) \barwedge (A_a, A_a', C, B)$. The P pencil cuts BC in the first of these three sets of points, and the A pencil cuts it in the third set. Thus the two pencils are projective. The proposer's solution also proves this theorem, as will appear below, but this method of proof is more complicated. For the remaining part of the proof the figure and the related proof vary with the position of P in the plane of ABC. The variation of the figure has already been discussed in the

Note to the solution of 3658, [1935, 256]. It was shown that, if P lies within the vertical angle of A formed by BA and CA produced, P' lies within the segment of the circumcircle of ABC opposite to vertex A; and conversely. In this case (I), P and P' lie outside the pedal circle (ω); the corresponding segments, such as $P\alpha$, $P\alpha_1$, have the same direction, and the circle of inversion (P) with center P is orthogonal to (ω). With respect to (P) the pole of the altitude $H_1\alpha_1$ lies on BC, since BC is the polar of α_1; it also lies on the perpendicular from P to $H_1\alpha_1$. This perpendicular is parallel to $\beta_1\gamma_1$ and $\beta'\gamma'$; it is therefore perpendicular to AP. Hence $H_1\alpha_1$ is parallel to PA. Thus the pole of $H_1\alpha_1$ is the intersection A_a' of BC with the perpendicular at P to AP. Since the three altitudes meet in H_1, the poles A_a', B_b', C_c' lie on a straight line Δ, and Δ is the polar of H_1. The rest of the proof easily follows for this case.

If P lies within the triangle so does also P'; if P lies outside the circumcircle and within the angle A so does also P'. In these cases (II) P and P' lie within the pedal circle (ω), and the segments, such as $P\alpha$, $P\alpha_1$, have opposite directions. In this case the above proof is subject to a slight change in order to obtain the circle of inversion (P). Let $\alpha_2\beta_2\gamma_2$ be the triangle symmetric to $\alpha_1\beta_1\gamma_1$ with respect to the center P; in other words $\alpha_2\beta_2\gamma_2$ is the translation of $\alpha'\beta'\gamma'$ by the vector $P'P$. The pairs of points, such as α, α_2, are inverse pairs with respect to the circle (P) with a radius of half the chord of (ω) perpendicular at P to PP'. We now consider H_2, the orthocenter of $\alpha_2\beta_2\gamma_2$, and prove in the same way that H_2 is the pole of Δ. Obviously, PH_2 is parallel to $P'H'$, where H' is the orthocenter of $\alpha'\beta'\gamma'$.

If P lies on a side the theorem becomes trivial. Suppose now that P lies on the circumcircle, but not at a vertex, say within the angle A; we take for (P) any convenient circle with center P. Let P_bB and P_cC be the reflections of PB and PC, respectively, in the corresponding internal bisectors of angles B and C. Then $\angle P_bBC+\angle P_cCB=B+C+\angle PBC+\angle PCB=A+B+C=180°$. Hence P' is at infinity in the direction CP_c, and the pedal circle (ω) of P and P' is a straight line s passing through α, β, γ. The circle through P which with (P) has s for the radical axis is the inverse of s. The inverses of α, β, γ are the intersections α_2, β_2, γ_2 of $P\alpha$, $P\beta$, $P\gamma$ with this new circle. The triangles $\alpha_2\beta_2\gamma_2$ and ABC are reciprocal polars, and they are also similar. We prove as before that PH_2 is perpendicular to Δ, where H_2 is the orthocenter of $\alpha_2\beta_2\gamma_2$. The radical axis s' of (P) and the circumcircle of ABC passes through A', B', C', the inverses of A, B, C; also the sets of three points, such as α_2, β_2, C', lie on a straight line perpendicular to PC. For a fixed P and a variable circle (P), the locus of H_2 is the perpendicular from P to Δ.

Also in the other cases we may take for (P) a circle of center P and with any radius k. If the inverses of α, β, γ are α_2, β_2, γ_2, then $P\alpha_2\cdot P\alpha=k^2$, and $P\alpha\cdot P\alpha'=c$; hence

$$\frac{P\alpha_2}{P\alpha'}=\frac{P\beta_2}{P\beta'}=\frac{P\gamma_2}{P\gamma'}=\frac{k^2}{c}$$

and the triangle $\alpha_2\beta_2\gamma_2$ is similar to $\alpha'\beta'\gamma'$, and similarly placed, with the center of similitude on and within the segments PP' in case I, and on the line of PP' but outside the same segment in case II. The proof follows in the same manner. The circumcircle of $\alpha_2\beta_2\gamma_2$ is the inverse of the pedal circle (ω) of P and P'. The polar of the circumcircle of $\alpha_2\beta_2\gamma_2$ is a conic with P as focus tangent to the sides of ABC, and therefore it must have the pedal circle of P and P' as auxiliary circle; hence P' is the other focus. Taking the polar of the circumcircle of ABC we obtain in the same way a conic with P as a focus, tangent to the sides of $\alpha_2\beta_2\gamma_2$, and with the pedal circle of the latter as auxiliary circle.

It may be of interest to restate the case where P lies on the circumcircle (O) of ABC in the form of a solution of 3658, l.c.; since this solution is different from others which have been received and yet is somewhat analogous to the one by Musselman.

We suppose that P lies within the angle A, and that its projections upon the sides of ABC are α, β, γ. We take for (P), the circle of inversion, a circle with center P and any convenient radius; let it have a common chord s' with (O). Then s' is the inverse of (O), and the inverses A', B', C' of A, B, C lie on s'. The inverses α_2, β_2, γ_2 of α, β, γ are the vertices of the polar reciprocal triangle of ABC, such that the projections of P upon sides of $\alpha_2\beta_2\gamma_2$ are A', B', C'. Then $\angle\beta_2P\gamma_2 \equiv \angle\beta P\gamma = 180° - A$, and $\angle BPC \equiv \angle B'PC = 180° - A = 180° - \alpha_2$. Hence $\angle\beta_2\alpha_2\gamma_2 = \alpha_2 = A = 180° - \angle\beta_2P\gamma_2$ and α_2, β_2, γ_2, P lie upon a circle (O_2), the circumcircle of $\alpha_2\beta_2\gamma_2$. The inverse of (O_2) is therefore a straight line s passing through α, β, γ. The polar of (O_2) is a parabola with P as its focus, tangent to the sides of ABC and having s as the tangent at its vertex.

In the same manner as above we show that $\beta_2 = B$, $\gamma_2 = C$; and that, by reflecting $\alpha_2\beta_2\gamma_2$ in the bisector of angle $CP\gamma_2$, it will be placed so that P is the center of similitude for it and ABC.

In Musselman's solution of 3658 the complex quantity t fixed a point on the unit circle (O). Let the circle (P) pass through O and have its center at $t=1$. Reflect the polar of circle (O) with respect to (P) in the vertical straight line at $t=1/2$, and we obtain the parabola given by $x=2/(1-t)^2$, whereas the point t_1 goes by this transformation into the tangent to the parabola with the equation $x=2/(1-t_1)\,(1-t)$.

3666 [1934, 113]. *Proposed by Martin Rosenman, Brooklyn, N. Y.*

Set up a one-to-one correspondence between the points in the open interval $0<x<1$ and the points in the closed interval $0 \leqq x \leqq 1$.

I. *Solution by H. J. Goldstein, New York.*

Let K represent the class of points in the open interval and K' the class of points in the closed interval. Select from K any denumerably infinite subclass $S=(s_1, s_2, \cdots)$ and from K' the denumerably infinite subclass S' which contains the members of S together with 0 and 1. Then the members of S are easily correlated with those of S', while the remainders $K-S$ and $K'-S'$ are identi-

cal. We therefore have the following 1-1 correspondence between K and K': s_1, s_2, s_{n+2} of S correspond respectively to 0, 1, s_n of S', $n=1, 2, \cdots$, and every other x of K corresponds to itself in K'.

In particular, we may take $s_n=(n+1)^{-1}$. Then if

$$\phi(x) = \lim_{m=\infty} (\cos \pi/x)^{2m},$$

$$f(x) = \frac{x}{1-2x\phi(x)} \quad (x \neq \tfrac{1}{2}), \qquad f(\tfrac{1}{2}) = 0,$$

$$F(y) = \frac{y}{1+2y\phi(y)} \quad (y \neq 0), \qquad F(0) = \tfrac{1}{2},$$

$f(x)$ and $F(y)$ exist and are one-valued for all points x of K and y of K'. Moreover, $y=f(x)$ is equivalent to $x=F(y)$, and y takes all values in K' when x takes all values in K. Therefore either f or its inverse F may be used to express directly a 1-1 correspondence between K and K'.

This solution may be readily extended to cover the corresponding problem for n dimensions; that is, to establish a 1-1 correspondence between the points of the open region $0<\rho<1$, and those of the closed region $0\leqq\rho\leqq 1$, where

$$\rho^2 = \sum_{i=1}^{n} x_i^2 .$$

II. *Solution by R. C. Stephens, Knox College.*

Denote by $A(x)$ the point x considered as a member of the set $0<x<1$ and by $B(x)$ the point x considered as a member of the set $0\leqq x\leqq 1$. Put $B(\frac{1}{2}[1\pm 3^{-n}])$ in correspondence with $A(\frac{1}{2}[1\pm 3^{-n-1}])$ for $n=0, 1, 2, \cdots$. For all other values of x, $0<x<1$, put $A(x)$ in correspondence with $B(x)$.

Solved also by M. F. Becker, C. P. Brady, Mannis Charosh, J. H. Edmonston, J. Rosenbaum, and H. D. Ruderman.

3667 [1934, 193]. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that $(-1)^{n-1}(n-1)2^{n-2}$ is the value of the n-rowed determinant for which $a_{ij}=|i-j|$.

Solution by John Williamson, The Johns Hopkins University.

The matrix of the given set of elements is a particular case of the matrix A for which $a_{ij}=a+|i-j|h$. The value of the determinant of A is easily determined by subtracting the elements of the $(j+1)$th column from the corresponding elements of the jth column, leaving the nth column unchanged. Repeat this process on the new determinant leaving the last two columns unchanged. After adding the last row to the first, we find that

$$|A| = (-1)^{n-1}2^{n-2}h^{n-1}[2a + (n-1)h]. \tag{1}$$

Since, however, A is symmetric, the following proof may be of greater interest; for it reduces the quadratic form associated with A to a very simple form. Let B denote the matrix with the elements

$$b_{n1} = b_{nn} = 1/2, \quad b_{ii} = 1, \quad b_{i,i+1} = -1, \quad i < n; \tag{2}$$

and with zeros for all the other elements. Let B' denote the transposed matrix of B, i.e., the rows of B' are the columns of B. Then a simple calculation shows that

$$BAB' = C,$$

where C is a diagonal matrix, i.e., all of its elements are zeros except those in its principal diagonal, and these in order are

$$-2h, -2h, \cdots, -2h, a + \tfrac{1}{2}(n-1)h.$$

Since the determinants of B and B' are each unity, we obtain again the result (1). This shows also that the quadratic form having the matrix A is reduced by a unimodular transformation to the form

$$-2h[x_1^2 + x_2^2 + \cdots + x_{n-1}^2] + [a + \tfrac{1}{2}(n-1)h]x_n^2.$$

Solved also by Frank Ayres, Jr., J. A. Bullard, Mannis Charosh, J. W. Clawson, E. W. Emery, Jeannette Fox, Hansraj Gupta, Roy MacKay, R. E. Moritz, W. V. Parker, A. Pelletier, B. D. Roberts, C. A. Rupp, E. P. Starke, E. E. Strock, C. W. Trigg, W. P. Udinski, F. Underwood, M. L. Vest, and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The important report on *The Training of Teachers of Mathematics* by the Commission on the Training and Utilization of Advanced Students of Mathematics, published in the May issue of the MONTHLY, can be obtained from the office of the Secretary of the Association at the price of ten cents per copy. The Association hopes to be able to send copies of the report without cost to non-mathematicians and educational authorities, since the report should be brought to the attention of many outside the group of mathematics teachers. Postcard suggestions of such addresses would be welcomed by Secretary W. D. Cairns, Oberlin, Ohio.

Dr. Richard Courant, visiting professor at New York University, was the speaker at the mathematics section of the New York Society for the Experi-

mental Study of Education at Columbia University March 2. His topic was "The Teaching of Mathematics and Physics."

Dr. J. L. Doob of Columbia University has been appointed associate professor of mathematics at the University of Illinois.

Professor Philip Franklin, of the Massachusetts Institute of Technology, has obtained leave of absence for advanced study and foreign travel during the academic year 1935–36.

Dr. E. L. Mackie has been promoted to a professorship of mathematics at the University of North Carolina.

Professor R. E. Langer of the University of Wisconsin has been appointed Lecturer in Mathematics in Harvard University for the year 1935–36.

Dr. I. J. Schoenberg has been appointed acting assistant professor of mathematics at Swarthmore College.

Professor Stephen Timoshenko, of the engineering mechanics department of the University of Michigan, has been appointed Hitchcock professor at the University of California.

Dr. J. L. Walsh has been promoted to a professorship of mathematics at Harvard University.

Dr. Hassler Whitney of Harvard University has been appointed assistant professor and tutor of mathematics in Harvard University, and has been granted a leave of absence for the first semester of the year 1935–36.

Associate Professor D. V. Widder of Harvard University has been granted a leave of absence for the year 1935–36.

Professor Norbert Wiener, of the Massachusetts Institute of Technology, has been granted leave of absence for the next academic year during which time he will be research professor of mathematics at the National Tsing Hua University in Peking, China.

Dr. R. C. Shook has been appointed to an instructorship at the College of the City of New York (tutor, evening session.)

Dr. E. A. Partridge, of the West Philadelphia Boys' High School, died March 22, 1934. He was a charter member of the Mathematical Association.

The American University at Cairo is seeking a man to teach mathematics for 1935–36. Board and room will be provided, but nothing is available for salary or travelling expenses to Egypt. It would be necessary for a man to invest from five to seven hundred dollars of his own money in the venture for expenses. Inquiries may be addressed to Mr. Hermann A. Lum, Executive Office of The American University at Cairo, 1000 Land Title Building, Philadelphia, Pa.

CONTENTS

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.
Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The Trustees have voted to accept the invitation of the Department of

Mathematics of Harvard University to hold the summer meeting of the Association there in September 1936, in connection with the celebration of the Harvard Tercentenary.

W. D. CAIRNS, *Secretary*

THE TWELFTH ANNUAL MEETING OF THE MICHIGAN SECTION

The twelfth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, on March 9, 1935, in conjunction with the Michigan Academy of Science. Seventy-three persons registered but the total attendance was about ninety, including the following forty-one members of the Association: W. L. Ayres, W. D. Baten, W. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, S. E. Crowe, Wayne Dancer, Albertus Darnell, C. M. Erikson, J. P. Everett, Peter Field, C. H. Fischer, W. B. Ford, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, H. S. Kaltenborn, Theodore Lindquist, C. E. Love, E. D. McCarthy, A. L. Nelson, H. H. Pixley, L. C. Plant, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, E. R. Sleight, G. G. Speeker, H. E. Vaughan, T. O. Walton, E. T. Welmers, R. L. Wilder.

There were sessions for the reading of papers both morning and afternoon, the Chairman of the Section, Professor L. S. Johnston, presiding at both times. A luncheon was held at noon at the Michigan Union attended by forty-eight persons and the annual business meeting was held at this time. The following were elected officers for the year 1935–36: Chairman, Professor J. B. Brandeberry, University of Toledo; Secretary-Treasurer, Professor W. L. Ayres, University of Michigan; Member of Executive Committee, Professor L. S. Johnston, University of Detroit. The continuation of the fall meeting was discussed and it was voted that the Section hold a fall meeting outside Ann Arbor in the even numbered years in addition to the annual meeting at Ann Arbor. The Secretary proposed the establishment of a prize contest in mathematics for undergraduates. This was discussed at length and a committee composed of the Executive Committee, Professor L. C. Emmons and Professor E. R. Sleight, chairman, was appointed to consider the matter further and report back to the Section.

The following program was presented:

1. "Entropy, strain and the Bose-Einstein statistics" by Professor W. S. Kimball and Mr. Max Wygant, Michigan State College, introduced by Professor V. G. Grove.

2. "The present status of the four-color problem" by Professor W. L. Ayres, University of Michigan.

3. "A modified Pascal triangle" by Professor W. Carl Rufus, University of Michigan, introduced by the Secretary (by title).

4. "On the normals to a certain type of surface in a space of four dimensions" by Professor V. G. Grove, Michigan State College.

5. "The formulas for plane motion" by Professor N. H. Anning, University of Michigan, introduced by the Secretary.

6. "The distributions of elements into sets" by Professor A. H. Copeland, University of Michigan.

7. "Interesting the superior student" by Professor E. R. Sleight, Albion College.

8. "Mathematics clubs" by Professor Wayne Dancer, University of Toledo.

9. "Examples of statistical problems in econometrics" by Professor H. H. Pixley, Wayne University.

Abstracts of some of the papers follow, numbered in accordance with their place on the program.

1. The three stages in the development of statistical mechanics are represented by three characteristic equations: (a) The first is Boltzmann's equation $S=k \log W$, giving entropy in terms of the probability. (b) The second defines probability as a statistical factor multiplied by the product of constant cells in phase space each raised to a power equal to the number of particles it contains, indicating extension in phase according to Gibbs. (c) The third stage developed at Michigan State College introduces a geometrical expression for weight characterized by variable cells each including a single particle.

2. In this paper Professor Ayres sketched the early attempts at the problem and exhibited the proof that any map may be colored in five colors. The attempts that have been made to lower this to four colors were discussed and some alternative statements of the problem were given.

3. The general term is usually expressed as $(m+n-2)!/(m-1)!(n-1)!$, rejecting terms in which $m<n$. The rth term of the mth diagonal is shown to be $(m+2r-3)!/(m+r-2)!(r-1)!$. Relationships between binomial coefficients of different orders follow directly. Professor Rufus showed also that formulas for the ratios in any regularly defined sequence of terms, such as rows, columns, diagonals, knight's moves, etc., may be computed.

5. In order to show a class in differential equations that the law of the inverse square implies motion in a conic, one must rapidly review the laws of motion. To say that the average student is already familiar with them is too fine a compliment. Professor Anning shows in this paper how by a mild use of vectors, differentiation, and interpretation, all necessary preliminary spade work can be done in one class period.

6. Professor Copeland's problem in this paper was to obtain formulas for the number of ways in which one can distribute n elements into r sets. The elements may be considered either different or indifferent and all or some of them may be distributed. Also the sets may be considered either different or indifferent and all or some of the sets may receive elements. Different arrangements of the elements in the sets may or may not be counted as different dis-

tributions. There are twenty-four cases to consider, twelve of which are discussed in Whitworth's *Choice and Chance*. In this paper a general method of obtaining such formulas is developed, and by means of this method all of the twenty-four cases can be treated.

7. In this paper Professor Sleight attempts to answer the question: To what extent should the spirit of research among undergraduates be encouraged? Methods for interesting the student of superior ability, and for developing in him a spirit of research are discussed. Attention is called to the fact that this process brings a student face to face with the need of thinking a problem through for himself, and of devising methods for its solution. Since individual initiative and resourcefulness are primary objects of education, this plan seems highly adequate for obtaining these ends.

8. Professor Dancer spoke briefly on the desirability of mathematics clubs for undergraduates, and gave many suggestions as to how to make the programs of such organizations interesting and instructive. He emphasized that great care should be taken to see that papers presented are suitable, and not beyond the students' understanding. He called attention to the fact that contests and recreations might be used to great advantage both in instruction and in maintaining enthusiasm.

9. Professor Pixley presented and discussed the statistical results of some of the recent work in econometrics. He examined the applications and limitations of the statistical analyses in the light of criteria involving the theoretical bases for the formulas proposed, the comparability of results yielded by the analyses when used on comparable sets of data, and various statistical tests of goodness of fit.

W. L. Ayres, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The thirteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at Agnes Scott College, Decatur, Georgia, on Friday and Saturday, March 22–23, 1935. Sessions were held in the afternoon and evening of the 22nd, and on the morning of the 23rd. The chairman of the Section, Professor W. P. Ott, presided, except Friday evening.

There were in attendance one hundred fifty-nine persons from thirty-two institutions including the following thirty-seven members of the Association: Martha E. Allen, D. H. Ballou, D. F. Barrow, Iris Callaway, J. B. Coleman, Forrest Cumming, U. P. Davis, B. F. Dostal, Floyd Field, H. K. Fulmer, Leslie J. Gaylord, W. E. Glenn, R. A. Hefner, Ruby Hightower, P. R. Hill, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, F. W. Kokomoor, Curtis Ledford, F. A. Lewis, W. N. Mebane, Jr., W. G. McGavock, S. W. McInnis, J. F. Messick, W. A. Moore, W. P. Ott, W. V. Parker, W. W. Rankin, H. A. Robinson, T. M. Simpson, W. B. Smith, F. H. Steen, R. P. Stephens, W. W. Weber, K. P. Williams, W. L. Williams.

On the evening of the 22nd a dinner was held in honor of the visiting speaker, Professor K. P. Williams, of Indiana University. At this time Professor H. A. Robinson presided.

The meetings of the Georgia Academy of Science and the Georgia Section of the American Chemical Society, and the initial meeting of the Southern Section of the American Physical Society were held at Agnes Scott College at the same time as the mathematics meeting. Over four hundred scientists including the mathematicians were present. This perhaps represents the largest group of the kind ever to convene in Georgia.

At the business session on the 23rd the following officers were chosen for 1935–36: Chairman, F. W. Kokomoor, University of Florida; Vice-Chairman, W. W. Rankin, Duke University; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was tentatively scheduled for March, 1936, at the University of South Carolina. A resolution was passed relative to the loss sustained by the Section in the passing of Professor M. D. Earle.

The following thirteen papers were read:

1. "Generalized step squares" by Professor R. A. Hefner, Georgia School of Technology.
2. "Mathematics and bacteriology" by Dr. H. H. Germond, University of Florida, introduced by the Secretary.
3. "Movement of Mercury's perihelion" by Professor K. P. Williams, Indiana University.
4. "The digits of π" by Professor P. R. Hill, University of Georgia.
5. "Some geometry associated with a certain collineation group" by Professor F. A. Lewis, University of Alabama.
6. "Integrating odd powers of sec x" by Professor W. V. Parker, Georgia School of Technology.
7. "The determination of the equation of the Pascal line" by Dr. B. G. Clark, University of Alabama, introduced by the Secretary.
8. "On mechanical integration" by Dr. F. H. Steen, Georgia School of Technology.
9. "A proof of the corner conditions in the calculus of variations" by Dr. J. D. Mancill, University of Alabama, introduced by the Secretary.
10. "Place of mathematics in secondary education" by Professor K. P. Williams, Indiana University.
11. "The background of collegiate mathematics" by Professor W. W. Rankin, Duke University.
12. "The attained and the unattained in the teaching of mathematics" by Dr. F. A. Beers, Examiner, University System of Georgia, by invitation.
13. "Psychological principles applied to the learning process in mathematics" by Professor B. P. Reinsch, Southern College, by title.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Hefner pointed out how the step method of constructing magic squares can be extended so that the order of the integers need not be the same as the order of the step used in placing them.

2. The ultimate object of mathematics in bacteriology is not the mere writing of an equation which describes the data, but the determination of the basic law underlying the phenomena. It was shown by Dr. Germond that the discovery of an equation describing certain results does not preclude the possible existence of other solutions. Various theories now held in bacteriology can be properly tested only when more precise methods of counting are developed.

3. In conformity with Newcomb's study of the transits of Mercury made in 1881, he published in 1895 tables which increased the centennial shift of the perihelion $43.37''$ above the requirements of the Newtonian law of gravitation. Professor Williams stated that these tables have not given very satisfactory results for the four transits that have occurred since 1900, and that Newcomb himself recognized the problem as still unsettled. The difficulties of the problem were discussed. Under the assumption that the only elements needing correction are the position and motion of Mercury's node and perihelion and the diameter of the sun, a solution by the method of least squares was shown to increase the centennial motion of the perihelion by $3.5''$ above that of Newcomb. The corrections obtained result in small residuals for the 15 transits involved, except the last two, those of 1924 and 1927.

4. Assuming the 707 known digits of π to be a sample, Professor Hill showed that π possessed the characteristics of a number formed by chance. Other irrationals were discussed.

5. Professor Lewis gave a preliminary report of a geometrical study of a collineation group in four variables which can be regarded as a generalization of the ternary group of Hesse.

6. Professor Parker derived a general integration formula for odd powers of $\sec x$.

7. A method for writing the equation of any one of the 60 Pascal lines which arise from a particular ordering of the six points was given by Dr. Clark and fifteen identities connected with the six points were derived. The results led to analytic proofs of theorems by Steiner, Kirkman and Salmon.

8. Dr. Steen discussed properties of a class of polynomials arising from the study of a certain integral to be a minimum. The investigation was suggested by a problem in mechanical integration.

9. A proof of the corner conditions which made no use of the extremal property of a minimizing curve was given by Dr. Mancill. The results were applicable to corners of minimizing curves lying on the boundary of the region in which admissible curves existed. The method used could be applied to any dimension and to non-parametric problems.

10. The recommendations of the National Committee on Mathematics Requirements as to making mathematics rigidly required were very specific. None beyond the ninth year was required; but some algebra, some numerical

trigonometry, some intuitive geometry and a little demonstrative geometry were to be required of all students in the ninth year. Professor Williams stated that these recommendations seem not to carry conviction; pupils are graduating from many high schools without having studied either algebra or geometry. It is imperative that those qualified to study it not only be given ample opportunity but be prevented from placing upon themselves a regrettable handicap by neglecting it. We not only should prepare thoroughly candidates for teaching positions and stimulate those that are in service, but also combat those unsound attacks on mathematics which too frequently mingle with the more reasoned criticisms of competent administrators and educators.

11. In addition to projecting the study of mathematics upon the background of utility, Professor Rankin presented the splendid opportunities in projecting it upon the background of fundamental principles. Through the efforts of David Eugene Smith, Archibald, Karpinski, Vera Sanford and others, a historical background is also available.

12. Since no adequate measures are available to indicate what opportunity students entering college have had to acquaint themselves with mathematics, and since high school grades vary widely as indices of achievement, Dr. Beers contended that students need opportunities under fairly uniform conditions to try themselves in the field so as to learn their relative degree of success as measured by a common yardstick.

13. Professor Reinsch showed how certain elementary principles of psychology may be applied to the learning process in mathematics.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twelfth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Louisiana College, Pineville, Louisiana, on Friday and Saturday, March 29–30, 1935. The chairman, Professor T. A. Bickerstaff of the University of Mississippi, presided, and introduced the guest speaker, Professor H. J. Ettlinger, University of Texas. Professor Ettlinger's subject at the dinner was "Mathematics as an experimental science."

The attendance was forty-eight, including the following twenty-four members of the Association: Nola L. Anderson, T. A. Bickerstaff, Leora Blair, H. E. Buchanan, D. S. Dearman, W. L. Duren, H. J. Ettlinger, Elizabeth Freas, May Hickey, J. R. Hitt, C. G. Killen, Dorothy McCoy, Elsie J. McFarland, A. C. Maddox, I. Maizlish, B. E. Mitchell, I. C. Nichols, S. T. Sanders, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, W. P. Webber.

Officers elected for the year 1935–36: Chairman, H. L. Smith, Louisiana State University; Vice-chairman for Louisiana, V. B. Temple, Louisiana Col-

lege; Vice-chairman for Mississippi, D. S. Dearman, State Teachers College, Hattiesburg; Secretary, Dorothy McCoy, Belhaven College.

The following six papers were read:

1. "Groups of space transformations resulting from inversions in spheres" by Professor May Hickey, Delta State Teachers College.

2. "The geometry of the complex triangle" by Professor B. E. Mitchell, Millsaps College.

3. "Some problems solved by Heaviside's directional calculus" by Dr. J. F. Thomson, Tulane University.

4. "A construction for the tangents at the nodes of the rational plane quintic" by Professor Elsie J. McFarland, Jones County (Mississippi) Junior College.

5. "A foundation for Riemannian geometry" by Professor H. L. Smith, Louisiana State University.

6. "Non-unique solutions of first order ordinary differential equations" by Professor H. J. Ettlinger, University of Texas.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In a previous paper Professor Hickey had considered a type of space transformation that consisted of an even number of inversions in spheres orthogonal to a unit sphere. In the present paper the spheres are situated anywhere, and one finds that the locus of points near which distances are unaltered is a sphere, the *isometric sphere* of the transformation. With the aid of the isometric spheres of the transformation and its inverse certain simpler sequences of inversions in spheres may be substituted for the original set. Again, the properly discontinuous group of such transformations was considered, its fundamental region constructed, and the boundary of the region shown to be similar to those of the same type of groups of linear transformations, with a greater variety of boundary points.

2. Professor Mitchell considered synthetically the triangle, consisting of one real and two conjugate imaginary vertices, immersed in the four-dimensioned complex plane. He showed that one side, the midpoint of one side, and one vertex of the pedal triangle are real, the other two in each case are conjugate imaginary. The following are also real: the circumcenter, the orthocenter, the nine-point center, the centroid, one altitude, one median, one bisector, the Euler line, the circumcircle, the nine-point circle, the self-conjugate circle. The study of the triangle with all vertices imaginary was facilitated by the introduction of three real points called Focal Points and three real lines called Mirror Axes, a Focal Point lying on each Mirror Axis. The Laguerre representation was employed for visualization.

3. Oliver Heaviside's work was devoted mainly to the study of disturbances in electrical networks. This work led him to linear differential equations with constant coefficients which he solved by two original methods: one his expan-

sion theorem, the other a method of expressing the operator in a power series—called algebrizing. Dr. Thomson discussed both methods, and applied them to some electric circuit problems. The advantages of Heaviside's methods over the classical method of solving linear differential equations were pointed out, and the wide application of Heaviside's calculus indicated.

4. In this paper Professor McFarland showed that when a plain quintic is determined by six nodes and the tangents at any one of them, then the tangents at any other node are the lines joining that point to the points of intersection of a readily determined straight line and conic.

5. A foundation for differential geometry in which the undefined terms are "preferred coordinate system" and "allowable coordinate system" has been given by O. Veblen. Professor Smith presented a foundation for Riemannian geometry in which the only undefined term is "distance between two points." The principal postulate affirms the existence of a coordinate system with certain properties expressed in terms of the distance notion.

6. Professor Ettlinger presented some recent work in the field of solutions of the first order ordinary differential equation, $y = f(x, y)$ in the strip $R: 0 \leqq x \leqq 1$, all values of y. Under more general hypotheses than heretofore given, the existence of Cauchy fields of "approximate" solutions starting from a given point were shown, as extensions of the classic Cauchy polygons, and the presence of absolutely continuous limit functions of those fields which are solutions was demonstrated. The conditions imposed on $f(x, y)$ are (1) Lebesgue integrability of $f(x, y)$ for a fixed y, (2) equi-absolute continuity of the Cauchy fields, and (3) continuity of $f(x, y)$ in y for x fixed. Under these hypotheses every Cauchy field has a solution and every solution has a Cauchy field. The existence of maximum and minimum solutions for a given point and of the complete band of solutions between them is proved. From the standpoint of two-dimensional analysis situs the totality of solutions through a point form a continuous curve. Further results were stated with respect to the continuity of the band solution as a function of the initial values and of $f(x, y)$. With respect to $f(x, y)$ it was pointed out that the solutions form an upper semi-continuous collection of curves in the sense of R. L. Moore.

DOROTHY McCOY, *Secretary*

THE TWENTY-FIRST ANNUAL MEETING OF THE KANSAS SECTION

The twenty-first annual meeting of the Kansas Section of the Mathematical Association of America was held in Topeka in the High School Building, on Saturday, March 16, 1935.

The session in the forenoon was a joint session with the Kansas Association of Mathematics Teachers. The luncheon, served by the staff of the High School cafeteria, has become a most pleasant social event of the meetings. In the afternoon, the two groups met separately. Professor G. W. Smith, chairman

A GENERALIZATION OF THE DERIVATIVE OF A FUNCTION*

By BEN DUSHNIK, University of Michigan

1. *Introduction.* The purpose of this note is to describe a certain process, a special instance of which is the process of obtaining the derivative function of a given differentiable function, as explained in any differential-calculus text. In addition, this process will be used in the solution of a specific problem, so as to indicate a type of problem to which the new notions may be applied.

2. *Definition.* Suppose that $y=f(x)$ is a uniquely defined function of the real variable x on the interval $a \leqq x \leqq b$, and $\alpha(x, z)$ is a uniquely defined function of the two independent variables x and z on the rectangular region $c \leqq x \leqq d$, $-h<z<h$ and the segment $x=0$, $-h'<z<h'$ $(h, h'>0)$. [Any one of these numbers a, b, c, d may be $\pm\infty$.] Suppose further that, for any x lying on the interval (c, d), $\alpha(x, z)$ is on the interval (a, b), $\alpha(0, z)$ does not vanish for sufficiently small values of z, and $\lim_{z\to 0}\alpha(0, z)=0$.

Consider now the ratio

$$\text{(A)} \qquad \frac{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]}{\alpha(0, z)}; \quad c \leqq x_0 \leqq d.$$

If the limit of this ratio exists as z decreases indefinitely in absolute value, we shall call it the α-derivative of the function $y=f(x)$ for $x=x_0$, and denote it by $f_\alpha(x_0)$. In symbols,

$$f_\alpha(x_0) = \lim_{z\to 0} \frac{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]}{\alpha(0, z)}.$$

If $f(x)$ has an α-derivative for every x_0 on the interval (c, d), we shall say that $f(x)$ is α-differentiable on this interval.

We may supplement the above definition by considering some special cases of the function $\alpha(x, z)$, which may be termed the "differentiator" function.

a) If the differentiator is $x+z$, the limit considered becomes

$$\lim_{z\to 0} \frac{f(x_0+z) - f(x_0)}{z}.$$

Obviously, this limit, when it exists, is merely df/dx at $x=x_0$. Hence, when the differentiator is $x+z$, the α-derivative of a function is merely the ordinary derivative, and it exists if, and only if, the ordinary derivative exists.

b) If the differentiator is $x+|z|$, the limit in question becomes

$$\lim_{z\to 0} \frac{f(x_0+|z|) - f(x_0)}{|z|},$$

* Presented to the Michigan Section of the Mathematical Association of America on March 18, 1933.

which is identical with

$$\lim_{z\to 0^+} \frac{f(x_0+z)-f(x_0)}{z}$$

(where z approaches 0 through positive values only). This last limit, however, is precisely the one used in defining the so-called right-hand derivative of $y=f(x)$ at $x=x_0$. In this case, then, the α-derivative reduces to the right-hand derivative. Similarly, if the differentiator is $x-|z|$, the α-derivative becomes the left-hand derivative.

c) Suppose $f(x)=e^x$, for all real x; $\alpha(x, z)=\log(x+1+z)$, for all $x\geqq t>-1$ and all z such that $|z|<1+t$ (or, when $x=0$, $|z|<1$). Here the limit in question becomes (for any $x\geqq t$)

$$\lim_{z\to 0} \frac{e^{\log(x+1+z)}-e^{\log(x+1)}}{\log(z+1)} = \lim_{z\to 0} \frac{z}{\log(z+1)} = 1.$$

Thus, for the differentiator $\log(x+1+z)$, the function e^x is α-differentiable on the interval $x>-1$, its α-derivative being identically equal to 1 there.

3. *Properties of the α-derivative.* The α-derivative possesses a number of properties which are analogous to those of the ordinary derivative. We proceed to mention here some of the more elementary ones in the several theorems that follow. It is to be understood that all the functions discussed hereafter have appropriate domains of definition, which will not be given explicitly, except where necessary for the sake of clarity.

THEOREM 1. *For a given $y=f(x)$ and differentiator $\alpha(x, z)$, define*

$$F(z) \equiv f[\alpha(x_0, z)];$$

then a necessary condition for $f(x)$ to have an α-derivative at $x=x_0$ is that $F(x)$ be continuous at $z=0$.

Proof. In the ratio whose limit defines the α-derivative, the limit of the denominator is 0 by hypothesis (see §2); hence, the ratio can have a limit only if the limit of the numerator likewise vanishes; i.e., if

$$\lim_{z\to 0} \{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]\} = \lim_{z\to 0} [F(z) - F(0)] = 0.$$

But the last equality states that $F(z)$ is continuous at $z=0$.

THEOREM 2. *For any differentiator, the α-derivative of a constant function is identically* 0.

THEOREM 3. *If $f(x)=g(x)+h(x)$, then, for any differentiator,*

$$f_\alpha(x_0) = g_\alpha(x_0) + h_\alpha(x_0),$$

provided the α-derivatives on the right exist.

THEOREM 4. *If* $f(x)=g(x)\cdot h(x)$, *then, for any differentiator,*

$$f_\alpha(x_0) = g[\alpha(x_0, 0)]\cdot h_\alpha(x_0) + g_\alpha(x_0)\cdot h[\alpha(x_0, 0)],$$

provided the α*-derivatives on the right exist.*

THEOREM 5. *If* $f(x)=1/g(x)$, *then, for any differentiator,*

$$f_\alpha(x_0) = -\frac{g_\alpha(x_0)}{\{g[\alpha(x_0, 0)]\}^2}$$

provided the α*-derivative on the right exists, and* $g[\alpha(x_0, 0)]\neq 0$.

Any one of these theorems is readily demonstrated by means of the fundamental definition. For example, theorem 5 may be demonstrated thus:

$$\frac{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]}{\alpha(0, z)} = \frac{\dfrac{1}{g[\alpha(x_0, z)]} - \dfrac{1}{g[\alpha(x_0, 0)]}}{\alpha(0, z)}$$

$$= -\frac{1}{g[\alpha(x_0, 0)]}\cdot\frac{1}{g[\alpha(x_0, z)]}\cdot\frac{g[\alpha(x_0, z)] - g[\alpha(x_0, 0)]}{\alpha(0, z)}.$$

We now notice that the right-hand product (of the three fractions) certainly has a limit as $z\to 0$; i.e.,

$$f_\alpha(x_0) = \lim_{z\to 0}\frac{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]}{\alpha(0, z)} = -\frac{g_\alpha(x_0)}{\{g[\alpha(x_0, 0)]\}^2}.$$

Theorem 4 yields the following corollary:

COROLLARY 1. *If* $f(x)=k\cdot g(x)$, *where* $k\neq 0$ *is a constant, then, for any differentiator,*

$$f_\alpha(x_0) = k\cdot g_\alpha(x_0),$$

provided the α*-derivative on either side exists.*

Theorems 4 and 5 together give:

COROLLARY 2. *If* $f(x)=h(x)/g(x)$, *then, for any differentiator,*

$$f_\alpha(x_0) = \frac{g[\alpha(x_0, 0)]\cdot h_\alpha(x_0) - h[\alpha(x_0, 0)]\cdot g_\alpha(x_0)}{\{g[\alpha(x_0, 0)]\}^2},$$

provided the α*-derivatives on the right exist, and* $g[\alpha(x_0, 0)]\neq 0$.

4. *An Application.*

For the purposes of this paragraph, the following lemma will be needed:

LEMMA. *Let* $y=f(x)$ *and* $\alpha(x, z)$ *be given, and suppose that, for* $x=x_0$, df/dx

*exists about** $x=\alpha(x_0, 0)$ *and* $d\alpha/dz$ *is defined both about** $x=x_0$, $z=0$, *and about** $x=0$, $z=0$; *finally, suppose that* $\alpha_z(0, 0)\neq 0$. *Then* $f_\alpha(x_0)$ *exists, and*

$$f_\alpha(x_0) = \frac{f'[\alpha(x_0, 0)]\cdot\alpha_z(x_0, 0)}{\alpha_z(0, 0)}.$$

[Here $f'[\alpha(x_0, 0)]$ means the value of df/dx at $x=\alpha(x_0, 0)$; $\alpha_z(x_0, 0)$ denotes the value of $\partial\alpha/\partial z$ at $x=x_0$, $z=0$, and similarly for $\alpha_z(0, 0)$].

Proof. By the hypothesis that $\lim_{z\to 0}\alpha(0, z)=0$ and theorem 1, we see that the ratio (A) is of the form $g(z)/h(z)$, with $\lim_{z\to 0}g(z)=\lim_{z\to 0}h(z)=0$. Hence, if the conditions of our lemma obtain, we can, in passing to the limit, apply L'Hopital's Rule, by first differentiating numerator and denominator separately with respect to z, and then setting $z=0$. But

$$\frac{d}{dz}\{f[\alpha(x_0, z)] - f[\alpha(x_0, 0)]\} = f'[\alpha(x_0, z)]\cdot\alpha_z(x_0, z),$$

and

$$\frac{d}{dz}\,\alpha(0, z) = \alpha_z(0, z).$$

Setting $z=0$, we obtain the conclusion of our lemma.

Consider now the problem of finding a solution for the functional equation

$$\text{(B)} \qquad \left.\frac{df}{dx}\right]_{x=q^2} = f(q).$$

In words, we want to find a differentiable function $f(x)$ of the real variable x such that, when we substitute q^2 for x in $f'(x)$, we obtain $f(q)$. Suppose that $g(x)$ is such a function and consider it in connection with the differentiator $\alpha(x, z) =x^2+z$. Here $\alpha(x, 0)=x^2$, $\alpha_z(x, z)\equiv 1$. By the lemma just demonstrated, we see that, for any x,

$$g_\alpha(x) = \frac{g'(x^2)\cdot 1}{1} = g(x),$$

the last equality being true because $g(x)$ is by hypothesis a solution of (B). In other words, in order to find a solution of (B), it is sufficient to find a differentiable solution of the functional equation

$$\text{(C)} \qquad g_\alpha(x) = g(x);\ \{\alpha(x, z) \equiv x^2 + z\}.$$

By analogy with the infinite series solution of a differential equation, we may attempt to represent a solution $g(x)$ of (C) in the form

$$g(x) = 1 + \sum_{n=1}^{\infty} a_n x^{\lambda_n},$$

* That is to say, in some neighborhood of.

where the constants a_n and λ_n are to be so chosen that the α-derivative of the function $a_n x^{\lambda_n}$ is to be $a_{n-1}x^{\lambda_{n-1}}$, the differentiator being x^2+z [in particular, the α-derivative of $a_1 x^{\lambda_1}$ is therefore to be $\equiv 1$]. A simple calculation—using again the lemma at the beginning of this paragraph—will show that

$$\lambda_n = \frac{2^n - 1}{2^{n-1}},$$

and

$$a_n = \frac{1}{\prod_{n=1}^{n} \lambda_n}.$$

The resulting function is thus

$$g(x) = 1 + x + \frac{x^{3/2}}{3/2} + \frac{x^{7/4}}{3/2 \cdot 7/4} + \cdots + a_n x^{\lambda_n} + \cdots .$$

Since $\lim_{n\to\infty}\lambda_n = 2$ and $a_n \leqq 2^n/3^n$, it can be readily proved* that the series defining $g(x)$ converges for all $x \geqq 0$, and that it converges uniformly in any finite interval (a, b), $b > a \geqq 0$. Hence, since the individual terms of the series defining $g(x)$ are differentiable, we can obtain $g'(x)$ (for any $x \geqq 0$) by differentiating termwise:

$$g'(x) = 1 + x^{1/2} + \frac{x^{3/4}}{3/2} + \cdots .$$

If we put here x^2 for x, we clearly obtain again the series for $g(x)$. We thus see that the function $g(x)$ is indeed a solution for the functional equation (C), and hence also a solution of (B).

[The fact that $g(x)$ is defined for non-negative values of x only should not be surprising, if one notices that equation (B) tells nothing about the values of $g'(x)$ for $x < 0$.]

The special example considered above suggests the following principle:

Suppose we have a functional equation of the form

$$\text{(D)} \qquad \left.\frac{df}{dx}\right]_{x=\phi(q)} = f[\psi(q)],$$

where ϕ and ψ are known functions of x, and f is the unknown function. We may let $\psi(q) = p$, and solve for q in terms of p (assuming this to be possible); substituting in $x = \phi(q)$, the above equation will be reduced to the form

* By using the ratio test for convergence, and the Weierstrass theorem concerning uniform convergence.

$$\left.\frac{df}{dx}\right]_{x=\theta(p)} = f(p). \tag{E}$$

Finally, if the function $\alpha(x, z) = \theta(x) + z$ satisfies the conditions laid down for a differentiator in §2, we can reduce (E) to

$$f_\alpha(x) = f(x), \tag{F}$$

using again the lemma demonstrated at the beginning of this paragraph. In many cases, solving (F) may be simpler than solving (D) directly; it is clear that any solution of (F) will also be a solution of (D).

ON INFINITE RADICALS

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Introduction. For approximately twenty-five years Professor Edward Kasner has periodically suggested to his classes at Columbia University the investigation of the problem of "infinite radicals." Thus it was proposed to find conditions for the convergence or divergence of the "right" infinite radical

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

and the "left" infinite radical

$$\cdots \sqrt{a_3 + \sqrt{a_2 + \sqrt{a_1}}}.$$

In particular, what are the properties of the number K (which we shall call the Kasner number)

$$K = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots}}}\ ? \tag{1}$$

Vieta is probably the first to have used finite radicals of an arbitrary number of root extractions. His famous infinite product, the first purely arithmetical process for calculating π, may be written*

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdot \cdots . \tag{2}$$

Before passing on to twentieth century work we may mention the following brief quotation from *The Life-Romance of an Algebraist* by George Winslow Pierce, 1891, p. 18:

"When a boy I discovered

$$\pi = 2^n\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots}}} \qquad (\text{with } n(=\infty)\sqrt{\ }\text{'s}), \tag{3}$$

$$\pi^{2^n} - \cdots = 2^{2^n \cdot n+1},$$

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* Glaisher, Messenger of Mathematics, vol. 2, (new series), 1873, p. 124.

and showed it to a friend, in the American Nautical Almanac office, Simon Newcomb,—an illustration of infinite IT."

1. *Ramanujan's Problem.* In April, 1911, Srinivasa Ramanujan published the following problem:*

Find the value of:

$$\text{(i)} \qquad \sqrt{[1+2\sqrt{\{1+3\sqrt{(1+\cdots)}\}}]},$$

$$\text{(ii)} \qquad \sqrt{[6+2\sqrt{\{7+3\sqrt{(8+\cdots)}\}}]}.$$

Ramanujan's solution† is incomplete, as we shall now show.

His solution of (i) is as follows:

$$f(n) \equiv n(n+2) = n\sqrt{1+(n+1)(n+3)} = n\sqrt{1+f(n+1)},$$
$$f(n) = n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} = \cdots \quad \cdots,$$
$$f(1) = 3 = \sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}.$$

This solution is incomplete, for we may write similarly

$$4 = \sqrt{1+2\cdot(15/2)} = \sqrt{1+2\sqrt{1+3\cdot(221/12)}} = \sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}},$$

and thus obtain the value 4 for the expression (i).

Thus Ramanujan has shown that

$$\text{(4)} \qquad \begin{aligned} 3 &= \sqrt{1+2\cdot 4} = \sqrt{1+2\sqrt{1+3\cdot 5}} = \cdots \\ &= \sqrt{1+2\sqrt{\cdots+n\sqrt{1+(n+1)(n+3)}}} \end{aligned}$$

and has concluded, without giving adequate justification, that the sequence

$$\text{(5)} \qquad \sqrt{1},\ \sqrt{1+2\sqrt{1}},\ \ldots,\ \sqrt{1+2\sqrt{1+\cdots+n\sqrt{1}}},\ \ldots$$

converges to the limit 3. But this can be proved as follows.

It is clear, upon comparing (4) and (5), that the sequence (5) is monotonic and bounded (<3). Hence it converges to a limit $\leqq 3$. To show that the limit is exactly 3 we shall prove that given an arbitrarily small positive $\epsilon<3$, there exists an integer $N=N(\epsilon)$ such that for all $n>N$,

$$\text{(6)} \qquad u_n \equiv \sqrt{1+2\sqrt{1+\cdots+n\sqrt{1}}} > 3-\epsilon.$$

Since $3-\epsilon=3r$, where $0<r=1-\epsilon/3<1$, we are to show that

$$u_n > 3r = r\sqrt{1+2\sqrt{1+\cdots+n\sqrt{1+(n+1)(n+3)}}},$$

that is,

* Journal of the Indian Mathematical Society, vol. 3 (1911), p. 90, problem 289.

† *Ibid.*, vol. 4 (1912), p. 226. Reproduced in G. H. Hardy's *Collected Papers of Srinivasa Ramanujan*, page 323.

$$\sqrt{1+2\sqrt{1+\cdots+n\sqrt{1}}} > \sqrt{r^2+2\sqrt{r^{2^2}+\cdots+n\sqrt{r^{2^n}[1+(n+1)(n+3)]}}}.$$

But $1>r^{2^i}$, $i=1, 2, \cdots$, and there exists an integer N_1 such that for all $n>N_1$

$$1 > r^{2^n}[1+(n+1)(n+3)] = r^{2^n}(n+2)^2,$$

since $r^{2^n}(n+2)^2 \to 0$ as $n\to\infty$. Hence we may take $N=N_1$ in inequality (6) and our proof is completed. In like manner we can complete Ramanujan's proof that the expression (ii) is equal to 4.

2. *Pólya's Criterion.* We return now to the problem of the convergence or divergence of the "right" infinite radical, i.e., the behavior of the sequence $\{u_n\}$ where

$$u_n = \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}.$$

This problem was proposed by G. Pólya* in the following reduced form:

Show that the sequence $\{u_n\}$

$$\left.\begin{matrix}\text{converges}\\ \text{diverges}\end{matrix}\right\} \text{ if } \varlimsup_{n\to\infty} \frac{\log\log a_n}{n} \begin{matrix}<\\ >\end{matrix} \log 2.$$

It is to be understood that $a_n \geqq 0$, all square roots are taken positive, and that for $a_n \leqq 1$, we shall adopt the convention of Pólya and Szegö that $(\log\log a_n)/n \equiv -\infty$.

Inasmuch as this statement of the problem says nothing concerning the case $\varlimsup_{n\to\infty}(\log\log a_n)/n = \log 2$, it is of interest to note that it is then necessary and sufficient for convergence that

$$\varlimsup_{n\to\infty} n\left\{\frac{\log\log a_n}{n} - \log 2\right\} < +\infty, \tag{7}$$

that is, that there exist an upper limit (7), either finite or $=-\infty$. This may be deduced simply from a necessary and sufficient condition for the convergence of the sequence $\{u_n\}$, which is apparently new and which we now derive.

THEOREM I. The sequence $\{u_n\}$ defined by

$$u_n \equiv \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}},$$

converges if and only if there exists a *finite* upper limit

$$\varlimsup_{n\to\infty} a_n^{2^{-n}} < +\infty.$$

Proof. First, suppose $\{u_n\}$ converges. Since

$$u_n \geqq \sqrt{0+\sqrt{a_2+\cdots+\sqrt{a_n}}} \geqq \cdots \geqq \sqrt{0+\sqrt{0+\cdots+\sqrt{a_n}}} = a_n^{2^{-n}}$$

the $\varlimsup_{n\to\infty} a_n^{2^{-n}}$ must be finite, i.e. $\{a_n^{2^{-n}}\}$ is bounded.

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* Aufgabe; Arch. d. Math. u. Phys. Serie 3, Bd. 24 (1916), p. 84: see also Pólya and Szegö *Aufgaben und Lehrsätze*, vol. 1, p. 30, problem 162.

Second, suppose $\overline{\lim}_{n\to\infty} a_n^{2^{-n}} < +\infty$. Then we may select a $G>0$ such that, for all $n>0$, $a_n^{2^{-n}} \leqq G$ and thus

$$a_n \leqq G^{2^n}.$$

Hence

$$\begin{aligned} u_n &\leqq \sqrt{G^2 + \sqrt{G^{2^2} + \cdots + \sqrt{G^{2^n}}}} \\ &= G\sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}. \end{aligned}$$

But

$$2 = \sqrt{2+2} = \sqrt{2+\sqrt{2+2}} = \cdots = \sqrt{2+\sqrt{2+\cdots+\sqrt{2+2}}} > \sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}$$

and thus

$$u_n < 2G, \text{ for all } n > 0.$$

However the sequence $\{u_n\}$ is non-decreasing; therefore it converges.

Let us now consider the criterion given by M. Pólya. In the first of the two cases $\overline{\lim}_{n\to\infty} (\log\log a_n)/n < \log 2$, that is, we may find an N such that for all $n>N$, $(\log\log a_n)/n < \log 2$, $a_n^{2^{-n}} < e$, and thus

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} \leqq e.$$

But in the second case $\overline{\lim}_{n\to\infty} (\log\log a_n)/n > \log 2$. Hence for some number $a>1$, $(\log\log a_n)/n > a\log 2$, i.e., $\log a_n > 2^{an}$, for an infinite number of values of n. That is, for these values of n,

$$a_n^{2^{-n}} > e^{2^{(a-1)n}},$$

and thus

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} = +\infty.$$

Finally there is the case $\overline{\lim}_{n\to\infty} (\log\log a_n)/n = \log 2$. Suppose $\{u_n\}$ converges in this case. Then $\overline{\lim}_{n\to\infty} a_n^{2^{-n}}$ is finite, by theorem I, i.e., there exists a $G>1$ such that $a_n^{2^{-n}} < G$ for all n. Hence $\log a_n < 2^n \log G$ and if $a_n > 1$

$$n\{(\log\log a_n)/n - \log 2\} < \log\log G$$

while if $a_n \leqq 1$ we have, by convention, $(\log\log a_n)/n = -\infty$. Consequently the condition (7) that $\overline{\lim}_{n\to\infty} n\{\log\log a_n)/n - \log 2\} < +\infty$ is necessary for convergence. It is also sufficient. For suppose the condition (7) holds but $\{u_n\}$ does not converge. Then $\overline{\lim}_{n\to\infty} a_n^{2^{-n}} = +\infty$ and we may find, given any $G>e$, an infinitude of integers n such that $a_n^{2^{-n}} > G$ and thus $\log a_n > 2^n \log G$ so that

$$n\{(\log\log a_n)/n - \log 2\} > \log\log G.$$

This contradicts our hypothesis that the condition (7) holds.

3. *Degree of Approximation.* It is easy to apply our method for completing the solution of Ramanujan's problem to the proof that $2=\sqrt{2+\sqrt{2+\cdots}}$. But we shall derive a more interesting relation which gives us an insight into the degree of approximation of $u_n=\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}$. Thus

$$2\cos(\theta/2)=\sqrt{2+2\cos\theta}=\sqrt{2+\sqrt{2+2\cos 2\theta}}=\cdots$$
$$=\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos 2^{n-1}\theta}}}$$

provided $\cos(\theta/2)\geqq 0$, $\cos\theta\geqq 0,\cdots$, $\cos 2^{n-2}\theta\geqq 0$, (we consider only non-negative square roots). But this condition is satisfied for $\theta=\pi/2^n$ and so

$$2\cos\frac{\pi}{2^{n+1}}=\sqrt{2+\sqrt{2+\cdots\sqrt{2+0}}}=u_n.$$

Hence $\lim_{n\to\infty}u_n=2$ and $2-u_n=2[1-\cos(\pi/2^{n+1})]$. Therefore

$$\lim_{n\to\infty}2^{2n}(2-u_n)=\pi^2/4,\qquad 2-u_n\sim\pi^2/(4\cdot 2^{2n}).$$

Since

$$\sqrt{2-u_{n-1}}=\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}\sim\pi/2^n,$$

where u_n has n radicals, it is clear that equation (3) of G. W. Pierce may be given a rigorous interpretation. But it is not as clear just what the second of Pierce's equations really means.

Consider now the more general case (where every $a_n>0$) and write

$$u_n=\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}$$
$$U_n=\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n+r_n}}}\qquad r_n\geqq 0.$$

We seek an approximate formula for the difference U_n-u_n. We shall make repeated application of the formula, easily proved,

$$\sqrt{a+x}\leqq\sqrt{a}+\frac{x}{2\sqrt{a}},\qquad a>0\leqq x.$$

Thus

$$\sqrt{a_n+r_n}\leqq\sqrt{a_n}+\frac{r_n}{2\sqrt{a_n}}$$

$$\sqrt{a_{n-1}+\sqrt{a_n+r_n}}\leqq\sqrt{a_{n-1}+\sqrt{a_n}+\frac{r_n}{2\sqrt{a_n}}}\leqq\sqrt{a_{n-1}+\sqrt{a_n}}$$
$$+\frac{r_n}{2^2\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}}$$

. . . .

$$\sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n+r_n}}} \leqq \sqrt{a_1+\sqrt{a_2+\cdots+\sqrt{a_n}}}$$

$$+\frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}\cdot\ \cdots\ \cdot\sqrt{a_1+\cdots+\sqrt{a_n}}}$$

$$U_n - u_n \leqq \frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}+\sqrt{a_n}}\cdot\ \cdots\ \cdot\sqrt{a_1+\cdots+\sqrt{a_n}}}. \tag{8}$$

This inequality is obviously sharper than that given by Pólya and Szegö,* namely

$$U_n - u_n < \frac{r_n}{2^n\sqrt{a_n}\cdot\sqrt{a_{n-1}}\cdot\ \cdots\ \cdot\sqrt{a_1}}.$$

In the particular case $u_n=\sqrt{2+\sqrt{\ \cdots\ +\sqrt{2}}}$, $r_n=2$, we have $U_n=2$. From the inequality in the *Aufgaben* we may infer merely

$$2-u_n = U_n - u_n < \frac{2}{2^n\cdot\sqrt{2^n}} = \frac{2}{2^{3n/2}}.$$

From inequality (8) however we get

$$2-u_n = U_n-u_n \leqq \frac{2}{2^n\sqrt{2}\cdot\sqrt{2+\sqrt{2}}\cdot\ \cdots\ \cdot\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}.$$

Now $u_n=\sqrt{2+\ \cdots\ +\sqrt{2}}=2\cos(\pi/2^{n+1})$ so that $\sqrt{2}\cdot\sqrt{2+\sqrt{2}}\cdot\ \cdots\ \cdot\sqrt{2+\ \cdots\ +\sqrt{2}}=2^n\cos(\pi/2^2)\cdot\cos(\pi/2^3)\cdot\ \cdots\ \cdot\cos(\pi/2^{n+1})$. Hence

$$2-u_n=U_n-u_n \leqq \frac{2}{2^n u_1 u_2\cdots u_n} \sim \frac{2}{2^{2n}\cdot 2/\pi} = \frac{\pi}{2^{2n}}$$

which gives the correct order of smallness of $2-u_n \sim \pi^2/4\cdot 2^{2n}$.

4. *Examples of Infinite Radicals.* Let us examine now the infinite radical defined by $u_n=\sqrt{x+\sqrt{x+\ \cdots\ +\sqrt{x}}}$, $x>0$. The sequence $\{u_n\}$ is monotone increasing. For $x\leqq 1$ it is bounded by the corresponding sequence for $x=1$ which, as we saw earlier, is bounded by 2. And when $x>1$,

$$u_n < \sqrt{x^2+\sqrt{x^{2^2}+\cdots+\sqrt{x^{2^n}}}} = x\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}} < 2x.$$

Hence we have established, independently of theorem I, the convergence of $\{u_n\}$. But, calling $\lim_{n\to\infty} u_n=u$, we may calculate u thus: $u_{n+1}^2=x+u_n$ and therefore

* *Aufgaben und Lehrsätze*, vol. 1, p. 30, problem 163.

$$u^2 = x + u, \qquad u = \frac{1 + \sqrt{1 + 4x}}{2}, \qquad x > 0.$$

However

$$u = 0 \quad \text{if} \quad x = 0.$$

We are now able to compute Kasner's number K which exists by Theorem I. Thus

$$K > \sqrt{1 + \sqrt{2 + \cdots + \sqrt{9 + \sqrt{10 + \sqrt{10 + \cdots}}}}} = 1\cdot 757933 \cdots$$

$$K < \sqrt{1 + \sqrt{\cdots + \sqrt{9 + \sqrt{10 + \sqrt{10^2 + \sqrt{10^{2^2} + \cdots}}}}}} = 1\cdot 757933 \cdots$$

$$K = 1\cdot 757933 \cdots .$$

In our calculation we used Bruns' seven place logarithms (1894).

Consider now the relation

$$x(2^n + x) = x\sqrt{2^{2n} + x(2^{n+1} + x)}.$$

Thus

$$x(2 + x) = x\sqrt{2^2 + x(2^2 + x)} = x\sqrt{2^2 + x\sqrt{2^4 + x(2^3 + x)}} = \cdots$$
$$= x\sqrt{2^2 + x\sqrt{2^4 + \cdots + x\sqrt{2^{2n} + x(2^{n+1} + x)}}}.$$

It is easy to prove (as in Ramanujan's problem) that

$$x(2 + x) = x\sqrt{2^2 + x\sqrt{2^4 + x\sqrt{\cdots}}} \tag{9}$$

and so

$$3 = \sqrt{2^2 + \sqrt{2^4 + \sqrt{\cdots + \sqrt{2^{2n} + \cdots}}}}.$$

From (9) we have, substituting $x/2$ for x,

$$2\left(1 + \frac{x}{4}\right) = \sqrt{2^2 + \frac{x}{2}\sqrt{2^4 + \frac{x}{2}\sqrt{2^6 + \cdots}}}$$

and

$$\left(1 + \frac{x}{4}\right) = \sqrt{1 + \frac{x}{2}\sqrt{1 + \frac{x}{2^2}\sqrt{1 + \cdots + \sqrt{1 + \frac{x}{2^n}\sqrt{\cdots}}}}}.$$

Left infinite radicals satisfy a characteristic difference equation of the second degree. Thus

$$u_n = \sqrt{a_n + \sqrt{\cdots + \sqrt{a_1}}}, \qquad u_{n+1} = \sqrt{a_{n+1} + \sqrt{a_n + \cdots + \sqrt{a_1}}}$$

$$u_{n+1}^2 = a_{n+1} + u_n.$$

On the other hand, in the case of right infinite radicals, if $\{u_n\}$ converges to a limit u,

$$u_n = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}$$

$$u = \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n + r_n}}}$$

$$r_n = \sqrt{a_{n+1} + r_{n+1}}, \qquad r_{n+1} = r_n^2 - a_{n+1}.$$

5. *Left Infinite Radicals.* We now prove

THEOREM II. The necessary and sufficient condition for the convergence of $\{u_n\}$,

$$u_n = \sqrt{a_n + \sqrt{a_{n-1} + \cdots + \sqrt{a_1}}},$$

is that there exist a limit a of the sequence $\{a_n\}$, i.e., that $a_n \to a$. When such a limit exists then

$$u_n \to \frac{1 + \sqrt{1 + 4a}}{2} \qquad \text{if } a > 0$$

$$u_n \to 1 \qquad \text{if } a = 0 \text{ and at least one } a_n > 0$$

$$u_n \equiv 0 \qquad \text{if } a = 0 \text{ and all } a_n = 0.$$

Proof. First suppose the radical converges. Observe that

$$u_{n+1}^2 = a_{n+1} + u_n \text{ and } a_{n+1} = u_{n+1}^2 - u_n \to u^2 - u$$

where u_n converges to the limit u. Hence the condition is necessary.

Next suppose that $a_n \to a > 0$, assuming that each $a_n \geqq 0$. Given any positive $\delta < a$ there exists an integer N such that for all $n > N$,

$$0 < a - \delta < a_n < a + \delta.$$

Hold δ fixed and choose a positive ϵ arbitrarily small. There exists an integer N_1 such that

$$(1 + \epsilon)^{2^{N_1}} > \frac{a + \delta + u_N}{a + \delta}, \qquad N = N(\delta) \text{ and } N_1 = N_1(\delta, \epsilon).$$

Hence for all $n > N + N_1$

$$\begin{aligned} u_n = \sqrt{a_n + \cdots + \sqrt{a_{N+1} + u_N}} &< \sqrt{a + \delta + \sqrt{\cdots + \sqrt{a + \delta + u_N}}} \\ &< (1 + \epsilon)\sqrt{a + \delta + \sqrt{\cdots + \sqrt{a + \delta}}} \\ &< (1 + \epsilon)\sqrt{a + \delta + \sqrt{a + \delta + \sqrt{\cdots}}}. \end{aligned}$$

Again,

$$u_n > \sqrt{a - \delta + \sqrt{\cdots + \sqrt{a - \delta + u_N}}} \geqq \sqrt{a - \delta + \sqrt{\cdots + \sqrt{a - \delta}}}$$

where the last expression involves $n-N$ radicals. Letting $n \to \infty$,

$$\sqrt{a-\delta+\sqrt{a-\delta+\sqrt{\cdots}}} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n$$
$$\leqq (1+\epsilon)\sqrt{a+\delta+\sqrt{a+\delta+\sqrt{\cdots}}}$$

i.e.,

$$\frac{1+\sqrt{1+4(a-\delta)}}{2} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq (1+\epsilon)\cdot\frac{1+\sqrt{1+4(a+\delta)}}{2}.$$

Now letting $\epsilon \to 0$,

$$\frac{1+\sqrt{1+4(a-\delta)}}{2} \leqq \varliminf_{n\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq \cdot\frac{1+\sqrt{1+4(a+\delta)}}{2}.$$

But $\delta>0$ is arbitrary and may be permitted to approach zero. Thus

$$\frac{1+\sqrt{1+4a}}{2} \leqq \varliminf_{m\to\infty} u_n \leqq \varlimsup_{n\to\infty} u_n \leqq \frac{1+\sqrt{1+4a}}{2}$$

$$\lim_{n\to\infty} u_n = \frac{1+\sqrt{1+4a}}{2}.$$

Let us now consider the case $a_n \to a = 0$. If every $a_n=0$ then every $u_n=0$ and so $u=0$. We may therefore suppose at least one $a_n>0$ and, without any loss of generality, we let $a_1>0$. Thus for any $n \geqq 1$

$$u_n = \sqrt{a_n+\sqrt{\cdots+\sqrt{a_1}}} \geqq a_1{}^{2^{-}}$$

and thus

$$\varliminf_{n\to\infty} u_n \geqq 1.$$

As before, given any arbitrarily small positive δ we may find an integer $N_1=N_1(\delta)$ such that for all $n>N_1$, $a_n<\delta$. Hold δ fixed and select an $\epsilon>0$ arbitrarily small. To ϵ there corresponds an integer N such that for $n>N$,

$$(1+\epsilon)^{2^n} > (1+\epsilon)^{2^N} > \frac{\delta+u_{N_1}}{\delta}, \qquad N = N(\delta,\epsilon).$$

Hence for all $n>N+N_1$

$$u_n < \sqrt{\delta+\sqrt{\cdots+\sqrt{\delta+u_{N_1}}}} < (1+\epsilon)\sqrt{\delta+\sqrt{\cdots+\sqrt{\delta}}}$$
$$< (1+\epsilon)\sqrt{\delta+\sqrt{\delta+\sqrt{\cdots}}}$$
$$= (1+\epsilon)\cdot\frac{1+\sqrt{1+4\delta}}{2}.$$

Letting $\epsilon \to 0$

$$\overline{\lim_{n\to\infty}}\, u_n \leqq \frac{1+\sqrt{1+4\delta}}{2}.$$

Since $\delta > 0$ is arbitrarily small we may let $\delta \to 0$ and so

$$1 \leqq \underline{\lim_{n\to\infty}}\, u_n \leqq \overline{\lim_{n\to\infty}}\, u_n \leqq 1, \qquad \lim_{n\to\infty} u_n = 1.$$

6. *Generalized Infinite Radicals.*

It is interesting to notice that if we generalize right infinite radicals by writing

$$u_n = \left\{a_1 + \left[a_2 + (a_3 + \cdots + a_n^{r_n})^{r_3}\right]^{r_2}\right\}^{r_1}$$

we may secure the ordinary series form by putting $r_1 = r_2 = \cdots = 1$, i.e.,

$$u_n = a_1 + a_2 + \cdots + a_n.$$

Also if we put all the exponents $= -1$ we obtain the continued fraction

$$u_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}.$$

Through a slightly broader generalization of right infinite radicals we may obtain as special cases the infinite product and the ascending continued fraction.

One may readily prove

THEOREM III. Let

(1) $$u_n = (a_1 + \{a_2 + \cdots + a_n^{r_n}\}^{r_2})^{r_1}$$

where $a_i \geqq 0$, $0 < r_i \leqq 1$, $i = 1, 2, \cdots$, and the series

(2) $$S = \sum_{i=1}^{\infty} r_1 r_2 \cdots r_i$$

converges. Then the necessary and sufficient condition for the sequence $\{u_n\}$ defined by (1) to converge is that

$$\overline{\lim_{n\to\infty}}\, a_n^{r_1 r_2 \cdots r_n} < +\infty.$$

It is also of interest to generalize infinite radicals to include negative or complex elements a_n. Convergence questions appear to become very difficult in such cases.

In addition to the references mentioned in this article the following may be noted:

1. "*Aufgaben und Lehrsätze,*"—G. Pólya and G. Szegö, vol. 1, p. 29, problem 161; vol. 1, p. 33, problems 183–5; the solutions refer to other literature.

2. Jahrbuch Fortschr. d. Math., vol. 39 (1908), p. 501; M. Cipolla, "*Intorno ad un radicale continuo*"; G. Candido, "*Sul numero* π."

3. Jahresbericht d. Deutsch Math. Ver., vol. 33 (1924), p. 69 and pp. 117–118; vol. 39 (1930), p. 6.

4. Zeitschr. f. Math. und naturw. Unterr., vol. 41 (1910), p. 161–186.

5. American Math. Monthly, Jan. 1917, problem 460.

6. National Mathematics Magazine, (Baton Rouge, La.), April 1935, problems 75, 78; May 1935, p. 247–8, 251–2.

ON SPHERES ASSOCIATED WITH THE TETRAHEDRON

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Notations. Let $ABCD$ be a tetrahedron with edges $BC=a$, $CA=b$, $AB=c$, $DA=a'$, $DB=b'$, $DC=c'$; (O) its circumsphere with center O and radius R; G its centroid; G_a, G_b, G_c, G_d the centroids of the faces BCD, CDA, DAB, ABC respectively; Ω the Monge point† (i.e., the symmetric of O with respect to G); M_a, M_b, M_c, M_d the medians of the tetrahedron (i.e., line segments which join the vertices A, B, C, D to the centroids G_a, G_b, G_c, G_d of the opposite faces).

1. We point out some little known formulas which are useful in the study of the tetrahedron.

We have first‡

$$
(1)\qquad
\begin{aligned}
M_a^2 &= \overline{AG_a}^2 = (a'^2+b^2+c^2)/3-(a^2+b'^2+c'^2)/9\\
M_b^2 &= \overline{BG_b}^2 = (a^2+b'^2+c^2)/3-(a'^2+b^2+c'^2)/9\\
M_c^2 &= \overline{CG_c}^2 = (a^2+b^2+c'^2)/3-(a'^2+b'^2+c^2)/9\\
M_d^2 &= \overline{DG_d}^2 = (a'^2+b'^2+c'^2)/3-(a^2+b^2+c^2)/9;
\end{aligned}
$$

from which we obtain§

$$
(2)\qquad M_a^2+M_b^2+M_c^2+M_d^2 = 4(a^2+a'^2+b^2+b'^2+c^2+c'^2)/9,
$$

and then

$$
(3)\qquad \overline{GA}^2+\overline{GB}^2+\overline{GC}^2+\overline{GD}^2 = (a^2+a'^2+b^2+b'^2+c^2+c'^2)/4.
$$

We shall call the line segments MN, ST, UV joining the midpoints of the

---

* Translated by J. E. LaFon, University of Oklahoma.

† Monge, Correspondance de l'Ecole Polytechnique de Paris, t. II.

‡ See Educational Times, 1890, p. 114. Journal de G. de Longchamps, 1890, p. 262. V. Thébault, Mathesis, 1930, p. 254.

§ V. Thébault, loc. cit.

opposite edges BC and DA, CA and DB, AB and DC bimedians of the tetrahedron. Then

$$
\begin{aligned}
\overline{MN}^2 &= (b^2 + b'^2 + c^2 + c'^2 - a^2 - a'^2)/4,\\
\overline{ST}^2 &= (c^2 + c'^2 + a^2 + a'^2 - b^2 - b'^2)/4,\\
\overline{UV}^2 &= (a^2 + a'^2 + b^2 + b'^2 - c^2 - c'^2)/4,
\end{aligned} \tag{4}
$$

from which*

$$
\begin{aligned}
\overline{MN}^2 + \overline{ST}^2 + \overline{UV}^2 &= (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/4\\
&= \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2.
\end{aligned} \tag{5}
$$

We have also proved†

$$
\begin{aligned}
\overline{\Omega A}^2 &= R^2 + (a'^2 + b^2 + c^2 - a^2 - b'^2 - c'^2)/4,\\
\overline{\Omega B}^2 &= R^2 + (a^2 + b'^2 + c^2 - a'^2 - b^2 - c'^2)/4,\\
\overline{\Omega C}^2 &= R^2 + (a^2 + b^2 + c'^2 - a'^2 - b'^2 - c^2)/4,\\
\overline{\Omega D}^2 &= R^2 + (a'^2 + b'^2 + c'^2 - a^2 - b^2 - c^2)/4,
\end{aligned} \tag{6}
$$

from which‡ we obtain

$$
\overline{\Omega A}^2 + \overline{\Omega B}^2 + \overline{\Omega C}^2 + \overline{\Omega D}^2 = 4R^2. \tag{7}
$$

Also, for example,

$$
2\overline{\Omega M}^2 + a^2/2 = \overline{\Omega B}^2 + \overline{\Omega C}^2,
$$

from which

$$
\overline{\Omega M}^2 = R^2 - a'^2/4, \qquad \overline{\Omega N}^2 = R^2 - a^2/4;
$$

then, and by analogy,

$$
\begin{aligned}
\overline{\Omega M}^2 + \overline{\Omega N}^2 &= 2R^2 - (a^2 + a'^2)/4,\\
\overline{\Omega S}^2 + \overline{\Omega T}^2 &= 2R^2 - (b^2 + b'^2)/4,\\
\overline{\Omega U}^2 + \overline{\Omega V}^2 &= 2R^2 - (c^2 + c'^2)/4;
\end{aligned} \tag{8}
$$

and finally

$$
\overline{\Omega M}^2 + \overline{\Omega N}^2 + \cdots + \overline{\Omega V}^2 = 6R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/4. \tag{9}
$$

Stewart's theorem applied to the triangle ΩMA, for example, gives

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* Cfr. Dostor, Nouvelles Annales de Mathématiques, 1867, p. 463. Journal de G. de Longchamps, 1878, p. 284. This MONTHLY, 1918, p. 122. V. Thébault, Mathesis, loc. cit.

† Gazeta Matematica, Bucarest, 1933, p. 86.

‡ *Note by translator.* From a property of centroids one has $\sum\overline{\Omega A}^2 = \sum\overline{GA}^2 + 4\overline{\Omega G}^2$ and $\sum\overline{OA}^2 = \sum\overline{GA}^2 + 4\overline{OG}^2$, from which $\sum\overline{\Omega A}^2 = 4R^2$. Using (5), $\overline{\Omega G}^2 = R^2 - \sum a^2/16$ which is found useful in verifying some of the formulas. The author introduces this expression later.

$$2\overline{\Omega M}^2 + \overline{\Omega A}^2 = 3\overline{\Omega G_d}^2 + 2\overline{AM}^2/3.$$

Thus we prove

$$(10)\quad \begin{aligned} \overline{\Omega G_a^2} &= R^2 - [3(a'^2 + b^2 + c^2) + (a^2 + b'^2 + c'^2)]/36, \\ \overline{\Omega G_b^2} &= R^2 - [3(a^2 + b'^2 + c^2) + (a'^2 + b^2 + c'^2)]/36, \\ \overline{\Omega G_c^2} &= R^2 - [3(a^2 + b^2 + c'^2) + (a'^2 + b'^2 + c^2)]/36, \\ \overline{\Omega G_d^2} &= R^2 - [3(a'^2 + b'^2 + c'^2) + (a^2 + b^2 + c^2)]/36, \end{aligned}$$

from which

$$(11)\qquad \sum \overline{\Omega G_a^2} = 4R^2 - 2(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/9.$$

Let G_a', G_b', G_c', G_d' be the midpoints of the medians of the tetrahedron (i.e., the symmetrics of the points G_a, G_b, G_c, G_d with respect to G). On applying Stewart's theorem to the triangles such as $\Omega G_d' G_d$, we prove

$$(12)\quad \begin{aligned} \overline{\Omega G_a'^2} &= R^2 - (a^2 + b'^2 + c'^2)/9 = \overline{OG_a^2}, \\ \overline{\Omega G_b'^2} &= R^2 - (a'^2 + b^2 + c'^2)/9 = \overline{OG_b^2}, \\ \overline{\Omega G_c'^2} &= R^2 - (a'^2 + b'^2 + c^2)/9 = \overline{OG_c^2}, \\ \overline{\Omega G_d'^2} &= R^2 - (a^2 + b^2 + c^2)/9 = \overline{OG_d^2}, \end{aligned}$$

from which

$$(13)\quad \sum \overline{\Omega G_a'^2} = \sum \overline{OG_a^2} = 4R^2 - 2(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/9.$$

2. According to the formulas (8), the powers of the Monge point with respect to the spheres described on a pair of opposite edges, BC, DA for example, as diameters are given by

$$(P_m) = \overline{\Omega M}^2 - a^2/4 = R^2 - (a^2 + a'^2)/4 = \overline{\Omega N}^2 - a'^2/4 = (P_n).$$

Similarly,

$$\begin{aligned} (P_s) &= R^2 - (b^2 + b'^2)/4 = (P_t), \\ (P_u) &= R^2 - (c^2 + c'^2)/4 = (P_v), \end{aligned}$$

from which

$$(14)\quad \begin{aligned} (P_m) + (P_n) &= 2R^2 - (a^2 + a'^2)/2, \\ (P_s) + (P_t) &= 2R^2 - (b^2 + b'^2)/2, \\ (P_u) + (P_v) &= 2R^2 - (c^2 + c'^2)/2, \end{aligned}$$

and

$$(15)\qquad \sum (P_m) = 6R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/2.$$

Thus we have:

The Monge point has equal powers with respect to the spheres described on two opposite edges of the tetrahedron as diameters.*

We conclude that Ω is the point common to the radical planes of the pairs of spheres† $(M, a/2)$ and $(N, a'/2)$; $(S, b/2)$ and $(S, b'/2)$; $(U, c/2)$ and $(V, c'/2)$.

N. A. Court has shown that two planes passing through the center O of the circumscribed sphere of the tetrahedron $ABCD$ perpendicular to two of the bimedians divide the third bimedian harmonically.‡

Thus the radical planes of the pairs of spheres $(M, a/2)$ and $(N, a'/2)$; $(S, b/2)$ and $(T, b'/2)$; $(U, c/2)$ and $(V, c'/2)$ divide harmonically the segment between the centers of the two remaining spheres.

3. The powers of Ω with respect to the spheres described on the bimedians MN, ST, UV as diameters have the values

$$
\begin{aligned}
(p_1) &= R^2 - (b^2 + b'^2 + c^2 + c'^2)/8, \\
(p_2) &= R^2 - (c^2 + c'^2 + a^2 + a'^2)/8, \qquad (16)\\
(p_3) &= R^2 - (a^2 + a'^2 + b^2 + b'^2)/8;
\end{aligned}
$$

from which

$$
\sum (p_1) = 3R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/4 = \sum (P_m)/2. \qquad (17)
$$

Hence the theorem:

The sum of the powers of the Monge point with respect to the spheres described on the edges of the tetrahedron as diameters is twice the sum of the powers of the same point with respect to the spheres described on the bimedians as diameters.

4. More generally, consider any point Q in space. We have

$$
\overline{QA}^2 + \overline{QB}^2 + \overline{QC}^2 + \overline{QD}^2 = \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2 + 4\overline{QG}^2
$$

and

$$
2(\overline{QM}^2 + \overline{QN}^2) = \overline{QA}^2 + \overline{QB}^2 + \overline{QC}^2 + \overline{QD}^2 - (a^2 + a'^2)/2,
$$

from which

$$
\overline{QM}^2 + \overline{QN}^2 = \sum \overline{GA}^2/2 + 2\overline{QG}^2 - (a^2 + a'^2)/4.
$$

Considering (3), and letting (P_m), (P_n), $\cdots$, denote the powers of the point Q for the spheres described on two opposite edges BC, DA, $\cdots$, as diameters

$$
\begin{aligned}
(P_m) + (P_n) &= (b^2 + b'^2 + c^2 + c'^2 - 3a^2 - 3a'^2)/8 + 2\overline{QG}^2, \\
(P_s) + (P_t) &= (c^2 + c'^2 + a^2 + a'^2 - 3b^2 - 3b'^2)/8 + 2\overline{QG}^2, \\
(P_u) + (P_v) &= (a^2 + a'^2 + b^2 + b'^2 - 3c^2 - 3c'^2)/8 + 2\overline{QG}^2;
\end{aligned}
$$

---

* J. Neuberg, Mathesis, 1925, p. 196.

† The notation $(M, a/2)$ denotes a sphere with center at M and radius $a/2$.

‡ This MONTHLY, 1933, p. 561.

from which

$$\sum (P_m) = 6\overline{QG}^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/8;$$

or, using (4),

$$(18) \qquad \sum (P_m) = 2(\overline{QG}^2 - \overline{MN}^2/4 + \overline{QG}^2 - \overline{ST}^2/4 + \overline{QG}^2 - \overline{UV}^2/4);$$

Thus we have:

The sum of the powers of any point in space with respect to the spheres described on the edges of a tetrahedron as diameters is twice the sum of the powers of the same point with respect to the spheres described on the bimedians as diameters.

In particular, when $Q = \Omega$, we have:

The sum of the powers of the Monge point with respect to the spheres described on three edges issued from the same vertex as diameters is constant for the four vertices of the tetrahedron.

Because of (15) this constant is

$$3R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/4 = \sum (p_1).$$

5. The formulas (1) and (12) give the expressions for the powers of the point O with respect to the spheres having for centers G_a, G_b, G_c. G_d and for diameters M_a, M_b, M_c, M_d. If (P_a), (P_b), (P_c), (P_d) denote these powers

$$(P_a) = R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/12 = (P_b) = (P_c) = (P_d).$$

Consequently:

The circumcenter of a tetrahedron is the radical center of the spheres having the centroids of the faces for centers and the corresponding medians of the tetrahedron for diameters. The common power of this point with respect to the four spheres is one third of the power of the Monge point for the circumsphere of the tetrahedron.*

In fact since†

$$(19) \qquad \overline{OG}^2 = \overline{G\Omega}^2 = R^2 - (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/16,$$

* *Note by translator.* This is a special case of a theorem by N. A. Court, this MONTHLY, 1932, p. 198. If the medians of a tetrahedron are divided in the same ratio and the points of division are taken for centers of four spheres whose radii are proportional to the respective medians, the radical center of these four spheres lies on the Euler line of the tetrahedron. He lets A' be such that $GA'/GA = u$; (A') be the sphere with A' as center, and $a' = v \cdot GA$ as radius; and (B'), (C'), (D') be the analogous spheres. He then shows that

$$OR/OG = (v^2 - u^2 + u)/u,$$

where R is the radical center of (A'), (B'), (C'), (D'). In our case $A' \equiv G_a$, $u = -1/3$, $v = 2/3$, $RO = 0$, i.e., $R \equiv 0$. The formula further lets one of u, v be assigned and the other so determined that $R \equiv 0$, with certain limitations for real spheres.

† V. Thébault, Nouvelles Annales de Mathématique, 1919, p. 424.

we have

$$(P_a) = (\overline{O\Omega}^2 - R^2)/3.$$

6. The same formulas (1), (12) give the values of powers (P_a'), (P_b'), (P_c'), (P_d') of the Monge point for the spheres described on the medians AG_a, BG_b, CG_c, DG_d as diameters, since the centers of the spheres coincide with the points G_a', G_b', G_c', G_d' and since

$$\Omega G_a = \Omega G_a', \cdots, \Omega G_d = \Omega G_d'.$$

Thus we meet this theorem due to N. A. Court:*

The Monge point is the radical center of the spheres described on the medians of the tetrahedron as diameters and the common power of this point with respect to the four spheres is one third of its power for the circumsphere of the tetrahedron.

The Monge point is common to the six planes each of which passes through the midpoint of one edge and is perpendicular to the opposite edge. The three planes which correspond to the edges issued from the same vertex are then perpendicular to the face opposite the vertex considered. As the tetrahedron $G_a' G_b' G_c' G_d'$ is homothetic to $ABCD$ and since Ω is the radical center of the spheres with centers G_a', G_b', G_c', G_d' described on the medians as diameters, the six planes drawn through the midpoints of the edges perpendicular to the respective opposite edges are the radical planes of the spheres (G_a'), (G_b'), (G_c'), (G_d') taken two by two.

7. Finally the sum of the powers (G_m), (G_n), $\cdots$, (G_v) of the centroid G with respect to the spheres described on the edges of the tetrahedron as diameters is given by

$$\sum (G_m) = -(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)/8,$$

for

$$\overline{GM}^2 = \overline{MN}^2/4 = (b^2 + b'^2 + c^2 + c'^2 - a^2 - a'^2)/16$$

and

$$(G_m) = \overline{GM}^2 - a^2/4 = (b^2 + b'^2 + c^2 + c'^2 - 5a^2 - a'^2)/16.$$

Also

$$\overline{GO}^2 - R^2 = -\sum a^2/16 = -\sum MN^2/4 = \sum (G_m)/2.$$

Thus we have:

The power of the centroid with respect to the circumsphere of the tetrahedron is numerically equal but opposite in sign to one fourth of the sum of the squares of the bimedians.

These relations and properties help one in the study of special tetrahedrons; notably the orthocentric and the isosceles tetrahedrons.

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* This MONTHLY, 1932, p. 196.

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

An Inquiry. Professor R. A. Johnson asks the question: "Has a computing machine been invented for evaluating determinants by direct process?" Some of our readers can perhaps furnish information on this point. EDITOR.

THE SOLUTION OF PROBLEMS IN MAXIMA AND MINIMA BY ALGEBRA

By RAYMOND GARVER, University of California at Los Angeles

The purpose of this note is to point out the wide applicability of two rather simple algebraic theorems in the solution of problems in maxima and minima. While both the theorems and their application to such problems are well-known, at least in the sense that they are treated in some of the older books,* I find nowhere any hint as to the real power of the algebraic methods. Not only do they lead to the ready solution of a large proportion of the maximum and minimum problems as stated in the ordinary calculus text, but they handle just as readily many problems involving several variables which the beginning calculus student cannot solve.

The theorems used are

THEOREM I. If $x_1, x_2, \cdots, x_n$, are n positive quantities with $k_1x_1+k_2x_2+\cdots k_nx_n=c$, $k_1, \cdots, k_n, c$, positive constants, then the product $x_1x_2\cdots x_n$ is a maximum when $k_1x_1=k_2x_2=\cdots=k_nx_n$.

THEOREM II. If $x_1, x_2, \cdots, x_n, k_1, \cdots, k_n, c$, have the same significance as above, and if $x_1x_2\cdots x_n=c$, then $k_1x_1+k_2x_2+\cdots+k_nx_n$ is a minimum when $k_1x_1=k_2x_2=\cdots=k_nx_n$.

The theorems are very easy to prove.† We may note here that Theorem I clearly still applies if we wish to maximize, not the product as there stated, but a positive constant times that product, or a positive integral power of that product, or the principal nth root of that product.

Some of the simpler applications are immediate; the maximizing of the area of a rectangle with given perimeter, and the minimizing of the perimeter of a rectangle with given area, are probably the simplest of all. The reader will recall cases where this type of problem is modified slightly. A concrete setting is secured by requiring the maximum rectangular area that can be enclosed with a given amount of fencing, and the modifications place the field along a river so that fence is needed only on three sides, or require fencing to divide

* See, for example, Chrystal, *Algebra*, v. II, 1889, 52–56.

† They are the corollaries of Chrystal, p. 55.

the field into two or more parts as well as to enclose the field, and so on. Theorem I also treats these cases immediately.

In many cases an obvious change of variable is required. If we wish to find the rectangle of maximum area inscribable in a given circle of diameter d, we must maximize xy subject to $x^2+y^2=d^2$. But, putting $X=x^2$, $Y=y^2$, we have to maximize the square root of XY with $X+Y=d^2$. We thus have $X=Y$, $x=y$, and the rectangle is a square. A more difficult problem from the standpoint of the calculus is that of finding the smallest ellipse that can be circumscribed about a given rectangle. If the equation of the ellipse is taken as $x^2/a^2+y^2/b^2=1$, and if one vertex of the rectangle is at (x_1, y_1), then we are to minimize πab with $x_1^2/a^2+y_1^2/b^2=1$. But putting $X=1/a^2$, $Y=1/b^2$, we must minimize $\pi/\sqrt{XY}$ with $x_1^2X+y_1^2Y=1$. The solution follows at once.

A little different situation arises in the problem of maximizing the area of a Norman window (rectangle surmounted by semi-circle) for a given perimeter. If $2x$ is the base, and y the height, of the rectangle, the problem is to maximize $2xy+\pi x^2/2$ subject to the condition $2x+2y+\pi x=c$. Since the quantity to be maximized is not a product, an obvious change of variable is $z=2y+\pi x/2$. We are then to maximize xz, with the auxiliary condition $x(2+\pi/2)+z=c$. Theorem I tells us that $z=x(2+\pi/2)$. Using the definition of z, we have $y=x$.

Of the same sort is the solution of the problem of maximizing the volume of a rectangular parallelopiped with the surface and altitude given.* We have $V=xyh$, with $xy+xh+yh=k$. But V can be written $h(k-xh-yh)$, and the substitution $X=x+h$, $Y=y+k$, reduces the problem to that of minimizing $X+Y$ with $XY=K$.

A slight extension of Theorems I and II is needed in many cases. Suppose we wish to make an open box of maximum volume from a square piece of cardboard 24 inches on a side by cutting equal small squares (say of side y) from the corners and then folding up the remaining piece to form the sides of the box. If the square base of the box has side x, we have $V=x^2y$, with $x+2y=24$. We may apply theorem I with $x_1=x$, $x_2=x$, $x_3=y$, provided the solution obtained from the theorem is consistent with the extra requirement $x_1=x_2$. However, if we write the auxiliary condition as $x/2+x/2+2y=24$, we see that x^2y is maximized when $x/2=2y$.

The same function x^2y, subject to a linear condition on x and y, appears if we attempt to maximize the volume of an ordinary box with square base† and a combined length and girth of 60 inches, if we wish to find the right circular cylinder of maximum volume that can be inscribed in a given right circular cone, and in various other problems.

If we are interested in minimizing the surface of a right circular cylinder of given volume, we have $S=2\pi(x^2+xy)$, with $x^2y=V/\pi=k$. Putting $x^2=X$,

* Most of the problems discussed here have been taken from Ford's *Calculus*. If the altitude is a variable as well, the solution may be found by the substitution $X=xy$, $Y=xh$, $Z=yh$. V becomes $\sqrt{XYZ}$.

† By the present methods, the problem is just as easy if the base is not assumed square.

$xy = Y$, we have to minimize $X+Y$ with $XY^2=k^2$. Employing the same device that was used in the open box problem, we think of minimizing $X+Y/2+Y/2$ with $XYY=k^2$, and find $X=Y/2$, or $y=2x$. A similar device is effective in maximizing the volume of a right circular cone, if the convex surface is constant.

A rather interesting problem is that of finding the isosceles triangle of maximum area that can be inscribed in a given circle. If r is the radius of the circle, y the altitude of the triangle, and x half its base, we have $A=xy$, subject to the condition $x^2=y(2r-y)$. The substitution of z for x^2/y reduces the auxiliary condition to $z+y=2r$, while A becomes $\sqrt{zy^3}$. The maximum area is thus obtained when $y=3z$, and it turns out that this requires the triangle to be equilateral.

I think the reader will agree that, in every case presented here, the substitutions made are obvious, provided we have the form of Theorems I and II in mind. If this be so, we have a natural and economical method of doing many problems in maxima and minima.

A Note on the Probability Function

By N. R. Wilson, University of Manitoba

The even probability function may be derived very simply by the method used in this Monthly, January, 1931, (page 25); using the principle that the probability of a compound event, made up of two independent events, is the product of their probabilities. We make the usual assumptions (1) that the function is positive, so that logarithms may be taken and (2) that these logarithms can be expanded in convergent Maclaurin's series.

Let $P(x, y:x', y')$ be a point in a plane, (x, y) being its co-ordinates with respect to a pair of orthogonal axes and (x', y') with respect to a second pair, bisecting the angles between the former. Let $\phi(t^2)$ be any even law of probability, and let the co-ordinates (x, y) be chosen under this law. By definition, the probability that x lies between x and $x+dx$ is $\phi(x^2)\,dx$; and that y lies between y and $y+dy$ is $\phi(y^2)\,dy$. The probability that P lies in the area having both these properties is by the principle of the introductory paragraph, $\phi(x^2)\,\phi(y^2)\,dx\,dy$; or $\phi(x^2)\,\phi(y^2)\,dA$, where A denotes the area.

By symmetry, the law of probability for the co-ordinates (x', y') with respect to the bisecting axes is even and the same for both axes; say $f(x'^2)$ and $f(y'^2)$. As in the preceding paragraph the probability that P lies in an area dA near P is $f(x'^2)\,f(y'^2)\,dA$. Equating these two results,

$$\phi(x^2)\phi(y^2)dA = f(x'^2)f(y'^2)dA\,.$$

Hence $\log\phi(x^2)+\log\phi(y^2)=\log f(x'^2)+\log f(y'^2)$.

Using the second assumption and writing $(x+y)/\sqrt{2}$ for x' and $(y-x)/\sqrt{2}$ for y' in the expansions,

$$(a_0+a_1x^2+a_2x^4+\cdots)+(a_0+a_1y^2+a_2y^4+\cdots)=$$

$$\{b_0 + b_1(x+y)^2 + b_2(x+y)^4 + \cdots\} + \{b_0 + b_1(y-x)^2 + b_2(y-x)^4 + \cdots\}$$

Comparing the coefficients of x^2y^2, x^4y^2, $\cdots$, we have that

$$b_2 = b_3 = b_4 = \cdots = 0.$$

Comparing the coefficients of x^4, x^6, $\cdots$, we have that

$$a_2 = a_3 = a_4 = \cdots = 0.$$

Hence $\log \phi(x^2)$ reduces to $a_0 + a_1x^2$; or, writing k for e^{a_0}, $\phi(x^2) = ke^{a_1x^2}$. The total probability for all values of x, viz.

$$\int_{-\infty}^{\infty} ke^{a_1x^2}dx,$$

must be 1. To make the integral finite, a_1 must be negative; $a_1 = -h^2$, say. Again, we have shown that the probability that $P(x, y; x', y')$ lies in dA near P is $\phi(x^2)\ \phi(y^2)\ dA$. The total probability for the whole plane must also be 1. Substituting the value just found for ϕ, writing r^2 for x^2+y^2, and using the polar form, $2\pi r\, dr$ for dA, this gives:

$$1 = \int_0^{\infty} 2\pi k^2 e^{-h^2r^2} r\, dr = \pi \frac{k^2}{h^2};$$

whence $k = h/\sqrt{\pi}$.

On Certain Applications of Autopolarity to the Theory of Algebraic Curves

By D. C. Duncan, Compton Junior College

An algebraic curve is said to be autopolar with respect to a conic section if the polar lines of all the points of the curve with respect to the conic are the totality of the tangents of the curve. The property of autopolarity with respect to one or more conics attaches to a very limited number of curves, but when it does, one has a very simple method of obtaining certain geometrical factors, to obtain which by standard methods is usually very tedious; *inter alia*, the equation of the locus in line coordinates, the bitangents, the inflexions, the foci; when, e.g., the point equation and point singularities are given. To illustrate we shall use the completely symmetric self-dual elliptic curve of order 8, and having, consequently, 8 cusps, 8 inflections, 12 nodes, and 12 bitangents. If the 8 cusps are distributed at equal distances around the unit circle, $\rho = 1$ (in polar coordinates), and have as cuspidal tangents the lines, $\theta = 0$, $\pm\pi/4$, $\pi/2$, one observes that the locus is *uniquely* determined and has for equation

$$27\rho^8 \sin^2 4\theta - 256(\rho^2 - 1)^3 = 0,$$

or in homogeneous rectangular coordinates,

$$27x^2y^2(x^2 - y^2)^2 - 16z^2(x^2 + y^2 - z^2)^3 = 0.$$

By use of notions of symmetry one finds that the curve has 8 crunodes on the circle $\rho=2$, corresponding to $\theta=\pm\pi/8$, $\pm 3\pi/8$, and 4 biflecnodes at infinity on the cuspidal tangents.

Now autopolarity pairs the points of the locus with the tangent lines of the locus, in particular, points of inflexion with cuspidal tangents, and inflexional tangents with cusps. Hence if one can find a conic section with respect to which a number of points and lines sufficient *uniquely* to determine the locus are pole and polar, one shall have established the autopolarity of the original locus. Frequently this is possible by inspection. This particular curve has a biflecnode at (1, 0, 0) [using homogeneous rectangular point coordinates], and cusps at $(0, \pm 1, 1)$. One readily obtains the tangents at these points, $27y^2-16z^2=0$, and $x=0$, or [in homogeneous rectangular line coordinates, u, v, w] $(0, \pm 3\sqrt{3}, 4)$ and (1, 0, 0). One observes that these elements are pole and polar with respect to the conics $3\sqrt{3}(x^2+y^2)\pm 4z^2=0$. By symmetry the other biflecnodal and cuspidal elements are correspondingly pole and polar with respect to these circles. Since these elements are sufficient to determine the locus uniquely the autopolarity follows. We now apply this property to obtain (1) the equation of the locus in line coordinates, (2) the bitangents, and (3) the foci.

(1). The line equation is immediately found by polarizing the line locus by means of the autopolarizing conic, $3\sqrt{3}(x^2+y^2)-4z^2=0$; i.e., by applying the transformation $x=4u$, $y=4v$, $z=-3\sqrt{3}w$, to the equation

$$27x^2y^2(x^2 - y^2)^2 - 16(x^2 + y^2 - z^2)^3z^2 = 0;$$

one obtains the desired equation

$$4096u^2v^2(u^2 - v^2)^2 - (16u^2 + 16v^2 - 27w^2)^3w^2 = 0.$$

(2). The 8 bitangents may be readily found by polarizing the 8 crunodes, $(\pm\sqrt{2\pm\sqrt{2}}, \pm\sqrt{2\mp\sqrt{2}}, 1)$, obtaining the line coordinates of the bitangents, $(\pm 3\sqrt{6\pm 3\sqrt{2}}, \pm\sqrt{6\mp 3\sqrt{2}}, -4)$.

(3). The foci are points of intersection of tangents to the curve which pass through the circular points at infinity, $(1, \pm i, 0)$. The pencil of lines with vertex at $(1, i, 0)$ has the equation $ix-y-imz=0$. The lines of the pencil which touch the locus are those whose poles with respect to the conic lie on the locus; i.e., the coordinates of the pole, $(-4i, 4, 3\sqrt{3}im)$, must satisfy the equation

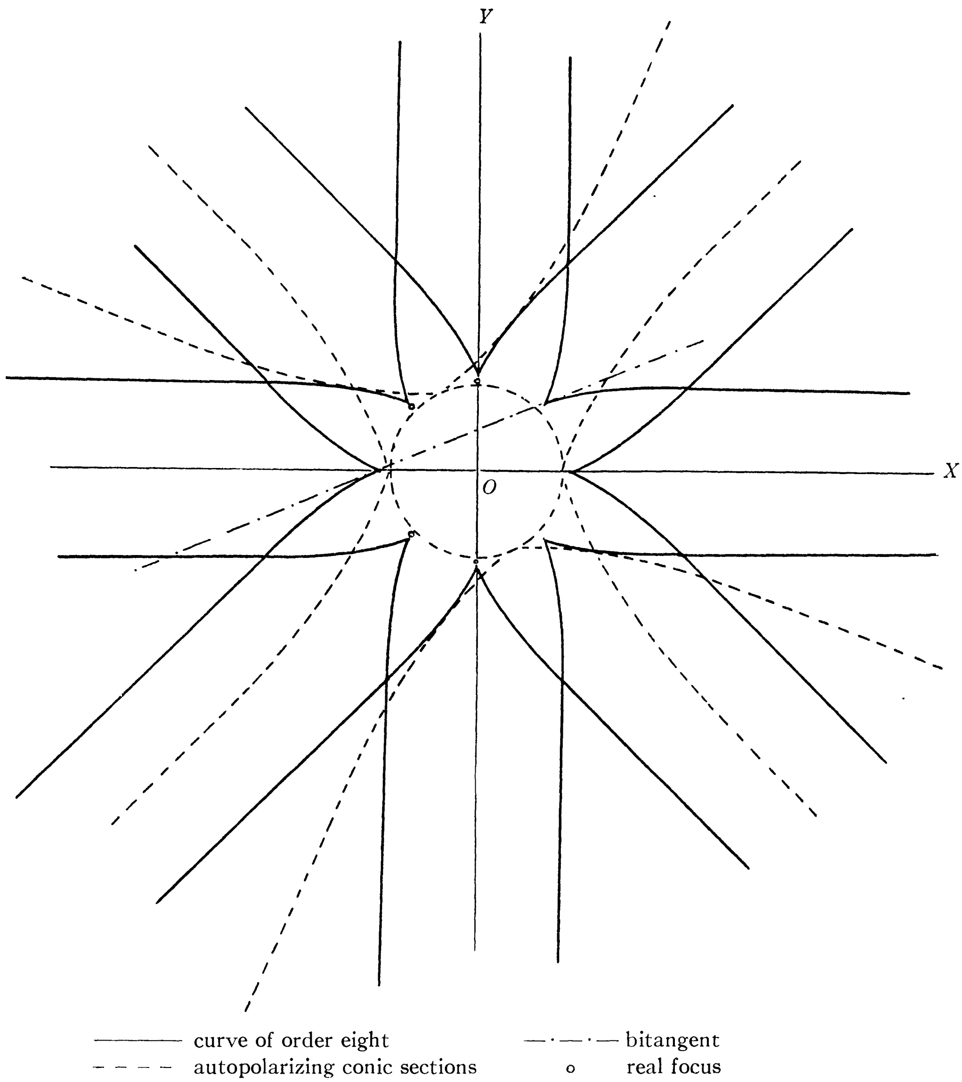
$$27x^2y^2(x^2 - y^2)^2 - 16(x^2 + y^2 - z^2)^3z^2 = 0,$$

whence one obtains $m^8=4(16/27)^3$. Using these 8 values of m, of which only 2 are real, with 8 correspondingly found with the other circular point, $(1, -i, 0)$, one may readily find the 64 foci by solving linear equations. [In this particular example, the expression for m contains no factor i; hence one has at once 8 foci at $(m, 0, 1)$ on the cuspidal tangent $y=0$, the m taking the values of the eight 8th roots of $4(16/27)^3$. By symmetry one finds that 24 other foci are corre-

spondingly situated on the other cuspidal tangents; the remaining 32 foci are distributed by 8's on the lines $\theta = \pm\pi/8$, $\pm 3\pi/8$, at distances from the origin corresponding to the eight 8th roots of $4(-16/27)^3$.]

When the coordinates of all the singular elements are known, one may obtain all the polarizing conics. In this example they are the two circles $3\sqrt{3}(x^2+y^2) \pm 4z^2 = 0$, the rectangular hyperbolas $3\sqrt{3}(x^2-y^2) \pm 4z^2 = 0$, $x^2 \pm 2xy - y^2 + (4/3)\sqrt{2/3}z^2 = 0$, and 4 other rectangular hyperbolas symmetrically placed, ten conics in all.

The sketch depicts the locus with the finite singularities and the real foci, and 3 autopolarizing conic sections.



RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Metodi Matematici—Essenza, Tecnica, Applicazioni. By Gino Loria. Hoepli, Milano, 1935. 276 pages, 51 figures. 20 lire.

In a time of feverish mathematical productivity, both in deepening the premises and extending the horizon, it is well to call an occasional halt and take an inventory of ground covered and of soundness of methods employed. This is what the author has attempted—and accomplished—in the present volume.

The work is divided into three roughly equal parts: methods in general; uses in geometry; uses in algebra and the theory of numbers. Although the preface states that only an elementary knowledge of mathematics is presupposed, this statement should be understood to include the calculus, projective geometry, theory of equations, and some knowledge of functions of a complex variable.

In the first part the real meaning of certain mathematical processes is explained; each step is analyzed carefully, and shown to be a natural part of a fairly homogeneous whole. The steps featured are the contrasts between synthetic and analytic procedures, reduction to absurdity, complete induction, uses of analogy, formal logic, generalization, counting of constants, transfer, and the concept of group.

In geometry, the processes are illustrated by particular problems, first by the methods and in the style of Euclid's Elements. It seemed to the reviewer that rather too much attention was paid to spherical triangles, but he must express his admiration for both method and result. The applications to projective and to descriptive geometry are almost entirely confined to problems of the first and second degree; particular emphasis is put on the three trial method, which is very successful.

The chapter on algebraic geometry features the Chasles principle of correspondence, mapping a rational surface on a plane, and sketches correspondence between dissimilar elements. The general theory of correspondence is mentioned but is so briefly treated as to be misleading. Of course adequate treatment of the problems or of the methods of this field would be impossible without a much broader assumption of prerequisites.

The chapter on algebra treats the concept of infinite descent, recurring formulas, complex and hypercomplex numbers, the schemes for separation of the roots of algebraic equations, with the theorems of Sturm and of Budan-Fourier, with recent refinements. Various methods for finding approximate numerical roots are explained and compared. Horner's method is not mentioned.

The discussion of the methods of the calculus is preceded by an excellent

presentation of the earlier methods employed for finding limits. The chapter itself is largely occupied by featuring various special devices in the integral calculus, and justifying their use. These include integration by parts, differentiation as to a parameter, use of complex variables, etc. All these are skilfully applied to the Boolean treatment of differential equations.

The book is provided with a comprehensive index which contains both names and subject matter. The printing and press work are excellent, and it is remarkably free of typographical errors.

Every college and high school teacher of mathematics should be familiar with the points raised and discussed in this volume. After reading it thoughtfully one could not help being a better teacher.

VIRGIL SNYDER

Gli Elementi d'Euclide e la Critica Antica e Moderna. Federigo Enriques. 3 vols., I. (Books I–IV), Rome, 1925, 323 pages, 25 lire; II. (Books V–IX), Bologna, 1930, 356 pages, 50 lire; III. (Book X), Bologna, 1932, 337 pages, 30 lire.

It is always entertaining to attend the wake at the ever-recurring death of Euclid—killed by the innumerable hands of innumerable educators. No mathematician of all time has been slaughtered more often than he, and none has had the satisfaction of a glorious resurrection so frequently. In considering this intellectual if not corporeal phenomenon, however, we have to bear in mind that the phenomenon is related to the spirit of *The Elements* (*Stoicheia*) and not to the body of the text. The thirteen "books" of these elements were written for scholars of university rank; they were too subtle for the adolescent mind. The idea of using the book in training schoolboys, which began in the seventeenth century and continued for two hundred years, was carried out with some approach to success so long as carefully selected boys were admitted to the schools of England, but it failed as soon as education became democratic, and even before that time in most of the Continental countries.

To summarize the situation succinctly, the renaissance period saw scores of books on geometry come from the press, and it was only in Great Britain that the text of Euclid held sway in the schools for the training of boys of the "upper class." In all the continental textbooks, however, the spirit of Euclid maintained its position. So it is today, in all highly civilized countries, that geometry is taught, usually to girls as well as boys, but that the textbooks lack the rigidity of proof which characterizes the work of the great scholar of Alexandria, being softened to fit the feebler mentality of high-school pupils of today.

It is encouraging, however, to observe that the greatest textbook that the world has ever seen—*The Elements* of Euclid— is by no means dead, and that in the region of sound scholarship it continues to stand as a monument to the achievements of the Greek mind. As evidence of this fact we now have a new edition of the work itself, and this from the hand of one of the outstanding Italian mathematicians of our time, Professor Federigo Enriques of the University of Rome.

The first volume, containing the first four Books of *The Elements*, appeared in 1925. Book I is the work of Professor Enriques (the general editor) and Maria Teresa Zapelloni; Book II, of the latter alone; Book III, of Adriana Enriques; and Book IV, of Amedeo Agostini. The later volumes are also the work of various authors—Guido Rietti and Ruth Struik, and particularly of Dr. Maria Teresa Zapelloni, one of the pupils of Enriques, an expert Greek scholar, and the one who had done most of the work on Books I and II. The series as a whole is therefore the work of Professor Enriques only as he is the general editor and as his scheme is carried out by assistants of ability vouched for by him.

As shown in the titles, the series covers only the first ten Books of *The Elements*—those relating to plane geometry. In its preparation the various authors had access to numerous standard editions of Euclid, including Sir Thomas Heath's first edition (3 vols., Cambridge, 1908), but unfortunately not the second one (Cambridge, 1926) although this appeared some time before the publication of volumes II and III.

The general purpose of the work was to render available to Italian readers a translation of the first ten Books of *The Elements*, together with notes upon the various definitions, axioms, postulates, and propositions. These notes are historical and are such as will supply fairly well the needs of teachers of plane geometry. It is natural, however, to compare this work with the monumental treatise of Sir Thomas Heath. Such a comparison shows, if we may judge by an examination of a random selection of portions of the text, that the translations have been made with care. When it comes to the notes, however, the English volumes show at once their supremacy. Had Professor Enriques himself taken the time that Sir Thomas Heath took in the preparation of his work, the result would have been different. This is especially manifest in the case of such basal definitions as those at the beginnings of the several Books. Heath has exhausted the subject, but the assistants of Enriques have—except in a few cases—merely touched upon the significance of the several terms. In the case of the assumptions a similar situation arises, the Italian work devoting but a single small page to the important and well-known Postulate 5, whereas the English treatise gives nineteen large pages or, by word count, about forty times as much material. For advanced students in the history of Greek mathematics, therefore, the work of Professor Enriques fails to approach the standard set by Sir Thomas Heath. Indeed, as we study the latter's treatises the more they stand out as the greatest contribution to Greek geometry that has been made in any language, not even excepting the late J. L. Heiberg's Latin and Greek versions of the most important manuscripts extant.

DAVID EUGENE SMITH

Analytical Geometry. By Vincent C. Poor. New York, John Wiley and Sons, Inc., 1934. v+244 pages. $2.25.

This textbook treats of both plane and solid analytical geometry. The plane

geometry includes a chapter on diameters, poles and polars of conics, and a chapter on higher plane curves, including a brief introduction to curve-fitting.

The book is written for a full year course, but is so arranged that material for a half-year course may be selected with little or no confusion. The book is well-written and the treatment is on an appropriately high level, except that perhaps more attention might have been given to certain extreme conditions and to certain converse theorems: e.g., certain fractional forms are developed and left without mention of the case of zero denominator (pp. 13, 33); the converse of Theorem II, p. 14 is omitted; and the equations of the conics are rationalized without any study of the effects of rationalization. However, those teachers who regard these omissions as defects can easily remedy them.

Some features of the book are:

Projections of directed line segments are introduced at the start and employed frequently.

Polar coordinates are introduced early in the book.

The conics are developed from the eccentricity definition.

With each conic there is study of tangents.

The general equation of the second degree is economically but effectively studied. This has been a confused chapter in many recent texts attempting brevity.

The important topic of parametric equations receives attention.

The exercises are numerous and good.

L. P. SICELOFF

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 165. *Proposed by V. F. Ivanoff, San Francisco.*

If $s_k = 1^k + 2^k + \cdots + n^k$, prove that $s_5 + s_7 = 2(s_3)^2$

E 166. *Proposed by W. E. Buker, Leetsdale High School, Pa.*

In the triangle PQR, M and N are points in PQ and PR respectively, such that $PM:MQ = a:b$, and $PN:NR = c:d$. In what ratio, in terms of a, b, c and d, is MN cut by the median of PQR drawn from P?

E 167. *Proposed by J. M. Feld, New York City.*

If $A + B + C = 180°$, and if $2S = \sin A + \sin B + \sin C$, prove that

$$\sin^2 A \sin^2 B \sin^2 C = 4S(S - \sin A)(S - \sin B)(S - \sin C).$$

E 168. *Proposed by V. Thébault, Le Mans, France.*

a, b and c are three consecutive digits in some order, and in some scale of notation the number $aabb = (cc)^2$. It is required to determine the base of the system of enumeration and the values of a, b and c.

E 169. *Proposed by A. A. Bennett, Brown University.*

Prove that n, the number of years in which a sum of money will double itself at the interest rate r, compounded annually, is given approximately by $n = 1/3 + 9/13r$, and determine the approximate error.

E 170. *Proposed by T. C. Fry, Bell Telephone Laboratories, New York City.*

It is a simple matter to erect equilateral triangles on the sides of any triangle ABC, thereby determining three points, P, Q and R, which constitute the new vertices of these equilateral triangles. It is here proposed to invert this process and construct the triangle ABC when given only the points P, Q and R.

SOLUTIONS

E 22 [1933, 110] *Proposed by R. M. Winger, University of Washington.*

As a western version of problem E7, find the digits represented by the various letters in the following problem in addition, and determine whether or not the solution is unique. (Except that obviously R and L, and S and G, are interchangeable.) No two different letters represent the same digit.

$$\begin{array}{rrrrr} & S & E & N & D \\ & M & O & R & E \\ & G & O & L & D \\ \hline M & O & N & E & Y \end{array}$$

Editorial Note. A solution of this problem by Simon Vatriquant [1933, 424] gave six basic solutions. A seventh fundamental solution has recently been found by Herbert Mansfield, a student at Bradley Polytechnic Institute, Peoria, Illinois. It is

$$\begin{array}{rrrrr} & 9 & 6 & 7 & 3 \\ & 1 & 5 & 0 & 6 \\ & 4 & 5 & 8 & 3 \\ \hline 1 & 5 & 7 & 6 & 2 \end{array}$$

This solution was overlooked by Vatriquant because of an error in the fourth line of his table at the bottom of page 424 in which the values of N and O should have been 7 and 5 respectively instead of 9 and 0.

E 108 [1934, 447]. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N.Y.*

Show how to construct a triangle when the orthocenter, the incenter and one vertex are given.

I. Solution by J. Rosenbaum, Hartford Federal College

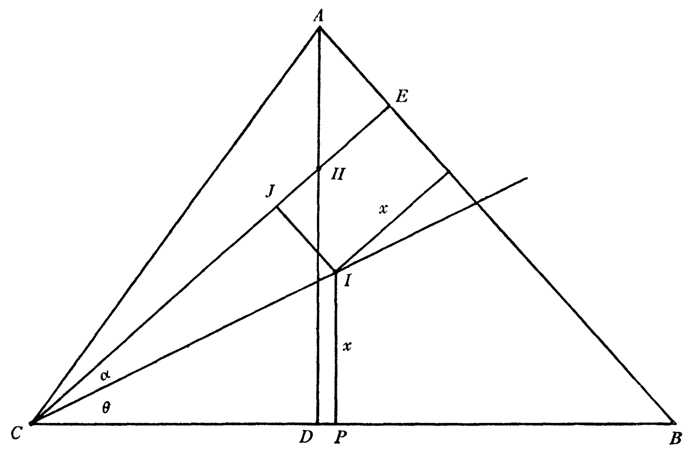


Fig. 1

In Fig. 1, let C be the given vertex, H the orthocenter, I the incenter, and ABC the required triangle. Denote the altitude CE by h, CH by p, CI by q, CD by m, the side CA by b, the in-radius by x, the angle HCI by α, and the angle ACB by 2θ.

Case I, α acute. From the right triangle CJI, $CJ=q\cos\alpha$, and hence

$$h = q\cos\alpha + x. \tag{1}$$

From the right triangles CEA, CDA, and CDH, we have respectively

$$b = \frac{h}{\cos(\theta-\alpha)} \tag{2}$$

$$m = b\cos 2\theta \tag{3}$$

$$m = p\cos(\theta+\alpha). \tag{4}$$

From (3) and (4)

$$b\cos 2\theta = p\cos(\theta+\alpha), \tag{5}$$

and using (1) and (2), (5) can be written

$$\frac{(q\cos\alpha + x)\cos 2\theta}{\cos(\theta-\alpha)} = p\cos(\theta+\alpha). \tag{6}$$

Clearing of fractions and expanding, we have

$$(7)\qquad (q \cos \alpha + x)(1 - 2 \sin^2 \theta) = p(\cos^2 \alpha - \sin^2 \theta).$$

Now from triangle CPI, $\sin \theta = x/q$; and substituting this in (7), one obtains the cubic equation in x,

$$(8)\qquad 2x^3 + (2q \cos \alpha - p)x^2 - q^2 x + pq^2 \cos^2 \alpha - q^3 \cos \alpha = 0.$$

Case II, *α obtuse*. A procedure similar to the above yields the equation

$$(9)\qquad 2x^3 - (2q \cos \alpha - p)x^2 - q^2 x - pq^2 \cos^2 \alpha + q^3 \cos \alpha = 0.$$

Equations (8) and (9) factor* respectively into

$$(10)\qquad (x + q \cos \alpha)(2x^2 - px + pq \cos \alpha - q^2) = 0$$

and

$$(11)\qquad (x - q \cos \alpha)(2x^2 + px + pq \cos \alpha - q^2) = 0.$$

It is seen that the linear factor in either (10) or (11) does not give a solution because in either case it gives a negative in-radius.

Choosing $CI = q$ as the unit of length, the roots of the second factor of (10) are

$$(12)\qquad x_1, x_2 = (p \pm \sqrt[3]{p^2 - 8p \cos \alpha + 8})/4,$$

where x_1 is the greater root. Considering (p, α) as the polar coordinates of a point, where C is the pole and CI the axis, the expression under the radical equated to zero is the equation of a circle with center $(4, 0)$ and radius $2\sqrt{2}$. Hence, for a real solution, the orthocenter H can not be within this circle. Next, it can be shown that when α is acute then 2θ is also acute; and since $x = q \sin \theta$, it follows, for $q = 1$, that we must have $x < \sqrt{2}/2$. Applying this restriction to x_1, it is seen from (12) that $p < 2\sqrt{2}$. With this restriction on p, x_1 has the value $\sqrt{2}/2$ when $\alpha = 45°$, and since $\cos \alpha$ is a decreasing function of α when α is acute, it follows that for x_1 to be admissible, α must be less than 45°.

The above restriction on the in-radius applied to x_2 reveals that either p must be less than $2\sqrt{2}$ and α any acute angle; or else α must be greater than 45°, in which case there is no restriction on p.

Finally, from the consideration that x_2 must be positive, it follows from (12) that $p \cos \alpha > 1$.

For case II, it can be shown geometrically that when α is obtuse then 2θ is also obtuse, so that here the in-radius must be greater than $\sqrt{2}/2$. As above, it appears from the expression for the positive root of (11) that α must be greater than 135°.

The above results are shown in Fig. 2. If C is the given vertex and I the given incenter, there will be *one* solution if the orthocenter H is anywhere in the single

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* Mr. Rosenbaum wishes to acknowledge that this factorization, as well as the geometrical conditions given below for the number of real solutions, were pointed out by W. B. Carver of Cornell.

shaded area, *two* solutions if H is in the double shaded area, and *no* solutions if H is in the unshaded area. The region $A+B$ admits x_1, $B+C$ admits x_2, while D admits the positive root for case II.

It is readily seen that when H is at C the triangle ABC is a right triangle, and that for this case there are an infinity of solutions.

After the in-radius is constructed, the construction for the triangle is obvious.

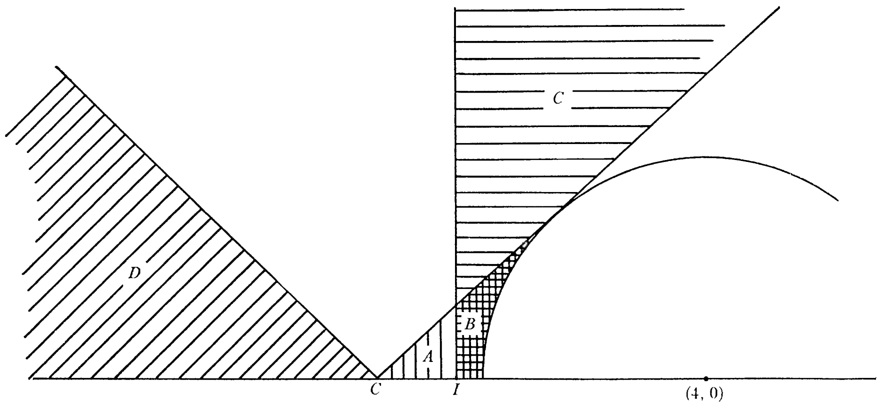

Fig. 2

II. *Solution by J. Balasundara Rao, Madras, India.*

If A is the given vertex, I the incenter and O the orthocenter (See Fig. 3);

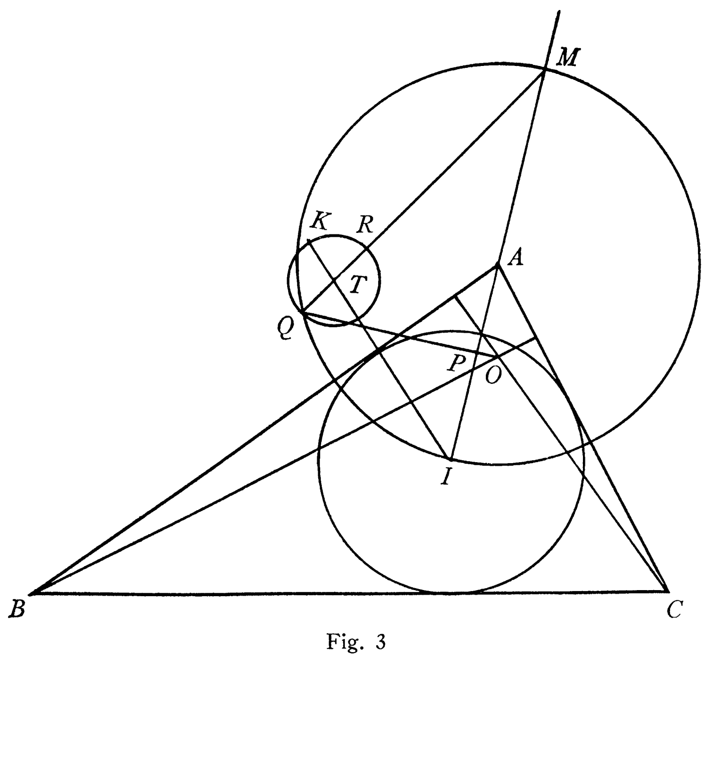

Fig. 3

1. With A as center and AI as radius draw a circle cutting IA produced at M.

2. Draw OP perpendicular to IA and produce it to cut the circle (1) at Q.

3. On QM lay off $QR=OA$ and construct a circle with diameter QR and center T.

4. Draw IT and produce it to cut circle (3) at K.

5. With I as center and $IK/2$ as radius draw a circle. It is the incircle of the desired triangle.

6. Draw the tangents from A to this circle, and the perpendiculars to these tangents from O, which perpendiculars produced back through O will intersect the tangents at the vertices B and C of the triangle ABC.

The justification of this construction is left to the reader.

Also solved by Leon Recht and Simon Vatriquant.

E 135 [1935, 44]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

While dealing regularly in an ordinary bridge game, South dropped some of the undealt cards onto the floor, but retained the rest in his hand. He then observed that the number of cards on the floor was two-thirds of the number he had already dealt to West, and that the number already dealt to East was two-thirds of the number of undealt cards still in his hand. How many cards had been dealt?

Solution by E. N. Yeager, Toledo, Ohio.

If x be the number of cards dealt, then

$$3E = 2[(52 - x) - 2W/3]$$

where E and W are the numbers of cards already dealt to East and West.

The following expressions may then be substituted for E and W:

$$W = (x + i)/4 \quad [i = 0, 1, 2 \text{ or } 3]; \quad E = (x - j)/4 \quad [j = -1, 0, 1 \text{ or } 2].$$

Substitution and reduction gives $37x=1248-4i+9j$. Of the possible values of i and j, only $i=j=2$ makes x an integer, which integer is 34. Hence South had dealt eight and a half rounds of cards when he dropped six, retaining his grasp on twelve.

Also solved by L. J. Adams, E. F. Allen, J. A. Benner, E. T. Browne, W. E. Buker, M. L. Constable, Daniel Finkel, D. W. Hall, E. H. Johnson, Roy MacKay, F. L. Manning, W. N. Mebane, Jr., N. W. Smith, Herbert Spiro, E. P. Starke, Ruth W. Stokes, C. W. Trigg, Simon Vatriquant and the proposer.

E 136 [1935, 109]. *Proposed by V. Thébault, Le Mans, France.*

With the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, used once each, form two numbers whose product is a maximum. Form two other numbers in the same way whose product is a minimum. None of the numbers is to start with a zero.

Solution by Mary L. Constable, Philadelphia, Pa.

It is obvious that if the factors are to produce a maximum product, their initial digits should be 9 and 8, and the following digits arranged in descending order.

In general, $a(b+1)<(a+1)b$ if $a<b$. So here $86(96+1)<(86+1)96$. Therefore $86\cdot 97<87\cdot 96$. Similarly $874\cdot 965<875\cdot 964$, $8752\cdot 9643<8753\cdot 9642$, and $87530\cdot 96421<87531\cdot 96420$. Consequently $87531\cdot 96420$, or 8,439,739,020 is the maximum product of factors of 5 digits each.

It can be proved that the product of two numbers of n digits each, arranged in descending order, is greater than the product of two numbers of $(n+r)$ and $(n-r)$ digits, formed by transferring the last r digits from the second number to the first and arranging the digits in descending order. (Note: The transfer of a terminal zero has no effect on the product.) Therefore the maximum product is

$$87531\cdot 96420 = 875310\cdot 9642 = 8{,}439{,}739{,}020.$$

The factors of the minimum product obviously begin with 2 and 1, and the digits are arranged in ascending order. Hence they are $1\cdot 203{,}456{,}789=203{,}456{,}789$; or, in case unity is not acceptable as a factor, the next smallest product is $2\cdot 103{,}456{,}789=206{,}913{,}578$.

Also solved by W. E. Buker, Daniel Finkel, E. P. Starke, C. W. Trigg, Simon Vatriquant, and the proposer.

E 137 [1935, 109]. *Proposed by M. J. Turner, Ball State Teachers College.*

Four lines, concurrent at M, are cut by a transversal in the points A, P, Q and B, in that order, with angles AMP and QMB equal. Prove that

$$MA:MB::(MP\cdot AQ):(MQ\cdot BP).$$

Solution by Irving Segal, Princeton University.

Since angles AMP and QMB are equal, angles AMQ and PMB are also equal, so that triangles AMQ and PMB have equal angles at M and equal altitudes from M. Hence their areas are proportional to their bases on the one hand, and to the products of their sides to M on the other hand. That is, $AQ:PB=(AM\cdot MQ):(PM\cdot MB)$, whence $MA:MB=(MP\cdot AQ):(MQ\cdot PB)$.

Also solved by W. E. Buker, W. B. Clarke, W. Douglas, Daniel Finkel, Abe Gelbart, R. A. Johnson, Sidney Kaplan, L. M. Kelly, H. R. Leifer, Morris Lieblich, Theodore Lindquist, Leon Recht, E. P. Starke, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

E 138 [1935, 110]. *Proposed by W. B. Campbell, Judson College, Rangoon, Burma, India.*

An ellipse of fixed area A and variable eccentricity e is rotated about its major axis, generating an ellipsoid of surface area S. Express S as a function of e. Determine the value of e for which S is maximum. Explain what happens when e approaches its limiting values, 0 and 1.

Solution by Simon Vatriquant, Athénée Royale d'Ixelles, Brussels, Belgium.

The surface area S of the ellipsoid is given by the integral

$$S = 2\pi \int_{-a}^{+a} yds = 4\pi \int_{0}^{a} yds.$$

A simple computation gives

$$yds = b(a^2 - e^2x^2)^{1/2}dx/a^2.$$

Hence

$$S = 2\pi ab[\arcsin e + e(1 - e^2)^{1/2}]/e$$
$$= 2Ae^{-1}[\arcsin e + e(1 - e^2)^{1/2}].$$

Computing dS/de, we obtain after reductions

$$dS/de = [e(1 - e^2)^{1/2} - \arcsin e]/e^2,$$

or, letting $e = \sin t$,

$$dS/de = (\sin 2t - 2t)/2 \sin^2 t.$$

This derivative decreases through zero when t increases through zero. Then S approaches its maximum value as t, and hence e, approach zero, and the ellipsoid then approaches a sphere.

If e approaches 1, then S approaches πA.

Consequently, of all the ellipses of given area rotating about their major axis, the circle generates the maximum surface area. When the major axis increases indefinitely, the ratio S/A of the generated surface area to the area of the generating curve approaches π.

Also solved by Paul Baldwin, Irving Segal, E. P. Starke, C. W. Trigg, and the proposer.

E 139 [1935, 110]. *Proposed by Raphael Robinson, University of California at Berkeley.*

By folding a rectangular sheet of paper three times, six superposed congruent triangles are obtained. Show that the ratio of the length and width of the rectangle is either 3:1 or $\sqrt{3}$:1.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

In a rectangle, the angles which may be divided by folding are 90°, 180°, and 360°. Since the final triangles are to be congruent, the three folds must divide the affected angles by 2, 2^2, 2^3, 3 or $2 \cdot 3$, except in the case of 360°, which cannot be divided into three parts by folding. Hence the angles formed must be 45°/4, 45°/2, 45°, 90°, 15°, 30° or 60°. From these angles the only combinations which total 180° are 45°+45°+90°, 30°+60°+90°, and 60°+60°+60°. Since 90° is not an admissible multiple of 60°, the third combination could not give one of the six congruent parts of a rectangle.

If congruent triangles be assembled into a rectangle so that they may be superposed by folding, the sum of the adjacent angles must total 90°, 180° or 360°; and adjacent triangles must be symmetrical with respect to their common side. The only possible arrangement of six congruent isosceles right triangles which meets these conditions is that resulting in a rectangle composed of three consecutive squares whose single diagonals join to form a broken line. The length and width of this rectangle are evidently in the ratio of 3:1. Such a rectangle may be folded into three superposed squares by two folds, and a third fold along a diagonal of the square yields the six congruent triangles.

The unique arrangement of six 30°–60° right triangles into a rectangle is that in which four of them are arranged into a rhombus which has sides equal to the hypotenuse of each triangle, and the rectangle completed with the other two triangles. If the hypotenuse be taken equal to 2, then the length and width of the rectangle are in the ratio $(2+1):\sqrt{3}$, or $\sqrt{3}:1$. Such a rectangle, $ABCD$, may be folded along the diagonal AC, again folded so that A coincides with C, and the third fold to bring the right angles of the trapezium into contact, gives the six congruent triangles.

Also solved by Simon Vatriquant, B. C. Zimmerman and the proposer.

E 140. [1935, 110] *Proposed by Maud Willey, Gulfport, Mississippi.*

In the following example in long division, six instead of ten was used as a number base. (Thus $2\times 3=10$, $2\times 4=12$, $4\times 5=32$, $5\times 5=41$, etc.) Then each digit was replaced by a code letter. Reconstruct the problem and show that the solution is unique.

$$\begin{array}{rl} a\,b\,)\,c\,d\,e\,f & (\,e\,d \\ c\,c\,d & \\ \hline e\,d\,f & \\ e\,a\,e & \\ \hline d & \end{array}$$

Solution by B. C. Zimmerman, Corozal, British Honduras.

From an examination of the partial products it is apparent that zero can not be represented by a, b, c, d or e. Therefore $f=0$. Similarly 1 can not be represented by a, b, d or e, so $c=1$.

From the last column, $d+e=10=(6)$, and from the second column $e<d$, so that $d=4$ and $e=2$. From the third column $a=d-1=3$, so $b=5$, and the divisor and dividend are respectively 35 and 1420, giving a quotient of 24 and a remainder of 4.

Editorial Note. Simon Vatriquant points out in his solution that much of the given data is superfluous, and states that the problem would still have a unique solution if it had been given in the form:

$$\begin{array}{rl} x\,x\,) & x\,d\,e\,x\,(\,x\,x \\ & \underline{x\,x\,d\;\;} \\ & \;\;\,e\,d\,x \\ & \;\;\,\underline{x\,a\,e} \\ & \;\;\;\;\;\;\;\;\;d \end{array}$$

where the x's may represent any digits, including those represented by a, d or e.

Also solved by W. E. Buker, Mary L. Constable, Daniel Finkel, Theodore Lindquist, F. L. Manning, J. Balasundara Rao, Leon Recht, B. D. Roberts, O. M. Rogers, E. P. Starke, Dorothy Stephenson, C. W. Trigg, E. T. Welmers and the proposer.

E 141 [1935, 110]. *Proposed by W. P. Udinski, University of Texas.*

Show that in every tetrahedron there must be at least one vertex at which each of the face angles is acute.

Solution by Roy MacKay, Eastern New Mexico Junior College.

If one face angle at one vertex of the tetrahedron is right or obtuse, the sum of the three face angles at this vertex exceeds π radians. If at least one face angle at each vertex is right or obtuse, the sum of the face angles at all the vertices would exceed 4π, which is impossible, since the sum of the interior angles of four triangles is just 4π radians. Consequently there must be at least one vertex of the tetrahedron at which all the face angles are acute.

Also solved by L. M. Kelly, Leon Recht, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3744. *Proposed by R. E. Gaines, University of Richmond.*

Three parallel tangents are drawn to the cardioid $\rho = a(1+\cos\theta)$, and also another set of three tangents perpendicular to these. The locus of three of the nine intersections of the tangents is a circle, and that of the other six is a limaçon whose equation is of the form $\rho = b + c\cos\theta$, with a suitable change of origin.

3745. *Proposed by R. E. Gaines, University of Richmond.*

The same problem as 3744 substituting normals for tangents.

3746. *Proposed by Paul Erdös, The University, Manchester, England.*

Given a triangle ABC, with the sides $a>b>c$, and any point O in its interior. Let AO, BO, CO cut the opposite sides in P, Q, R. Prove that

$$OP + OQ + OR < a.$$

3747. *Proposed by Frank Irwin, University of California.*

Find the single condition that all the roots of the secular equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix} = 0$$

should be equal, the a's being real and $a_{ji}=a_{ij}$; and hence determine the cases in which all the roots are equal.

3748. *Proposed by Harry Langman, Brooklyn, N.Y.*

If we set

$$\sum_{i=0}^{n-1} (-1)^i \frac{{}_{n-1}C_i}{(i+1)^{t+1}} = a_t,$$

where the C's are binomial coefficients, show that

$$\begin{vmatrix} a_1 & -a_2 & a_3 & \cdots & (-1)^{n-1}a_{n-2} & (-1)^n\ a_{n-1} & (-1)^{n+1}a_n \\ -a_0 & a_1 & -a_2 & \cdots & (-1)^{n-2}a_{n-3} & (-1)^{n-1}a_{n-2} & (-1)^n\ a_{n-1} \\ 0 & -a_0 & a_1 & \cdots & (-1)^{n-3}a_{n-4} & (-1)^{n-2}a_{n-3} & (-1)^{n-1}a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -a_0 & +a_1 & -a_2 \\ 0 & 0 & 0 & \cdots & 0 & -a_0 & a_1 \end{vmatrix} = \frac{1}{n!n^n}.$$

SOLUTIONS

3668 [1934, 193]. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N.Y.*

Construct a triangle ABC, given the circumcenter, the foot of the altitude from A, and the point common to BC produced and the bisector of the exterior angle at A.

I. *Solution by Roy MacKay, Eastern New Mexico Junior College.*

Denote the three given points by O, D and E, where O is the circumcenter, D the foot of the altitude from A, and E the foot of the bisector of the exterior angle at A.

Drop OM perpendicular to ED. Construct a circle on OE as diameter to cut

the perpendicular bisector of DM at P_1 and P_2 where P_1 is to be taken on the side of ED remote from O when E, D and O are not collinear. Let EP_1 and EP_2 cut the perpendicular to ED through D at A_1 and A_2 respectively. With center O and radii OA_1 and OA_2 determine B_1, C_1, B_2 and C_2 on ED. Triangles $A_1B_1C_1$ and $A_2B_2C_2$ satisfy the conditions.

It is sufficient to prove that A_1E and A_2E bisect the exterior angles at A_1 and A_2, respectively. Since the proof is the same in both cases, the subscripts are omitted.

Extend EA to cut OM at F. From the congruent right triangles OPA and OPF, $OA=OF$. Hence F is the midpoint of the arc CAB of the circumcircle of the triangle ABC. Therefore FAE bisects the exterior angle of the triangle ABC at A.

There is no solution if angle EDO is less than or equal to $\pi/2$. There is always one solution $(A_1B_1C_1)$ if angle EDO is greater than $\pi/2$. There are two solutions if OF_2 is greater than OM. It is easy to show that this last inequality may be stated in terms of the given elements as

$$OD \cos(\pi - ODE) > 4ED \tan^2 OED.$$

II. *Solution by Rufus Crane, Ohio Wesleyan University.*

Call these three points O, D, and R, respectively. Draw OR, DR, and a perpendicular to DR at D, cutting OR at Q. Then A lies on this perpendicular. In any triangle the internal bisector of the angle A bisects the angle OAD. Hence the internal and external bisectors form with AO and AD a harmonic pencil which determines on OR a harmonic range. Therefore, on OR find P, the harmonic conjugate of R with respect to O and Q. On PR as a diameter draw a semicircle. This cuts DQ at A. Draw the circle with center O and radius OA. This cuts DR at B and C, and the triangle is complete.

Solved also by L. M. Bauer, J. W. Clawson, H. G. Diebel, D. L. MacKay, J. S. Miller, A. Pelletier, and H. D. Ruderman.

3669 [1934, 193]. *Proposed by W. E. Buker, Leetsdale, Pa.*

Given a triangle with a point P on one side and line l through P which bisects the area of the triangle. Find the envelop of l as P describes the perimeter of the triangle.

I. *Solution by B. D. Roberts, New Mexico Normal University.*

In rectangular coordinates let the triangle have vertices $(0, 0)$, $(a, 0)$ and (b, c). Taking one third of the situation: let the point P be $(d, 0)$ and let l pass also through the point (e, f) on the other side through the origin. Now $df=ac/2$, $bf=ec$, and the equation of the line l is $(e-d)\,y=f\,(x-d)$. Eliminating e and f from these three equations we get as the equation of the family of lines l

$$y(ab - 2d^2) = ac(x - d),$$

where d is the parameter.

The envelope of this family, as found by the usual method, is

$$8y(cx - by) = ac^2,$$

showing the required locus to be a hyperbola whose asymptotes are the two sides under consideration.

The complete envelope is then a three-cusp locus composed of parts of three hyperbolas. Each hyperbola has two sides of the triangle, extended, as asymptotes, and is tangent to the medians to these two sides. The parts involved in the envelope are the parts between the points of tangency to the medians.

II. *Solution by E. P. Starke, Rutgers University.*

We recall from Analytic Geometry (See Osgood & Graustein, p. 152) the following property of the hyperbola: the tangents to a hyperbola cut off from the asymptotes triangles having equal areas; the point of tangency is the midpoint of the segment of the tangent intercepted by the asymptotes.

A hyperbola can be constructed having two sides, say a and b, as its asymptotes and tangent to the medians to a and b at their midpoints. By the property noted above, the hyperbola tangent to one median at its midpoint will also be tangent to the other: moreover, if any point on the hyperbola between these two points is chosen, the corresponding tangent will intersect a and b internally and cut off half the area of the triangle.

Construct the segments of the two hyperbolas similarly related to the other pairs of sides, and the required locus is complete: a curvilinear triangle composed of segments of three hyperbolas, each tangent to two medians at their midpoints and having two sides of the triangle as asymptotes.

Solved also by J. W. Clawson, A. Pelletier, F. Underwood, and S. Vatriquant.

Editorial Note. This problem furnishes a simple illustration of geometrical limits. Let b and c be two given straight lines cutting in A; and let a variable straight line cut b and c in N and M so that the area AMN is constant. Let $M'N'$ be another position of the moving line cutting MN in P'. Since the areas $NP'N'$ and $MP'M'$ are equal, $P'N\cdot P'N'=P'M\cdot P'M'$. Hence, as M' approaches M, P' approaches the middle point P of MN, and the envelope of MN touches it at its middle point P. Draw through P parallels PU and PV to b and c forming the parallelogram $AUPV$: the area of this parallelogram is one-half the given area of AMN. Hence the envelope is the locus of the vertex P of such a parallelogram having a constant area; and it is easily traced after locating one point P by using this fact.

3670 [1934, 193]. *Proposed by N. A. Court, University of Oklahoma.*

The variable line equidistant from two given points in space and passing through a third fixed point generates a cone of the second degree.

I. *Solution by J. E. LaFon, University of Oklahoma*

Let the line PQ pass through the fixed point P and be equidistant from the

fixed points A, B. Let the mid-point of AB be C and denote by M, N, Q the feet of the perpendiculars from A, B, C upon PQ. For PQC to be a right angle, Q must lie on the sphere with PC for diameter. Since $AM=BN$, it follows that Q, the mid-point of MN, must lie on the perpendicular bisector of AB. Thus the locus of Q is a circle and PQ generates a quadric cone.

II. *Solution by L. Richardson, The University of British Columbia.*

Let the variable line l pass through the fixed point O, which is chosen as origin of vectors; and let $\mathbf{a}$ and $\mathbf{b}$ denote the vectors from O to the first two fixed points A and B. If $\mathbf{d}$ is the unit vector along l, then the squares of the distances of A and B from l are $\mathbf{a}^2-(\mathbf{d}\cdot\mathbf{a})^2$ and $\mathbf{b}^2-(\mathbf{d}\cdot\mathbf{b})^2$. From the conditions of the problem, we have

$$(\mathbf{d}\cdot\mathbf{a})^2 - (\mathbf{d}\cdot\mathbf{b})^2 = \mathbf{a}^2 - \mathbf{b}^2 :$$

hence l generates a cone of the second degree with vertex at O.

Solved also by C. E. Buell, J. W. Clawson, Rufus Crane, A. V. Richardson, E. P. Starke, W. P. Udinski, F. Underwood, S. Vatriquant, and Maud Willey.

Editorial Note. The solutions by Crane and Miss Willey used synthetic geometry with the same results as in I above; Udinski used vectors; while the remaining solvers used analytical geometry. The vector equation in II may be interpreted by writing it in the form

$$\left(\mathbf{r} - \frac{\mathbf{a}+}{2}\right)\cdot(\mathbf{a} - \mathbf{b}) = 0, \qquad \mathbf{r} = \left(\frac{\mathbf{a}+}{2}\cdot\mathbf{d}\right)\mathbf{d}.$$

Let C be the mid-point of AB; its vector is $(\mathbf{a}+\mathbf{b})/2$. A sphere with OC as diameter cuts l in P, and the second equation shows that $\mathbf{r}$ is the vector OP. The first equation shows that P lies in the plane perpendicular to AB at its mid-point C. The second expression shows also that P is the mid-point of the projections of A and B on l.

If $\mathbf{a}^2=\mathbf{b}^2$, O lies in the plane which is the perpendicular bisector of AB; and the cone degenerates into the two planes

$$(\mathbf{a} - \mathbf{b})\cdot\mathbf{d} = 0, \qquad (\mathbf{a} + \mathbf{b})\cdot\mathbf{d} = 0.$$

Also, if A and B are distinct and collinear with O, then $(\mathbf{d}\cdot\mathbf{a})^2=\mathbf{a}^2$, and the cone reduces to the straight line AB. These facts were noted in several solutions.

3671 [1934, 193]. *Proposed by J. Rosenbaum, Milford, Conn.*

In a tetrahedron, if any two of the three centers, the circumcenter, the incenter, and the centroid, are coincident, then all three are coincident.

Solution by H. D. Ruderman, James Madison High School, Brooklyn, N.Y.

Let the vertices of the tetrahedron be denoted by A_1, A_2, A_3, A_4; the circumcenter, the incenter, the centroid, by O, I, C, respectively. We prove first:

Lemma. If for each pair of faces having an edge in common the two altitudes to that edge for the corresponding triangles are equal, then $C\equiv O$.

Let A_1H_1 and A_3H_3 be altitudes of faces $A_1A_2A_4$ and $A_3A_2A_4$, and let them be equal; and let H and M be the mid-points of H_1H_3 and A_1A_3. Then the right triangles A_1H_1H and A_3H_3H are congruent; hence $A_1H=A_3H$, and HM is perpendicular to A_1A_3. Let K be the mid-point of H_1A_3; then HK and KM determine a plane HKM, since A_1A_3 and A_2A_4 are skew. Since HK and KM are perpendicular to H_1H_3, it follows that HM also is perpendicular to H_1H_3, i.e., it is the common perpendicular to A_1A_3 and A_2A_4. If we interchange the rôles of these two opposite edges, it follows that the straight line bisecting opposite edges must be their common perpendicular. This shows that the mid-point of this perpendicular, C, is equidistant from any two vertices, and therefore it is at the same distance from all the vertices. Hence $C \equiv O$.

Case I; $O \equiv I$. Since the faces can all be inscribed in equal circles, the face angles subtending the same edge are either equal or supplementary. The circumcenter of each face is the projection of I on that face, and therefore this center lies within the triangle. Hence the two angles must be equal. It then follows that the sum of the three angles at each vertex is 180°. From this it can be shown that the faces are congruent triangles. From the lemma we now have $O \equiv I \equiv C$.

Case II; $O \equiv C$. Here the centroid C is at the same distance from the four vertices, and hence the opposite edges are equal. The faces are congruent triangles and so their planes are equidistant from C. Hence $O \equiv I \equiv C$.

Case III; $C \equiv I$. The distance of C from any face is one-fourth the altitude of the tetrahedron to that face. Since $C \equiv I$, the four altitudes are equal. Therefore, the faces have equal areas, and the altitudes to the common edge of two faces are equal. The lemma shows that $O \equiv I \equiv C$.

Solved also by J. E. LaFon, and the proposer.

Editorial Note. The solution of LaFon contains the remark: The symmetric of the circumcenter with respect to the centroid is the Monge point; the problem may be extended to the following: If any two of the points, the circumcenter, the incenter, the centroid, the Monge point, of a tetrahedron coincide, the tetrahedron is isosceles, i.e., opposite edges are equal, and the four points coincide.

The proposer refers to problem 3512 [1932, 552], which shows that, if the circumcenter and the incenter are the same point, this point is also the centroid. If the circumcenter and the centroid coincide, he states that problem 3624 [1934, 401] shows that the faces are congruent triangles; and from this he infers that the three points coincide.

A solution by Leon Recht was received after the above had been sent in for printing. A lemma was first proved stating that if the faces of the tetrahedron are congruent triangles the three centers O, I, C coincide. It was then proved that if any two of these centers coincide the faces are congruent. Independent proofs were given and also references to E 86 [1934, 520], 3482 [1932, 364] and to those already cited above. A number of interesting properties of

such tetrahedrons were given with references to E 66 [1934, 329] and E 29 [1933, 493].

3672 [1934, 193]. *Proposed by W. V. Parker, Mississippi Woman's College.*

Prove that the sum of the primitive roots of $x^n-1=0$ is zero if n has a repeated prime factor and is $(-1)^k$ if n is the product of k distinct primes.

Solution by Maud Willey, Long Beach, Miss.

If $n=ab$, a prime to b, and if the primitive roots of $x^a-1=0$ are $a_1, a_2, a_3, \cdots, a_{\phi(a)}$ and the primitive roots of $x^b-1=0$ are $b_1, b_2, b_3, \cdots, b_{\phi(b)}$, then the sum of the primitive roots of $x^n-1=0$ is

$$(a_1+a_2+a_3+\cdots+a_{\phi(a)})(b_1+b_2+b_3+\cdots+b_{\phi(b)});$$

for each of the $\phi(a)\phi(b)$ products a_ib_j is a primitive root of $x^n-1=0$ and every primitive root of this equation can be expressed as one and only one of these products.

The sum of all the roots of $x^n-1=0$ is 0. If n is a prime, then the equation has $n-1$ primitive roots and the root $x=1$. In this case the sum of the primitive roots is -1. Using mathematical induction, it can be shown that the sum of the primitive roots is $(-1)^k$ if n is the product of k distinct primes.

If $a=p^r$, p a prime, then all the roots of $x^a-1=0$ are primitive except those that are also roots of $x^{p^{r-1}}-1=0$. Hence in this case the sum of the primitive roots of $x^a-1=0$ is $0-0=0$. Therefore if n has a repeated prime factor, $a=p^r$, the sum of the primitive roots of $x^n-1=0$ has a zero factor, $a_1+a_2+a_3+\cdots+a_{\phi(a)}$.

Solved also by Jeanette Fox, and E. P. Starke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Southern Intercollegiate Mathematics Association held its second annual meeting in Shreveport, Louisiana, at Centenary College, May 11, 1935. The schools having won the right to participate in the finals by virtue of winning in the preliminary contest in their respective regions were:

Millsaps College, Region I
Centenary College, Region II
McMurry College, Region III
College of the Ozarks, Region IV

Centenary College was awarded the Association Cup for having the highest average in the finals.

The following officers were elected: President, Dr. I. Maizlish, Centenary College; Vice-President, Dean Mitchell, Millsaps College; Secretary-Treasurer,

Jennie Tate, McMurry College; Members of Executive Council; Region I, Dr. G. A. Baker, Mississippi Women's College; Region III, Professor J. E., Burnam, Hardin-Simmons University. The term of office for members from Region II and Region IV had not expired. They are Dr. Karl A. Maring, Loyola University, and Dr. H. M. Hosford, University of Arkansas.

The Trustees of Columbia University, at their meeting on March 4, accepted the resignation of Dr. T. S. Fiske as professor of mathematics, to take effect at the end of the academic year 1934–35, and voted to bestow upon him the title of professor emeritus of mathematics. After his release from active service he will retain a study and office at Columbia University.

Professor Frederick H. Bailey who has been a member of the staff of the department of mathematics at the Massachusetts Institute of Technology since 1891 and professor since 1907 will retire at the end of the academic year with the title of professor emeritus.

Professor Emmy Noether, of Bryn Mawr College, died April 14, 1935.

The following ninety-four doctorates with mathematics or mathematical physics as major subject were conferred during 1934 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics) and title of dissertation are given in each case if available.

E. B. Allen, Rensselaer Polytechnic Institute, June, minor in physics. *Extension of a theorem due to M. D'Ocagne to partial differential equations in n variables.*

A. E. Andersen, Harvard, June. *Topics in the theory of binary forms.*

Frances E. Baker, Chicago, March. *A contribution to the Waring problem for cubic functions.*

D. H. Ballou, Harvard, June. *A class of completely monotonic functions every positive power of which is also completely monotonic.*

J. L. Barnes, Princeton, June. *On the Laplace-Stieltjes transformation I.*

J. J. Barron, Wisconsin, June. *The application of asymptotic forms to an expansion problem of the Sturm-Liouville type where the coefficient of the parameter changes sign.*

Miriam F. Becker, Yale, June. *On relative fields.*

A. F. Bernhard, Michigan, June. *The mechanics of the top.*

Solomon Bilinsky, Washington University (St. Louis), June, minor in physics. *A theory of functions of an abstract variable.*

Marion T. Bird, Illinois, June, minors in statistics and physics. *On generalizations of sum formulas of the Euler-Maclaurin type.*

W. M. Borgman, Chicago, August. *Intersector varieties in hyperspace.*

F. L. Brooks, Ohio State, August. *Various properties of the metric limit.*

G. M. Brown, Michigan, February. *On sampling from compound populations.*

J. A. Clarkson, Brown, June, *Riemann-Stieltjes double integrals and bounded variation for functions of two variables.*

Nancy Cole, Harvard, June. *The index form associated with an extremaloid.*

Byron Cosby, II, Chicago, March. *Fields for multiple integrals in the calculus of variations.*

G. F. Cramer, Missouri, August. *Rings and ring-ideals in a relative quadratic field.*

R. H. Downing, West Virginia, June, minor in mathematical physics. *Flatsphere geometry in non-euclidean n-space.*

T. L. Downs, Jr., Harvard, June. *On the planar points of an analytic surface.*

Mary J. Fisher, Toronto, March. *A new method in the theory of algebraic functions of one variable.*

B. C. Getchell, Michigan, June. *Integration of interval functions.*

H. A. Giddings, Massachusetts Institute of Technology, June, minor in chemistry. *On the extension of the notion of developable surfaces to V_2 and V_3 in R_4.*

Margaret Gurney, Brown, June. *Some general existence theorems for partial differential equations of hyperbolic type.*

Mary E. Haller, Washington, June, minor in physics. *Self-projective rational octavics.*

E. K. Haviland, Johns Hopkins, June. *On the theory of absolutely additive distribution functions.*

M. A. Heaslet, Stanford, June, minor in physics. *Concerning the development coefficients of an equianharmonic function.*

Mrs. V. F. Hopper, (Grace M.) Yale. June. *New types of irreducibility criteria.*

Olive Hughes, Bryn Mawr, June, minor in physics. *A certain mixed linear integral equation.*

Antoinette Killen Huston (Mrs. R. E.), Chicago, June. *The integral bases of all quartic fields with a group of order eight.*

J. A. Hyden, Cornell, June, minor in physics. *The Weddle and Kummer surfaces for restricted positions of six base points.*

Nathan Jacobson, Princeton, January. *Non-commutative polynomials and cyclic algebras.*

Evan Johnson, Jr., Chicago, August. *Dynamics of variable masses.*

H. S. Kaltenborn, Michigan, June. *On Stieltjes mean integrals.*

Rosella Kanarik, Pittsburgh, June. *Fundamental regions in S_4 for the Hessian group.*

Nathan Kaplan, Massachusetts Institute of Technology, June, minor in physics. *A study of three-dimensional Riemann surfaces in six-dimensional Euclidean space.*

Gertrude S. Ketchum, (Mrs. P. W.) Illinois, June, minor in English. *On certain generalizations of the Cauchy-Taylor expansion theory.*

Irving Kittell, New York University, June, minor in physics. *On the classification and coloring of spherical maps.*

S. C. Kleene, Princeton, January. *A theory of positive integers in formal logic.*

J. C. Knipp, Pittsburgh, June. *Frégier surfaces.*

R. L. Kruger, Marquette, June, minor in physics. *Plane quintic curves, a classification and projective construction.*

W. S. Lawton, Pennsylvania, February. *A method of deriving a homogeneous linear differential equation of the second order satisfied by a certain class of orthogonal polynomials.*

Marjorie Leffler, Ohio State, August. *A lemma in potential theory.*

Jack Levine, Princeton, June. *Projective scalar differential invariants.*

F. A. Lewis, Ohio State, August. *Some properties of an infinite class of collineation groups.*

Marie Litzinger, Chicago, August. *A basis for residual polynomials in n variables.*

A. N. Lowan, Columbia, May. *Application of the Laplace transformation to certain problems in physics.*

S. T. Ma, California (Berkeley), May. *The relations between the solutions of the linear differential equation of the second order having four regular singular-points.*

L. A. MacColl, Columbia, May. *On the distribution of the zeros of sums of exponentials of polynomials.*

C. W. MacGregor, Pittsburgh, June, minors in physics and mechanics. *Selected problems in the theories of flat plates and plane stress.*

J. D. Mancill, Chicago, December. *The minimum of a definite integral with respect to unilateral variations.*

M. F. Manning, Massachusetts Institute of Technology, June, major in physics. *Exact solutions of the Schrödinger equation.*

Louis Marick, Wisconsin, June, major in physics. *An investigation of the temperature effect upon the resistance and crystal structure of cobalt.*

W. T. Martin, Illinois, June, minor in astronomy. *On expansions in terms of a certain general class of functions.*

B. B. Murdock, Yale, June. *Skew frequency curves.*

E. P. Northrop, Yale, June. *An operational solution of the Maxwell field equations.*

Rufus Oldenburger, Chicago, March. *Composition and rank of n-way matrices and multilinear forms.*

J. W. Querry, State University of Iowa, February, minors in applied mathematics and education. *A general theory of mechanical quadrature.*

W. C. Randels, Brown, June. *Some problems in a theory of Fourier series.*

J. F. Randolph, Cornell, September, minor in astronomy. *Caratheodory measure and a generalization of the Gauss-Green lemma.*

Louis Rauch, California (Berkeley), August. *The almost periodic solution of a problem in forced vibrations by a process of iteration.*

R. F. Rinehart, Ohio State University, August. *Some properties of the discriminant matrices of a linear associative algebra.*

M. I. S. Robertson, Princeton, June. *On the theory of univalent functions.*

W. J. Robinson, Ohio State, August. *Rings with one prime.*

J. B. Rosser, Princeton, June. *A mathematical logic without variables.*

M. F. Rosskopf, Brown, June. *Some inequalities for non-uniformly bounded ortho-normal polynomials.*

J. M. Rowat, Toronto, June. *Existence theorems for the differential equation* $dy-dx=f\ (x, y)$ *under conditions of Perron integrability.*

O. K. Sagen, Chicago, June. *The integers represented by sets of positive ternary quadratic non-classic forms.*

G. E. Schweigert, Johns Hopkins, June. *The analysis of certain curves by means of derived local separating points.*

Nathan Schwid, Wisconsin, June, minor in applied mathematics. *The asymptotic forms of the Hermite and Weber functions.*

C. H. W. Sedgewick, Brown, June. *Generalized Lambert series.*

Robert Serber, Wisconsin, June, major in physics. *Some optical properties of molecules.*

J. A. Sharpe, Wisconsin, June, major in physics. *Surface motion in the compressional phase of a deep-focus earthquake and the effects of a layered crust.*

Sister Ann Elizabeth Shea, Wisconsin, October. *Regular Cremona transformations in* S_4.

R. C. Shook, Chicago, June. *Concerning Waring's problem for sixth powers.*

A. J. Smith, Pennsylvania, June. *Concerning upper semi-continuous collection on curves and two-dimensional manifolds.*

F. C. Smith, Michigan, June. *Contributions to the study of the a symptotic developments of analytic functions.*

F. H. Steen, Harvard, June. *On a class of polynomials which minimize definite integrals.*

J. K. Stewart, West Virginia, June, minor in mathematical physics. *Some birational transformations associated with a class of ruled varieties in n-space.*

J. W. T. Suckau, Ohio State, June. *Uniform convergence and the Lebesgue theory.*

Sister Helen Sullivan, Catholic University minor in physics and mechanics. June. *The non-self-symmetric quadrilaterals in and circumscribed to the rational cuspidal quartic with a line of symmetry.*

W. R. Talbot, Pittsburgh, August. *Fundamental regions in* S^6 *for the simple quaternary* G^{60} *Type I.*

Henrietta P. Terry, Illinois, June, minor in chemistry. *Abelian subgroups of order* P^m *of the I-groups of the Abelian groups of order* P^n *type* $1, 1, 1, \cdots$.

R. W. Thomas, Pittsburgh, June, minor in physics. *Four-space representation of chain congruences in the plane of two complex variables.*

G. R. Trott, Johns Hopkins, June. *On the canonical form of a non-singular pencil of Hermitian matrices.*

P. L. Trump, Wisconsin, October. *On a reduction of a matrix by the group of matrices commutative with a given matrix.*

J. L. Vanderslice, April, Princeton. *Non-holonomic geometries.*

Henry Van Engen, Michigan, June. *On the asymptotic behavior of analytic functions.*

J. I. Vass, Wisconsin, June, minor in applied mathematics. *A class of boundary problems of highly irregular type.*

W. J. Wagner, Pittsburgh, June, minor in astronomy. *Reciprocals of some elementary figures with respect to a space cubic.*

R. J. Walker, Princeton, June. *Reduction of the singularities of an algebraic surface.*

G. C. Webber, Chicago, June. *Waring's problem for cubic functions.*

K. W. Wegner, Wisconsin, June. *The equivalence of pairs of Hermitian matrices.*

C. B. Wright, Pittsburgh, June. *Fundamental regions in* S_4 *for the collineation group* $G_{3:360}$.

C. R. Wylie, Jr., Cornell, June, minor in physics. *Curves whose tangents belong to a linear complex.*

Granted in 1933, not already recorded:

F. G. Dressel, Duke, June. *A boundary value problem for the heat equation.*

Mabel Griffin, Duke, June. *Invariants of pfaffian systems.*

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Nineteenth Summer Meeting of the Association, Ann Arbor, Mich., Sept. 9-10, 1935.
Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4; Beaver Falls, Pa., Oct. 26.
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11; College Park, Md., Dec. 7.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

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THE APRIL MEETING OF THE OHIO SECTION

The twentieth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 4, 1935, with an afternoon session, a dinner, and an evening session. Professor Henry Blumberg, Chairman of the Section, presided at these sessions. The Section was honored by the presence of Professor Gabriel Szegö of Washington University, Professor E. J. Moulton of Northwestern University, and Dr. B. O. Skinner, Director of the Department of Education of the State of Ohio, who were guests of the Section and contributed to the program.

Eighty-three persons registered attendance, forty-nine of whom were members of the Association, namely: R. B. Allen, W. E. Anderson, A. H. Bailey, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, Henry Blumberg, J. B. Brandeberry, O. E. Brown, C. E. Buell, C. T. Bumer, R. S. Burington, E. H. Clarke, T. F. Cope, Rufus Crane, Wayne Dancer, O. L. Dustheimer, T. M. Focke, Harris Hancock, R. C. Hildner, E. M. Justin, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, Florentina Mathias, C. J. McGee, C. C. Morris, E. J. Moulton, J. R. Overman, Jesse Pierce, H. S. Pollard, Tibor Radó, S. E. Rasor, C. E. Rhodes, Hortense Rickard, S. A. Rowland, G. W. Spenceley, H. E. Stelson, Gabriel Szegö, C. F. Thomas, C. C. Torrance, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, Jesse Pierce, Heidelberg College; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, C. C. MacDuffee, Ohio State University; Member of Program Committee, Mary E. Sinclair, Oberlin College.

It is expected that the next meeting will be held on Thursday, April 2, 1936, at the Ohio State University.

The following papers were read:

1. "On the change of form" by the Chairman of the Section, Professor Henry Blumberg, Ohio State University.
2. "Some recent applications of Sturm's oscillation method" by Professor Gabriel Szegö, Washington University, by invitation.
3. "On the theory of linear circuits from the standpoint of the theory of matrices" by Professor R. S. Burington, Case School of Applied Science.
4. "On the teaching of mathematics in the secondary schools" by Dr. B. O. Skinner, Director of Education, State of Ohio, by invitation.
5. "Statistics relative to the mathematical preparation of mathematics teachers and of freshmen in the University of Cincinnati" by Professor Harris Hancock, University of Cincinnati.
6. "Mathematics clubs" by Professor Wayne Dancer, Toledo University.
7. "The training and utilization of advanced students of mathematics" by Professor E. J. Moulton, Northwestern University, by invitation.

Abstracts of these papers follow:

1. Professor Blumberg analyzed the role of change of form (which is the same as substitution, if understood in a properly abstract sense) in the mathematical thought process. The interest was not in the mere exposition of an idea but in suggestions for improving the technic of teaching and learning. Among the more important types of change of form discussed were: decomposition, abstraction, logical changes of form, passage from descriptive to structural properties, from differential to integral properties, from absolute to relative properties, from problems to relationships, from the infinite to the finite, and various others. These changes are reversible. Changes of form may be regarded as co-extensive with the whole of mathematics in its differential aspect.

2. Using Sturm's method, a new and very short way was given by Professor Szegö for obtaining the inequalities for the zeros of Legendre polynomials $P_n(\cos \theta)$ due to Bruns, as well as certain more exact inequalities due to Markoff and Stieltjes. Denoting by $\theta_1, \theta_2, \cdots, \theta_n$ the zeros of $P_n(\cos \theta)$ on the range $0, \pi$ in increasing order, the estimates

$$\frac{\nu - 1/4}{n + 1/2}\pi < \theta_\nu < \frac{\nu}{n+1}\pi \qquad \nu = 1, 2, \cdots, (n/2).$$

are proved. The upper bound is identical with that of Markoff-Stieltjes, the lower bound is even better. The same method yields estimates for the zeros of some ultraspherical polynomials as well as of some Bessel functions.

3. Professor Burington discussed, from the standpoint of the theory of matrices, certain elementary mathematical and physical concepts which underlie the recent work of Cauer, Howitt, Gewertz, Kron, Burington and others, in the field of linear electrical circuit theory. He defined a total matric algebra, reviewed certain elementary physical laws used in linear network theory, discussed the matric differential equations of a network, introduced the concept of a network matrix and considered its relation with the matrix of instantaneous power. He closed with the theory of equivalent networks, using well known theorems in matric algebra.

4. Dr. Skinner urged that the junior high school be used as a proving ground for those who desire to study mathematics further. The teacher is the most important element in the teaching-process; especially in such a cold, exact subject should he be warm-hearted and sympathetic. Besides the practical benefits that accrue to the student, the following five outstanding values of mathematics for high school pupils have been suggested: (a) it demands a proof; (b) it combines simple, direct thinking with the simplest set of fundamental concepts; (c) the value of exact expression is emphasized; (d) it plays a fundamental role in making possible the things we enjoy in our present day civilization; (e) geometry can instill the sense of form.

5. Professor Hancock undertook to show that under present conditions the only way to have students properly prepared in mathematics when entering our universities and colleges is through the introduction of honor courses into

the public schools. Results of a questionnaire answered by several hundred prominent men in Cincinnati and elsewhere showed at least 95% wanted mathematics emphasized in the high schools. (See School and Society, vol. 1, 1915, pp. 893–900.) Some 4500 examinations of freshmen as they entered twenty of the leading colleges and universities of Ohio showed scarcely over half sufficiently prepared in mathematics. Statistics show a gradual drift on the part of teachers from the study of mathematics to that of methods in education and finally to the total neglect of mathematical courses. The results of mingling boys of best genius (the Jeffersonian word for ability) with 50 mediocre pupils and placing them under a teacher who knows virtually no mathematics are disastrous.

6. See the abstract, elsewhere in this issue, of Professor Dancer's paper at the meeting of the Michigan Section.

7. Professor Moulton reported activities of the Association's Commission on the Training and Utilization of Advanced Students of Mathematics, with regard to (a) the employment of Doctors of Philosophy in mathematics (see the April number of the Monthly), (b) the employment bureau under his direction, (c) statistical information concerning doctorates granted in mathematics since 1862 and papers published by the Doctors, (d) Master's degrees granted in mathematics since 1920, (e) trend of mathematical registrations in the leading universities, (f) the training of teachers of mathematics (published in the May number of the Monthly). It appears that the number of undergraduates in the universities during the last fifteen years has been essentially constant in spite of the growth of the universities, but that the number of graduate students has increased greatly. The demand for well trained teachers in secondary schools should for the future still further increase the number of graduate students.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-fourth meeting of the Iowa Section was held at Grinnell College, Grinnell, Iowa, on Friday and Saturday, April 19–20, 1935, in conjunction with the meetings of the Iowa Academy of Science. The Section chairman, Professor M. E. Graber, presided at both sections, relieved for a time by Professor Cornelius Gouwens.

The attendance was forty, including the following twenty-five members of the Association: F. A. Brandner, L. M. Coffin, Julia T. Colpitts, A. T. Craig, Marian E. Daniells, C. W. Emmons, Cornelius Gouwens, M. E. Graber, O. C. Kreider, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Sigurd Mundhjeld, I. F. Neff, Arthur Ollivier, J. F. Reilly, H. L. Rietz, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, C. W. Strom, J. S. Turner, L. E. Ward, C. W. Wester.

Dinner was enjoyed together with the members of the Physics Section of the Iowa Academy of Science on Friday evening. The officers elected for 1935–

36 are as follows: Chairman, Julia T. Colpitts, Iowa State College; Vice-Chairman, R. B. McClenon, Grinnell College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution was adopted expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by their host, Grinnell College, and especially by Professors W. J. Rusk and R. B. McClenon, of the Department of Mathematics.

The following ten papers were read:

1. "Continuation formulas for $u^2=x^2+y^2+z^2$" by Professor W. J. Rusk, Grinnell College.

2. "A short solution of the Diophantine equation $2x^4-y^4=z^2$" by Professor J. S. Turner, Iowa State College.

3. "Note on the stereographic projection of the unit sphere" by Professor J. V. McKelvey, Iowa State College.

4. "The influence of the French Revolution on mathematics" by Professor Helen F. Smith, Iowa State College, introduced by Professor E. R. Smith.

5. "Elementary geometry from the standpoint of Castelnuovo and Peano" by Professor M. E. Graber, Morningside College.

6. "On the resolution of $(x^p-1)/(x-1)$ for p a prime less than 100" by Professor Cornelius Gouwens, Iowa State College.

7. "A criterion for prime numbers" by Professor C. W. Strom, Luther College.

8. "Computation of the roots of the polynomials of Legendre" by Professors E. R. Smith and Archie Higdon, Iowa State College.

9. "A note on first order interpolation in a logarithm table" by Professor J. F. Reilly, University of Iowa.

10. "A set of cubic equations" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The formulas used by Professor Rusk are as follows:

If

$$u_1^2 = x_1^2 + y_1^2 + z_1^2$$

then

$$u_r^2 = x_r^2 + y_r^2 + z_r^2 \qquad (r = 2, 3, 4, \cdots)$$

$$u_r = 2u_{r-1} \pm x_{r-1} \pm y_{r-1} \pm z_{r-1}$$

$$x_r = u_{r-1} \pm y_{r-1} \pm z_{r-1}$$

$$y_r = u_{r-1} \pm x_{r-1} \pm z_{r-1}$$

$$z_r = u_{r-1} \pm x_{r-1} \pm y_{r-1}.$$

The corresponding signs are taken all through the formulas and the x, y, z are taken positively. The proof is immediate by mathematical induction.

2. The solution given by T. Pepin (see *Dickson's History*, vol. 2, p. 626) is obtained by a more direct procedure. The fifth numerical solution is (42422452969, 9788425919, 2543305831910011724639).

3. When the sphere defined by $x^2+y^2+(z-k)^2=1$ is projected from $(0, 0, 1+k)$ upon a uv-plane coincident with the xy-plane, the element of arc on the surface of the sphere is given by the equation: $ds^2=\lambda(u, v)(du^2+dv^2)$. Professor McKelvey shows that the portions of the uv-plane for which the functions $\lambda(u, v)$ and $[\lambda(u, v)]^{1/2}$ are subharmonic vary with k, but that the corresponding portions of the sphere are independent of k.

4. Professor Smith advised a study of the early volumes of the Journal de L'École Polytechnique for the light they throw upon the relationship between the history of the times and the rapid development of French mathematics. The founding of L'École Polytechnique held together the most prominent mathematicians of the country, it trained a remarkable group of younger men, it set a high standard of excellence in teaching and research and furnished an avenue for publication. The great educators practically all held very influential offices in the government and were able to control, to a degree, plans for public education.

5. Professor Graber told of the results of his teaching a course in elementary analytic geometry where both plane and solid analytic geometry were taken together. The concepts of two-space were immediately carried into three-space. This seems to be a common practice in Italian schools.

6. In this paper, Professor Gouwens displayed the resolution of

$$4X = y^2 - (-1)^{(p-1)/2}pz^2$$

for p a prime greater than 71 and less than 100, when $X=(x^p-1)/(x-1)$.

7. Let $\int n$ denote the sum of the divisors of the positive integer n and $\int(n-a)$, $a<n$, similarly denote the sum of the divisors of the positive integer $(n-a)$. Then Professor Strom showed that

$$\int n = \sum_k (-1)^{k+1}\left[\int\left(n - \frac{3k^2-k}{2}\right) + \int\left(n - \frac{3k^2+k}{2}\right)\right], \qquad (k = 1, 2, 3, \cdots).$$

Where the sum on the right is to be extended to include all positive values of $n-(3k^2-k)/2$ and $n-(3k^2+k)/2$ and $\int(n-n)$, if it occurs, is defined to be equal to n. A necessary and sufficient condition that n be prime is that the sum on the right be equal to $(n+1)$.

8. This paper included a list of the roots of the polynomials of Legendre, $P_n(x)$, for values of n up to and including 25 correct to six decimal places.

9. The differences in a table of logarithms are often somewhat irregular due to the fact that the logarithms are cut to a certain number of figures before the differences are taken. This irregularity in the differences leads at times to errors in interpolated values. In his paper, Professor Reilly showed one method of eliminating such errors.

10. Professor Allen considered equations of the type

$$xyz(y - z)(z - x)(x - y) = c.$$

If a certain selection of eight such equations, involving six unknowns, are given, and they are consistent, they can be solved by the computation of cross ratios of the unknowns, and the reduction of three of them, by linear transformation, to 1, 0, and ∞, respectively.

CORNELIUS GOUWENS, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The third annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on Saturday, May 4, 1935. The Chairman of the Section, Professor H. P. Pettit, presided.

The attendance was seventy-three, including the following twenty-eight members of the Association: Kathryn P. Bartlett, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, H. H. Conwell, L. A. V. DeCleene, Henry Ericson, H. P. Evans, W. W. Hart, R. C. Huffer, Elizabeth E. Knight, R. E. Langer, Peter Luteyn, C. C. MacDuffee, Morris Marden, Sister Mary Felice, J. S. McNair, E. J. Moulton, R. E. Norris, G. A. Parksinson, H. P. Pettit, Irene Price, J. H. Rose, W. E. Roth, I. S. Sokolnikoff, P. L. Trump, J. I. Vass.

At the close of the morning session Professor R. E. Langer, of the University of Wisconsin, spoke in memory of Professor E. B. Skinner who devoted a beautiful and useful life to the interests of mathematics. His last service to the Association was to arrange this program of the Wisconsin Section immediately before his death on April 3.

A dinner was held at noon at Hotel Aberdeen. At the business meeting of the session, the following officers were elected for the year 1935–36: Chairman, H. H. Conwell, Beloit College; Secretary, Ethelwynn R. Beckwith, Milwaukee-Downer College; Program Committee, H. P. Evans, University of Wisconsin, and Henry Ericson, Washington High School, Milwaukee. The fourth annual meeting was set for May 1936, to be held at the University of Wisconsin, Madison.

The following papers were read during the morning and afternoon sessions:

1. "The solution of problems in modern geometry by means of vectors" by Professor L. A. V. DeCleene, St. Norbert College.

2. "Concerning a special type of polynomial" by Professor Irene Price, Oshkosh State Teachers College.

3. "Some applications of probability and statistics" by Professor H. P. Evans, University of Wisconsin.

4. "Some comments, criticisms, and suggestions concerning certain phases

of the teaching of elementary mathematics" by Professor I. N. Warner, Platteville State Teachers College.

5. "Training of high school teachers of mathematics" by Dean E. J. Moulton, Northwestern University.

6. "The Wisconsin plan for training teachers of secondary mathematics" by Professor Curtis Merriman, University of Wisconsin, by invitation of the Program Committee.

Abstracts of these papers follow, the numbers corresponding to the numbers of the titles:

1. The paper of Professor DeCleene shows the simplicity with which vectors can be used in the solutions of problems in modern geometry. The vector equations bear a resemblance to those of analytic geometry. Because equations of this type do not involve the coordinates equally, they lack symmetry. However, symmetric functions can be introduced. Since circles and lines can be easily represented by vector equations, all the solutions of problems of modern geometry can be found readily by using vectors.

2. Professor Price's paper deals with the properties of a set of polynomials which are obtained by using the same generating function as that used for the Legendre and Hermite polynomials but by making a different substitution for the variable in the generating function. The polynomials thus obtained have many interesting properties, some analogous to those of the Legendre and Hermite polynomials. A very unusual property is the breaking up of the set into two distinct subsets, each possessing different properties.

3. The paper of Professor Evans is an exposition of the statistical application of certain fundamental theorems of probability. The recent work of Pearson and Clopper in the December 1934 issue of Biometrika relating to "Confidence belts" and "Confidence coefficients" is discussed in connection with sampling theory. As another application of sampling theory and the principle of maximum likelihood, the problem of estimating the number of fish in a lake by means of sample catches at various times is discussed. The coefficient of correlation between two functions is discussed from the standpoint of the theory of probability and an application of its use is given.

4. Among many college teachers there is a marked indifference toward, if not a genuine hostility to, any procedure which considers methods of instruction and presentation. Results of student accomplishment in courses where better methods of instruction are used show this attitude to be one of gross error. Methods of instruction cannot be ignored in the high school nor in the grades below it. Mechanical drills upon abstract symbols, meaningless formulae, lack of simple, clear and concise definitions soon suffice to kill the love for mathematics that a child naturally has when he enters school. Students in college are badly handicapped in performing operations involving the more simple common fractions because the knowledge of the common fraction has been neglected and a limited knowledge of the decimal fraction substituted in its stead. Examples to demonstrate these criticisms were given in Professor Warner's paper.

5. Professor Moulton reported the recommendations concerning the preparation of high school teachers of mathematics which were prepared by a Commission of the Association, and which were published in the MONTHLY for May 1935.

6. Professor Merriman explained the new experimental plan the University of Wisconsin is using in training teachers of secondary mathematics. For three semesters the students with mathematics as a major receive their professional training in a class which is under the joint instruction of a professor from the general field of educational theory and a special methods professor from the department of mathematics. Both professors attend all class sessions, the actual teaching changing from one to the other according to the nature of the topic under consideration. Close correlation between theory and practice is secured by so arranging the work that a general method or psychological principle is immediately applied to teaching procedures in mathematics. The methods professor is in general charge of the teaching of mathematics at the Wisconsin High School. This makes it easy to arrange for demonstration lessons and follow-up discussions. During the second and third semesters the students have actual teaching experience at the high school. A fourth semester of work is given by the Dean of the College of Education and is planned to relate the preceding three semesters of specialized work to the larger field of educational practice and theory. This insures that the teacher can participate in the professional discussions of his co-workers in other subjects and corrects the errors of over-intensive specialization.

MAY M. BEENKEN, *Secretary*

READINGS IN THE LITERATURE ON TEACHING WITH SPECIAL REFERENCES TO MATHEMATICS

In the May, 1935, issue of the AMERICAN MATHEMATICAL MONTHLY, the Commission on the Training and Utilization of Advanced Students of Mathematics presented a *Report on the Training of Teachers of Mathematics.* One of the recommendations of the Commission involved guided reading in books and periodicals related to the theory of teaching, testing methods, and educational research. "The guided reading described . . . is not intended to justify a requirement of course work in Education. In fact, we are convinced that the objective . . . would not be attained if the guided reading were replaced by typical general courses in educational theory. The reading described . . . should prepare the candidate to evaluate intelligently or criticize constructively conclusions and methods with which he may later be involved in the teaching profession."

The Chairman of the Commission, E. J. Moulton, thinking that it might be of distinct service to mathematicians to have at hand a list of readings recommended by a competent group of mathematics teachers, asked the following Committee to prepare an appropriate bibliography: J. O. Hassler, Chairman,

University of Oklahoma; William Betz, Rochester Public Schools and the University of Rochester; Ralph Beatley, Harvard University; W. W. Hart, University of Wisconsin; W. D. Reeve, Columbia University; E. W. Schreiber, Western Illinois State Teachers College. This Committee has submitted the following report:

This Committee was appointed to prepare minimum and adequate lists of books and magazine articles that could be used in guiding graduate students in the reading recommended in the *Report on the Training of Teachers of Mathematics*, this MONTHLY, vol. 42, 263–277, 1935.

Every member of the Committee is of the opinion that any adequate course of training for mathematics teachers must include some study of the history, philosophy, and fundamental concepts of mathematics. In view of the fact that the report mentioned above recommends courses in these subjects, the Committee deems it unnecessary to submit a bibliography covering them, but wishes to add its approval to the recommendations of the report. In case a department offers no such courses then a series of directed readings should be given to include material found in such books as J. W. Young's *Fundamental Concepts of Algebra and Geometry*, Cassius J. Keyser's *Pastures of Wonder*, or *Thinking about Thinking*, Felix Klein's *Famous Problems of Elementary Geometry* (translated by Beman and Smith), or *Elementary Mathematics from an Advanced Standpoint* (translated by Hedrick and Noble), J. W. A. Young's *Monographs on Modern Mathematics*, Tobias Dantzig's *Number, the Language of Science*, E. T. Bell's *The Queen of the Sciences*, Vera Sanford's *A Short History of Mathematics*, and D. E. Smith's *History of Mathematics*. For the reason that a comprehensive view of the science of mathematics such as should be had by a teacher of the subject is not always gained by the student studying mathematics course by course in the college or university, the Committee emphasizes these topics.

The following recommendations consider primarily the prospective high school teachers, but part D refers to prospective college teachers.

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A. *Psychology and Methods of Teaching*

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A minimum list can be had by selecting one reference from each of the four groups above, but if the total is to be reduced it would be better to use selections from all items. Under the title "Supplementary Reading" will be given other desirable references on the topics listed above.

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Respectfully submitted, June 8, 1935.

Ralph Beatley
William Betz
W. W. Hart
W. D. Reeve
E. W. Schreiber
J. O. Hassler, *Chairman*.

EQUATIONS OF POLYGONS*

By C. O. OAKLEY, Haverford College

Consider the equation

$$x - 1 + |x + y - 1| = 0. \tag{1}$$

Now a straight line, $u=0$, and in particular $u \equiv x+y-1=0$, divides the plane into two regions, one region where the function u is plus and one where u is minus. In the $+$ region, where $x+y-1>0$, we may drop the absolute-value signs in equation (1) which then reduces to

$$+, \quad 2x + y - 2 = 0.$$

Similarly for the $-$ region, the absolute-value signs may be dropped provided we write $x-1-(x+y-1)=0$, which reduces to

$$-, \qquad y = 0.$$

* Presented, in modified form, to the Association, September 3, 1934 under the title: "Semi-linear Equations."

Hence the graph of (1) is the broken line (Fig. 1) which breaks or turns about the line $x+y-1=0$ at the point (1, 0).

Consider next the equation

$$-1+|x|+|x+y-1|=0. \tag{2}$$

Here we have two lines, $x=0$ and $x+y-1=0$, about which we might expect turnings in the graphical representation of this equation; and, inasmuch as these two lines divide the plane into four regions, we might also expect to find four line-segments making up the graph since in any one region equation (2) reduces to a linear equation and hence represents a straight line (which, if it is to make up part of the graph, must lie in or cut that region). The sign-sets (the two signs go respectively with the two absolute-value terms) and the reduced equations are

$$\begin{array}{ll} ++, & 2x+y-2=0, \\ -+, & \quad\;\; y-2=0, \\ --, & -2x-y\quad\;\, =0, \\ +-, & \quad\; -y\quad\;\, =0. \end{array} \tag{3}$$

We note (Fig. 2) that the figure is closed and is a parallelogram the equations of whose diagonals are $x=0$ and $x+y-1=0$.

Definitions. An equation of the form

$$u_0+m_1|u_1|+\cdots+m_n|u_n|=0, \tag{4}$$

where the m's are constants and where

$$u_i \equiv a_ix+b_iy+c_i, \qquad i=0, 1, 2, \cdots, n,$$

we define as a semilinear equation of order $p(\leqq n)$ if there are p and only p non-vanishing coefficients among the m's. It is supposed that the u_i's $(i=1, 2, \cdots, n)$ are distinct linear functions; throughout we shall deal only with the case of real coefficients, variables, functions, etc. Equations such as (3) we shall call with Söderblom* the auxiliary equations or the equations associated with (2). Connected with these auxiliary equations there will always be the sign-sets of the corresponding regions. The graph of an auxiliary equation in its associated region we shall call a solution (in that region) of the semilinear equation. Regions in which there is no part of the graph, i.e., in which the equation has no solution, we shall call vacuous. Vacuous regions and their associated equa-

* Axel Söderblom, *Sur l'Emploi de Valeurs Absolues dans la Géométrie Analytique*, 174 pp., Göteborg, Wald. Zachrisson, Boktryckeri A. B., 1906. See also two articles in the Bulletin de L'Institut Aérodynamique de Koutchino, Fascicule V, 1914, and a mimeographed lecture given in 1919 at the University of Copenhagen, by D. P. Riabouchinsky who treats among other things such questions as differentiation of absolute values. For example $D_x|x|=|x|/x$; etc. These are, so far as the author is aware, the only references to the subject of semilinear equations.

tions play an important role in the theory of semilinear equations. The lines $u_i=0, i=1, 2, \cdots, n$ we shall call the diagonals.

Putting a minus sign before the second absolute-value term in (2) "opens" the polygon (Fig. 3). Even in the case of (1) we might think of $x+y-1=0$ as the equation of the "diagonal" of this (open) "polygon." For convenience of reference to these broken-line graphs we shall adopt this convention of nomenclature and shall refer to them as open and closed polygons. We shall also refer indifferently to the diagonals of the polygon or to the diagonals of the equation.

Again the semilinear equation of order three,

$$-x+|x|-|x-1|+|y|=0, \tag{5}$$

has three diagonal lines that divide the plane into (only) six regions (since two of the lines are parallel). The graph is pictured in Fig. 4. The diagonals of the equation $|x-1|+|x-y-1|=0$ intersect in the point (1, 0); and it will be noticed that the equation associated with any one of the four regions is satisfied by only the one point (1, 0) in that region assuming that this (a boundary) point lies in each region. Hence the graph is the point (1, 0). The equation $x+1+|x|+|y|=0$ is satisfied by no point. The graph of the equation

$$-1+|x|+|x-1|+|y|=0 \tag{6}$$

is of interest. The three diagonals are $x=0$, $x-1=0$, and $y=0$; and the sign-sets and auxiliary equations are:

$$\begin{array}{ll}
+\,+\,+, & 2x+y-2=0,\\
+\,-\,+, & y=0,\\
-\,-\,+, & -2x+y=0,\\
-\,-\,-, & -2x-y=0,\\
+\,-\,-, & -y=0,\\
+\,+\,-, & 2x-y-2=0.
\end{array}$$

The graph of (6) is, therefore, the unit segment on the x-axis from 0 to 1 (Fig. 5).

The semilinear equation of a particular configuration need not be unique. For example the graph of the equation

$$f_1(x, y) \equiv -1+|x|+|x-1|+k|y|=0, \tag{7}$$

is the same unit segment as that of (6) for every value of $k>0$. Equations (6) and (7) are of the same order but even this is not necessary; the semilinear equation, of order five,

$$f_2(x, y) \equiv -3+|x|+|x-1|+|y+1|+|y|+|y-1|=0 \tag{8}$$

again represents the segment (0, 1). Or the equation may be based upon other diagonals; for example

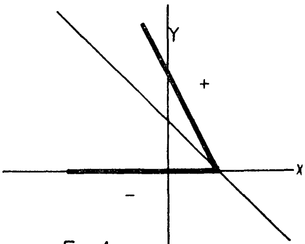


FIG. 1

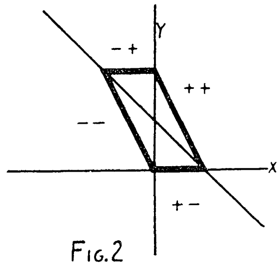


FIG. 2

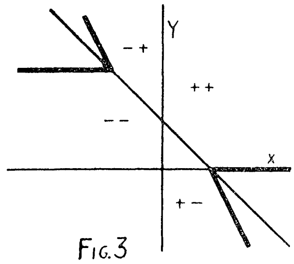
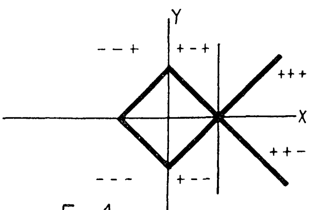


FIG. 3



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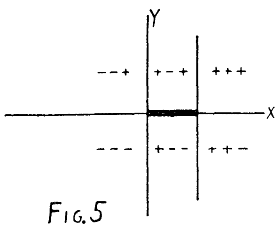

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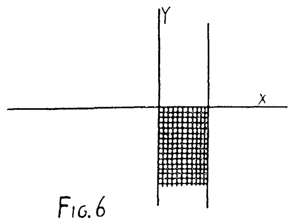

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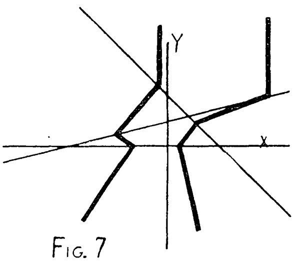

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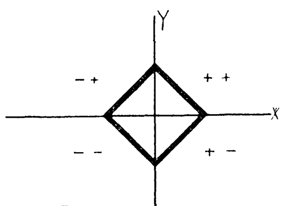

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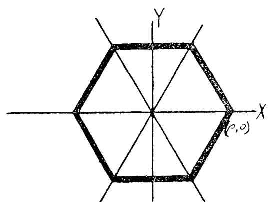


FIG. 9

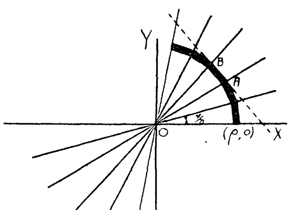


FIG. 10

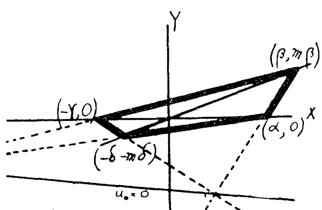


FIG. 11

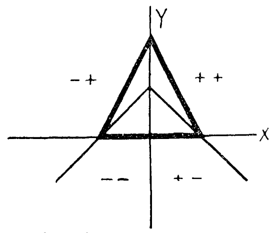


FIG. 12

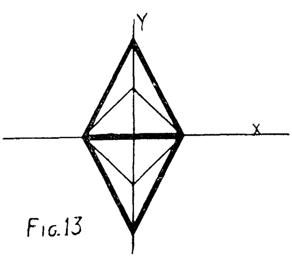


FIG. 13

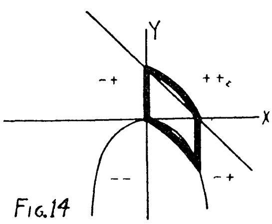


FIG. 14

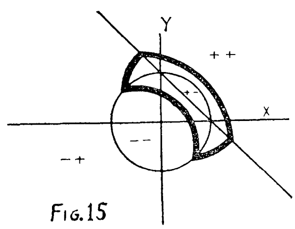


FIG. 15

$$(9) \qquad f_3(x, y) \equiv y + 1 - |x| - 3|y| - |x + y - 1| = 0,$$

where the three diagonals divide the plane into seven regions, is represented graphically by the same segment.*

If, for any particular region, the associated equation reduces to the identity $0 \equiv 0$, then every point in the region is a solution. Thus the equation $x - |x| = 0$ represents the right half-plane and $x + y + |x| + |y| = 0$ the third quadrant; the graph of $y - a + |x| + |x - a| + |y| = 0$ is a half-strip as indicated in Fig. 6. The equation, of order four,

$$x - y + 2 + 3|y| - |x - 4y + 2| - 5|x| + 2|x + y - 1| = 0$$

has four diagonals which divide the plane into eleven regions in three of which there is no graphical representation (Fig. 7). The equation $|x| + |y| = 1$ represents the square whose diagonals coincide with the axes and whose vertices are $(\pm 1, 0)$, $(0, \pm 1)$; the regions are the quadrants themselves (Fig. 8).

It is thus seen by these examples that equations of the type (4) may represent a wide variety of broken-line graphs such as polygons, line-segments, areas, etc. For an equation (4) of order p, there are p diagonal lines $u_i = 0$. If these lines are as linearly independent as possible, i.e., no three concurrent and no two parallel, they divide the plane into $p(p+1)/2+1$ (a maximum number of) regions of which $(p-1)(p-2)/2$ are finite ($p \geqq 2$). Since there would be 2^p sign-sets, it follows that in general there are more sign-sets than regions. For any given sign-set and hence for any given region, equation (4) reduces to $u \equiv ax + by + c = 0$, (or $u \equiv 0$) which is the equation of a straight line. This line may or may not contain points of the particular given and associated region.

The converse problem, namely that of finding the semilinear equation of a given configuration, presents many difficulties, chief among them being the choice of diagonals. In the first place, because of the *linearity* of the auxiliary equations, it is necessary to isolate into separate regions of the plane each line-segment of the figure whose equation is sought by a wise and proper choice of lines $u_i = 0$. Thus it is immediately seen that no semilinear representation is possible for a figure or configuration for which there exists no set of diagonal lines possessing this isolating property. And since in general the given figure will not occupy every region formed by this set of lines, there is also the problem of keeping such regions vacuous. (Conceivably it might happen that extraneous solutions would turn up in these regions thus spoiling the problem; and this is exactly what does occur with many choices of diagonals.) And there are other troublesome factors which we shall not go into at this time. For convex polygons, considered both as curves and as areas, and for certain other configurations the problem of semilinear representation is solvable and the theory is not too difficult. We turn now to the proof of the following

* Note that even though f_1, f_2, and f_3 give rise to the same unit segment, i.e., reduce essentially to the same function in the regions of the graph, yet they are distinct functions in the other regions of the plane. See C. Fox, The Mathematical Gazette, v. 15, Dec. 1931, for an interesting example in polar coordinates.

THEOREM. *A semilinear equation of order n exists such that the geometric representation is a regular polygon of $2n$ sides.**

Proof. We choose axes so that the center of the polygon is at (0, 0) and one vertex at $(\rho, 0)$ (Fig. 10). The coordinates of the vertices are therefore (α_i, β_i) where

$$\alpha_i = \rho \cos (i - 1)\frac{\pi}{n},$$

$$\beta_i = \rho \sin (i - 1)\frac{\pi}{n}.$$

And the equations of the diagonals are

$$u_i = \cos (i - 1)\frac{\pi}{n}\, y - \sin (i - 1)\frac{\pi}{n}\, x = 0.$$

We seek an equation of the form

$$\sum_{i=1}^{n} m_i \left| u_i \right| = K \tag{10}$$

and wish to determine m_i and K so that (10) is the equation of the polygon.

Substituting the coordinates of one vertex-point† of each region from $+ - - \cdots -$ to $+ + \cdots +$ respectively in the upper half-plane into (10), we get

$$\begin{array}{llll}
m_1 \cdot 0 & + m_2\rho \sin \dfrac{\pi}{n} & + \cdots + m_n\rho \sin (n-1)\dfrac{\pi}{n} & = K, \\
m_1\rho \sin \dfrac{\pi}{n} & + m_2 \cdot 0 & + \cdots + m_n\rho \sin (n-2)\dfrac{\pi}{n} & = K, \\
m_1\rho \sin \dfrac{2\pi}{n} & + m_2\rho \sin \dfrac{\pi}{n} & + \cdots + m_n\rho \sin (n-3)\dfrac{\pi}{n} & = K, \\
\cdots & \cdots & \cdots & \cdots \\
m_1\rho \sin (n-1)\dfrac{\pi}{n} & + m_2\rho \sin (n-2)\dfrac{\pi}{n} & + \cdots + m_n \cdot 0 & = K.
\end{array}$$

The determinant of this system is

---

* This theorem was given without proof by Söderblom.

† One vertex is sufficient since each vertex may be thought of as lying in two regions and if both were used we would get two identical systems of equations.

$$\Delta = \rho^n \begin{vmatrix} 0 & \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} \cdots & \sin(n-1)\frac{\pi}{n} \\ \sin\frac{\pi}{n} & 0 & \sin\frac{\pi}{n} \cdots & \sin(n-2)\frac{\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{\pi}{n} & 0 \quad \cdots & \sin(n-3)\frac{\pi}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \sin(n-1)\frac{\pi}{n} & \sin(n-2)\frac{\pi}{n} & \cdot \quad \cdots & 0 \end{vmatrix},$$

and if $\Delta \neq 0$, the above system can be solved for the ratios m_i/K. By bringing the last row to the second row, the next to the last row to the third row, etc., and by writing $\sin(\pi/n) = \sin\{(n-1)\pi/n\}$, etc., we have (noting that the sign of the determinant in so doing has been changed $(n-2)+(n-3)+\cdot\quad+1 = (n-1)(n-2)/2$ times):

$$\Delta = \rho^n(-1)^{(n-1)(n-2)/2}\cdot \begin{vmatrix} 0 & \sin\frac{\pi}{n} \cdots & \sin(n-1)\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} \cdots 0 & \\ \sin\frac{2\pi}{n} & \cdot \quad \cdots & \cdot \\ \cdots & \cdots & \cdots \\ \sin(n-1)\frac{\pi}{n} & 0 \quad \cdots & \sin(n-2)\frac{\pi}{n} \end{vmatrix}.$$

From this form we see at once that Δ is a cyclic determinant and*

$$\Delta = \rho^n \cdot f(x_1)\cdot f(x_2)\cdot \cdots \cdot f(x_n),$$

where

$$f(x_j) = 0 + \sin\frac{\pi}{n}x_j + \sin\frac{2\pi}{n}x_j^2 + \cdots + \sin(n-1)\frac{\pi}{n}x_j^{n-1},$$

and where x_j is an n-th root of unity, i.e.,

$$x_{k+1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n-1.$$

Now

* G. Kowalewski, *Einführung in die Determinantentheorie*, Leipzig, 1909, p. 117.

$$\Delta = \rho^n \prod_{j=1}^{n} f(x_j)$$

and hence is zero when and only when one of the factors $f(x_j)=0$. We have

$$f(x_{k+1}) = \sin\frac{\pi}{n}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) + \sin\frac{2\pi}{n}\left(\cos\frac{4k\pi}{n} + i\sin\frac{4k\pi}{n}\right)$$
$$+ \cdots + \sin(n-1)\frac{\pi}{n}\left[\cos(n-1)\frac{2k\pi}{n} + i\sin(n-1)\frac{2k\pi}{n}\right].$$

If $f(x_{k+1})=0$, then the real part vanishes as does the imaginary part; this gives

$$\sin\frac{\pi}{n}\cos\frac{2k\pi}{n} + \cdots + \sin(n-1)\frac{\pi}{n}\cos(n-1)\frac{2k\pi}{n} = 0,$$
$$\sin\frac{\pi}{n}\sin\frac{2k\pi}{n} + \cdots + \sin(n-1)\frac{\pi}{n}\sin(n-1)\frac{2k\pi}{n} = 0.$$

These are respectively equivalent to

$$\sin\theta+\sin 2\theta+\cdots+\sin(n-1)\theta-[\sin\phi+\sin 2\phi+\cdots+\sin(n-1)\phi]=0,$$
$$\cos\phi+\cos 2\phi+\cdots+\cos(n-1)\phi-[\cos\theta+\cos 2\theta+\cdots+\cos(n-1)\theta]=0,$$

where $\theta=(2k+1)\pi/n$ and $\phi=(2k-1)\pi/n$; and these in turn are, by means of the identities,

$$\sin\alpha + \sin 2\alpha + \cdots + \sin(n-1)\alpha \equiv \sin\frac{n\alpha}{2}\sin(n-1)\frac{\alpha}{2}\csc\frac{\alpha}{2},$$

$$\cos\alpha + \cos 2\alpha + \cdots + \cos(n-1)\alpha \equiv \cos\frac{n\alpha}{2}\sin(n-1)\frac{\alpha}{2}\csc\frac{\alpha}{2},$$

equivalent to

$$\cot(2k+1)\frac{\pi}{2n} = \cot(2k-1)\frac{\pi}{2n}, \tag{11}$$

and

$$\begin{aligned}&\cos(2k-1)\frac{\pi}{2}\sin(n-1)(2k-1)\frac{\pi}{2n}\csc(2k-1)\frac{\pi}{2n}\\ &\quad = \cos(2k+1)\frac{\pi}{2}\sin(n-1)(2k+1)\frac{\pi}{2n}\csc(2k+1)\frac{\pi}{2n}.\end{aligned} \tag{12}$$

Relation (11) is impossible while (12) is identically true since

$$\cos(2k-1)\frac{\pi}{2} = \cos(2k+1)\frac{\pi}{2} = 0.$$

Therefore, denoting by $\Re f(x_j)$ the real part of $f(x_j)$, we have

$$\Delta = \rho^n \prod_{j=1}^{n} \Re f(x_j) \neq 0,$$

and the ratios m_i/K can be determined.

We cannot stop at this point, however, and say that therefore an equation (10) exists since for such existence it is not sufficient that the m_i can be determined—every coefficient m_i must be different from zero since otherwise some absolute-value term would disappear and the order of the equation would thus be reduced. Geometrically it is clear that if a diagonal is needed, its absolute-value term must appear in the equation with a coefficient $m_i \neq 0$ for otherwise it would be impossible to treat this particular configuration (of polygon *and* diagonals) by semilinear methods.

Now Δ is not only a cyclic determinant but it is also a semi-sigma determinant, i.e.,

$$\sum_{j=1}^{n} a_{ij} = \sigma \text{ for } i = 1, 2, \cdots, n;$$

and by a certain theorem concerning such determinants,* $\Delta = \sigma\Delta'$ where Δ' is the value of one of the determinants (they are all equal) formed from Δ by replacing the elements of some one column by units. Combining these results we have, for the value of the coefficients m_i,

$$m_1 = m_2 = \cdots = m_n = K/\rho\sigma = K/\rho \cot \frac{\pi}{2n}\cdot$$

Setting $m_i = 1$ we get $K = \rho \cot(\pi/2n)$ and the equation (10), which is the equation of the regular $2n$-gon with center at $(0, 0)$ and one vertex at $(\rho, 0)$, becomes

$$\sum_{i=1}^{n} \left| \cos (i-1)\frac{\pi}{n} y - \sin (i-1)\frac{\pi}{n} x \right| = \rho \cot \frac{\pi}{2n}\cdot \tag{13}$$

Checking this for the square we set $n=2$ and get $|x| + |y| = 1$. For $n=3$ the equation of the regular hexagon turns out to be (Fig. 9)

$$|y| + |(y - \sqrt{3}x)/2| + |(-y - \sqrt{3}x)/2| = \rho\sqrt{3}.$$

Now equation (13) has an interesting and important geometric interpretation. Since the u_i are in normal form, the $\sum |u_i|$ represents the sum of the distances (taken positively) from a point on the perimeter of the polygon to the (diametral) diagonals. And this sum is a constant.† This fact can be proved very readily from simple geometric considerations and can then be made the basis of the derivation of equation (13). We proceed to do this for this geometric

* See the author's paper, *A Note on Determinants*, This MONTHLY, 39 (1932) pp. 589–593.

† This was known to Söderblom and is, no doubt, to be found in other places in the literature.

treatment throws light on the problem of possible extensions of the results obtained so far. Through a point in the plane, say the origin, let us draw n rays which divide 2π into $2n$ equal angles (a system of polar-coordinate lines with equal angular spacings) (Fig. 10). These lines shall constitute the diagonals of the polygon. The sum of the distances from every point P on a side AB of the polygon to the diagonals passing through the vertices A and B is a constant. For let $u_1=0$, $u_2=0$ be the equations, in normal form, of these two diagonals which form an isosceles triangle with AB. Then, since the sum of the distances to these diagonals is obviously the same for the points A and B, and since this sum is a linear function for all points in the sector AOB, it follows that for every point P on AB the sum is a constant and equal to the perpendicular distance from A to the diagonal passing through B. The same kind of argument would apply to all other pairs of diagonals which form isosceles triangles with AB extended. If n is even, there will be an integral number of such pairs; if n is odd, there will be a last diagonal, which is parallel to AB, and in either case it follows that S, the sum of the distances from every point P on the perimeter of a regular polygon of an even number of sides to the diametral diagonals, is a constant K, namely, one-half the sum of the projections of the vertices onto a diameter. Since any point not on the perimeter of this $2n$-gon would lie on another concentric (and parallel) $2n$-gon, and since K depends upon the size of the polygon, it follows that for no point P not on the polygon is $S=K$. From these facts we can write down equation (13) immediately.

Similarly theorems exist for a regular polygon of any number of sides where other sets of diagonals are used, i.e., the j-th set of diagonals: the set of every diagonal joining two vertices separated by j other vertices. In the case $j=0$, the theory yields the equation of all points interior to as well as on the boundary of the polygon.

The equation of the square can be so modified as to represent certain quadrilaterals. Thus the graph of $|x|/a+|y|/b=1$ is the parallelogram with vertices at $(\pm a, 0)$, $(0, \pm b)$; in each region this equation reduces to the intercept form of the straight line. For the general convex quadrilateral we have the following

THEOREM. *A semilinear equation of order two exists for every quadrilateral whose diagonals intersect in an interior point.*

With diagonals, vertices, etc., chosen as in Fig. 11, the equation of the quadrilateral is

$$\begin{aligned}\frac{\gamma-\alpha}{2\alpha\gamma}x+\frac{\gamma\delta(\alpha-\beta)-\alpha\beta(\gamma-\delta)}{2m\alpha\beta\gamma\delta}y-1+\frac{\beta+\delta}{2m\beta\delta}|y| \\ +\frac{\alpha+\gamma}{2m\alpha\gamma}|y-mx|=0.\end{aligned}\tag{14}$$

In case the diagonals are at right angles the equation takes the form

$$\frac{\gamma - \alpha}{2\alpha\gamma} x + \frac{\delta - \beta}{2\beta\delta} y - 1 + \frac{\beta + \delta}{2\beta\delta} | y | + \frac{\alpha + \gamma}{2\alpha\gamma} | x | = 0. \tag{15}$$

It is of interest to point out with Söderblom that if we set $u_0 = 0$ in equation (14), namely,

$$\frac{\gamma - \alpha}{2\alpha\gamma} x + \frac{\gamma\delta(\alpha - \beta) - \alpha\beta(\gamma - \delta)}{2m\alpha\beta\gamma\delta} y - 1 = 0, \tag{16}$$

we get the equation of the exterior diagonal of the quadrilateral. The same is true, of course, for u_0 in equation (16) where, if the figure is a parallelogram ($\alpha = \gamma, \beta = \delta$), we get simply $-1 = 0$ which indicates that the exterior diagonal is the line at infinity.

No semilinear equation of order two exists for a quadrilateral whose diagonals intersect in an exterior point. This is apparent at once (although Söderblom erred in this regard) since no two lines can be drawn which would isolate each of the four sides into separate regions of the plane. Likewise, and for other reasons, there is no semilinear equation of order less than six for the triangle. For a more general discussion of these and other more complicated configurations, see a paper by the author, *Semi-linear equations*, in The Tôhoku Mathematical Journal, vol. 44, Part I, August, 1935.

In passing we point out some directions of possible extensions of the theory of semilinear equations:

Absolute-value symbols might be piled up. Consider the equation $-1 + |x| + ||x| + y - 1| = 0$. Here the turning of the graph takes place not only on $x = 0$ and $x + y - 1 = 0$ but also on $-x + y - 1 = 0$. We note (Fig. 12) that these last two are merely half-lines and that the graph is a triangle. Figure 13 gives the graph of $-1 + |x| + ||x| + |y| - 1| = 0$.

Or equations of the form

$$P_{0q} + \sum_{i=1}^{n_1} m_{1i} | P_1 | + \sum_{i=1}^{n_2} m_{2i} | P_2 | + \cdots + \sum_{i=1}^{n_q} m_{qi} | P_q | = 0, \tag{17}$$

where P_{0q} is a polynomial of not more than the q-th degree and P_j is a polynomial of the j-th degree, might be considered. Figures 14 and 15 are respectively the graphs of $-1 + |x + y - 1| + |y + x^2| = 0$ and $-1 + |x + y - 1| + |x^2 + y^2 - 1| = 0$. In both examples it will be seen that the graph in one of the regions $(- +)$ is made up of two non-connected arcs. Functions other than polynomials might be used; if $f = 0$ is the equation of a simple closed curve, then $f \pm |f| = 0$ will represent the interior or exterior of the region bounded by $f = 0$.

The theory could be developed for n dimensions. For example $|x| + |x - 1| + |y| + |z| = 1$ is the equation of the unit segment in three-space from 0 to 1 on the x-axis. Again $\sum_{i=1}^{n} |x_i| = 1$ is the equation of the regular hyper 2^n-gon in n-space.

Systems of semilinear equations are of interest. Examine the system

$$-x+|x|+|y|=0, \tag{18}$$

$$x-\lambda+|x-\lambda|+|y|=0. \tag{19}$$

Equation (18) plots the positive x-axis while (19) represents the half-line (on x-axis) from $-\infty$ to λ. Adding we get

$$-\lambda+|x|+|x-\lambda|+2|y|=0, \tag{20}$$

which operation yields, in this case, the equation of the part common to (18) and (19). Equation (20) therefore plots the segment $(0, \lambda)$. We have thus found one semilinear equation which is equivalent to the given system and in this sense we have solved the system. For certain systems it is true that there is always one equation that is equivalent to the system.

The equation

$$|x|+|y|+|x+y-1|=1$$

is satisfied by every point within and on the boundary of the triangle whose vertices are $(0, 0)$, $(0, 1)$, $(1, 0)$, and by no others. This equation is equivalent to the system of linear inequalities

$$\begin{aligned} x &\geqq 0, \\ y &\geqq 0, \\ 1-x-y &\geqq 0. \end{aligned}$$

In a system of inequalities, the sign of equality must necessarily appear in each inequality if there is to be an equivalent semilinear equation for a point-set which satisfies the latter is always a closed set. Since there exists a semilinear equation representing all points interior to and on the boundary of an arbitrary convex polygon, this one equality (a very special semilinear equation) is geometrically equivalent to the system of linear inequalities which represents the same set of points.

A TYPE OF OSCILLATION WITHIN THE HELIUM ATOM

By J. F. THOMSON, Tulane University

Introduction. In a recent paper* Professor H. E. Buchanan works out the small oscillations of the electrons near the equilateral triangle positions, considering the electrical forces only. His article is one of a rather large number that have been published recently on the subject of the neutral helium atom.†

* American Mathematical Monthly, vol. 38 (1931), pp. 511–521.

† Rawles, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 618–630; Van Vleck, Bulletin of the National Research Council, vol. 10, part 4, (1926), Chapter 7; D. Buchanan, Transactions of the Royal Society of Canada, Series 3, vol. 23 (1929), pp. 227–245; U. Crudeli, Rendiconti Del Circolo Matematico Di Palermo, vol. 51 (1926), pp. 1–20; Langmuir, Physical Review, vol. 17 (1921), pp. 339–353.

This paper gives the variations of Professor Buchanan's solution when both gravitational and electrical forces are taken into account.

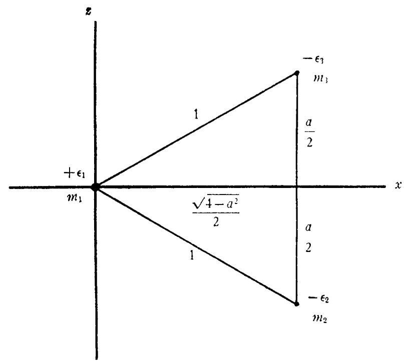


FIG. 1

Given the positive nucleus ϵ_1 with mass m_1 and the two electrons with masses m_2 and m_3 and charges $-\epsilon_2$ and $-\epsilon_3$ respectively, as shown in Fig. 1. Assuming Coulomb's Law for the forces due to electrical charges, and Newton's Law for the gravitational attraction, the relation $Ma = F$ gives the differential equation:

$$
\begin{aligned}
&m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \\
(1) \qquad &m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \\
&m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \qquad i = 1, 2, 3;
\end{aligned}
$$

where

$$U = \frac{k^2\epsilon_1\epsilon_2}{r_{12}} + \frac{k^2\epsilon_1\epsilon_3}{r_{13}} - \frac{k^2\epsilon_2\epsilon_3}{r_{23}} + \frac{K^2 m_1 m_2}{r_{12}} + \frac{K^2 m_2 m_3}{r_{23}} + \frac{K^2 m_3 m_1}{r_{31}},$$

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \qquad i, j = 1, 2, 3, \qquad j \neq i;$$

and where

k^2 = proportionality factor in Coulomb's Law,
K^2 = gravitational constant.

Let the motion of the bodies be referred to a new system of axes having the same origin as the old, and rotating in the $\xi\eta$-plane in the direction in which the electrons move with uniform angular velocity ω.

Substituting

$$
\begin{aligned}
x_i &= \xi_i \cos \omega t - \eta_i \sin \omega t, \\
y_i &= \xi_i \sin \omega t + \eta_i \cos \omega t, \qquad i = 1, 2, 3, \\
z_i &= \zeta_i,
\end{aligned}
$$

in (1), the equations become

$$
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} - \omega^2\xi_i - \frac{1}{m_i}\frac{\partial U}{\partial \xi_i} = 0,\\
(2)\qquad &\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} - \omega^2\eta_i - \frac{1}{m_i}\frac{\partial U}{\partial \eta_i} = 0, \qquad i = 1, 2, 3.\\
&\frac{d^2\zeta_i}{dt^2} - \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i} = 0,
\end{aligned}
$$

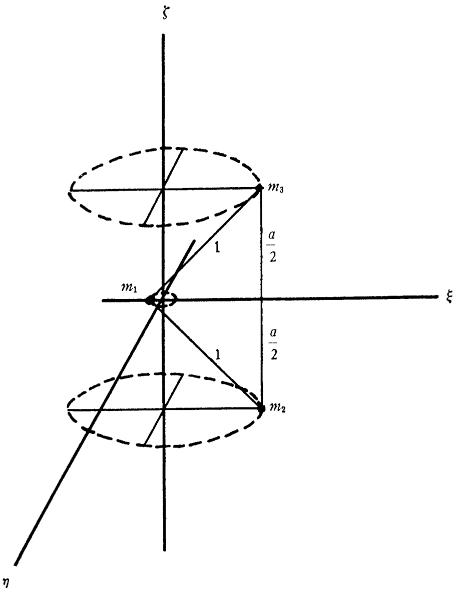


FIG. 2

Triangular Solutions. We wish a solution of equations (2) in which the bodies remain at the vertices of a triangle and rotate in circles, as shown in Fig. 2. ξ_i, η_i, ζ_i are constants and their derivatives are zero. We choose $r_{13}=r_{12}=1$, $r_{32}=a$. A solution will exist if (2) can be satisfied, that is, if we can have simultaneously

$$
\begin{aligned}
&-m_1\omega^2\xi_1 + k^2\epsilon_1\epsilon_2(\xi_1-\xi_2)\\
&+k^2\epsilon_1\epsilon_3(\xi_1-\xi_3) + K^2m_1m_2(\xi_1-\xi_2) + K^2m_1m_3(\xi_1-\xi_3) = 0,\\
&-m_2\omega^2\xi_2 + k^2\epsilon_1\epsilon_2(\xi_2-\xi_1)\\
(3)\qquad &-\frac{k^2}{a^3}\epsilon_2\epsilon_3(\xi_2-\xi_3) + K^2m_1m_2(\xi_2-\xi_1) + \frac{K^2}{a^3}m_2m_3(\xi_2-\xi_3) = 0,\\
&-m_3\omega^2\xi_3 + k^2\epsilon_1\epsilon_3(\xi_3-\xi_1)\\
&-\frac{k^2}{a^3}\epsilon_2\epsilon_3(\xi_3-\xi_2) + K^2m_1m_3(\xi_3-\xi_1) + \frac{K^2}{a^3}m_3m_2(\xi_3-\xi_2) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-m_1\omega^2 n_1 + k^2\epsilon_1\epsilon_2(\eta_1-\eta_2)\\
&+k^2\epsilon_1\epsilon_3(\eta_1-\eta_3)+K^2m_1m_2(\eta_1-\eta_2)+K^2m_1m_3(\eta_1-\eta_3)=0,\\
&-m_2\omega^2\eta_2+k^2\epsilon_1\epsilon_2(\eta_2-\eta_1)\\
(4)\qquad &-\frac{k^2\epsilon_2\epsilon_3}{a^3}(\eta_2-\eta_3)+K^2m_1m_2(\eta_2-\eta_1)+\frac{K^2m_2m_3}{a^3}(\eta_2-\eta_3)=0,\\
&-m_3\omega^2\eta_3+k^2\epsilon_1\epsilon_3(\eta_3-\eta_1)\\
&-\frac{k^2\epsilon_2\epsilon_3}{a^3}(\eta_3-\eta_2)+K^2m_1m_3(\eta_3-\eta_1)+\frac{K^2m_2m_3}{a^3}(\eta_3-\eta_2)=0,
\end{aligned}
$$

$$
\begin{aligned}
&k^2\epsilon_1\epsilon_2(\zeta_1-\zeta_2)+k^2\epsilon_1\epsilon_3(\zeta_1-\zeta_3)+K^2m_1m_2(\zeta_1-\zeta_2)+K^2m_1m_3(\zeta_1-\zeta_3)=0,\\
(5)\qquad &k^2\epsilon_2\epsilon_1(\zeta_2-\zeta_1)-\frac{k^2\epsilon_2\epsilon_3}{a^3}(\zeta_2-\zeta_3)+K^2m_2m_1(\zeta_2-\zeta_1)+\frac{K^2m_2m_3}{a^3}(\zeta_2-\zeta_3)=0,\\
&k^2\epsilon_3\epsilon_1(\zeta_3-\zeta_1)-\frac{k^2\epsilon_3\epsilon_2}{a^3}(\zeta_3-\zeta_2)+K^2m_3m_1(\zeta_3-\zeta_1)+\frac{K^2m_3m_2}{a^3}(\zeta_3-\zeta_2)=0.
\end{aligned}
$$

In order that (3), (4), and (5) have a solution other than trivial ones it is necessary that each of the three third order determinants composed respectively of the coefficients of $\xi_i, \eta_i, \zeta_i, i=1, 2, 3$, should vanish. The determinant of coefficients of ζ_i does not contain ω and vanishes identically. The determinant of coefficients of η_i is the same as that of the coefficients of ξ_i. We equate this determinant to zero, and let $m_2=m_3=m$, $2\epsilon=\epsilon_1=2\epsilon_2=2\epsilon_3$. After some reductions, $-\omega^2$ appears as a common factor of one of the rows. Removing it, the equation becomes

$$
\begin{vmatrix}
m_1 & -4k^2\epsilon^2-2m_1mK^2 & 0\\
m & -m\omega^2+2k^2\epsilon^2+m_1mK^2 & \dfrac{m\omega^2}{2}-k^2\left(\epsilon^2-\dfrac{\epsilon^2}{a^3}\right)-K^2\left(\dfrac{m_1m}{2}+\dfrac{m^2}{a^3}\right)\\
m & -m\omega^2+2k^2\epsilon^2+m_1mK^2 & -\dfrac{m\omega^2}{2}+k^2\left(\epsilon^2-\dfrac{\epsilon^2}{a^3}\right)+K^2\left(\dfrac{m_1m}{2}+\dfrac{m^2}{a^3}\right)
\end{vmatrix}=0
$$

which may be written as the product of two factors, and either one may be equal to zero. Thus

$$
(6a)\qquad \frac{m\omega^2}{2}-k^2\left(\epsilon^2-\frac{\epsilon^2}{a^3}\right)-K^2\left(\frac{m_1m}{2}+\frac{m^2}{a^3}\right)=0,
$$

or

$$
(6b)\qquad \begin{vmatrix}
m_1 & -4k^2\epsilon^2-2m_1mK^2 & 0\\
m & -m\omega^2+2k^2\epsilon^2+K^2m_1m & 1\\
m & -m\omega^2+2k^2\epsilon^2+K^2m_1m & 1
\end{vmatrix}=0.
$$

From the first factor

$$\omega = \pm \sqrt{\frac{2k^2\epsilon^2}{m}\left(1 - \frac{1}{a^3}\right) + 2K^2\left(\frac{m_1}{2} + \frac{m}{a^3}\right)}. \tag{7}$$

This will be a solution of the problem, i.e. a possible value of the velocity of rotation, as long as the expression inside the radical is positive. The plus or minus sign in front of the radical indicates rotation in one or the other direction; clockwise or counter-clockwise. (If $K=0$ and $a=1$, this reduces to zero, one of the values found by Professor Buchanan.)

Hence

$$\frac{2k^2\epsilon^2}{m}(a^3 - 1) + K^2(m_1a^3 + 2m) \leqq 0$$

$$\left(\frac{2k^2\epsilon^2}{m} + m_1K^2\right)a^3 > \frac{2k^2\epsilon^2}{m} - 2K^2m$$

$$a^3 \geqq \frac{2k^2\epsilon^2 - 2K^2m^2}{2k^2\epsilon^2 + K^2mm_1}. \tag{7a}$$

Now
$$\epsilon = 4.77 \times 10^{-10} \text{ e.s.u.}$$
$$k = 1$$
$$K = 6.69 \times 10^{-8}$$
$$m = 8.99 \times 10^{-28} \text{ g.}$$
$$m_1 = 6.62 \times 10^{-24} \text{ g.}$$

Substituting these values in the right-hand side of the inequality it is seen that the second member of both numerator and denominator has practially no influence on the value of the fraction. The right-hand side is therefore approximately unity. The conclusion is that a must be greater than a value slightly less than one, otherwise the expression inside the radical in (7) will be negative, making the velocity imaginary, and hence there would be no solution.

Now consider the second factor, namely, (6b). Add the third row to the second, then expand the determinant and solve for ω. We have

$$\omega = \pm \sqrt{\frac{(2k^2\epsilon^2 + m_1mK^2)(m_1 + 2m)}{mm_1}} \tag{8}$$

which, if K is put equal to zero, reduces to the value of ω found by Professor Buchanan.

We consider first the variations from the equilateral triangle positions due to the gravitational forces. If in (3) we put $K^2=0$ and $a=1$, ξ_1 and ξ_2 can be found in terms of ξ_3. There results ξ_2 ξ_3, $\xi_1 = -2m\xi_3/m_1$. The values of η_1, η_2

and η_3 could be found in the same way from equations (4), but it would be easier to satisfy (4) by $\eta_1=\eta_2=\eta_3=0$. Equations (5) are satisfied by $\zeta_1=0$, $\zeta_2=-\zeta_3=\frac{1}{2}$. The motion is then as shown in the second figure. The equilateral triangle is in the $\xi\zeta$-plane, and rotates about the ζ axis. The electrons and the nucleus move in circles whose planes are parallel to the $\xi\eta$-plane.

The equations for small oscillations. In (2) let

$$\begin{aligned} \xi_i &= \xi_i^{(0)} + x_i, \\ \eta_i &= y_i, \qquad\qquad i = 1, 2, 3, \\ \zeta_i &= \zeta_i^{(0)} + z_i, \end{aligned}$$

where x_i, y_i, and z_i represent the variations from the equilateral triangle solutions $\xi_i^{(0)}$, 0, $\zeta_i^{(0)}$. By means of the center of gravity equations

$$m_1x_1 + m(x_2 + x_3) = 0, \qquad m_1y_1 + m(y_2 + y_3), \qquad m_1z_1 + m(z_2 + z_3) = 0,$$

x_2, y_2, and z_2 are eliminated. Equations (2) then become

$$\begin{aligned}
&\frac{d^2x_1}{dt^2} - 2\omega\frac{dy_1}{dt} = \omega^2(\xi_1^{(0)} + x_1) \\
&\qquad - \frac{H}{m_1}\left(\frac{\xi_1^{(0)} + x_1 - \xi_2^{(0)} - x_2}{r_{12}^3} + \frac{\xi_1^{(0)} + x_1 - \xi_3^{(0)} - x_3}{r_{13}^3}\right) \\
&\frac{d^2x_3}{dt^2} - 2\omega\frac{dy_3}{dt} = \omega^2(\xi_3^{(0)} + x_3) \\
&\qquad - \frac{H}{m}\left(\frac{\xi_3^{(0)} + x_3 - \xi_1^{(0)} - x_1}{r_{31}^3}\right) + \frac{J}{m}\left(\frac{\xi_3^{(0)} + x_3 - \xi_2^{(0)} - x_2}{r_{32}^3}\right) \\
(9)\quad &\frac{d^2y_1}{dt^2} + 2\omega\frac{dx_1}{dt} = \omega^2y_1 - \frac{H}{m_1}\left(\frac{y_1 - y_2}{r_{12}^3} + \frac{y_1 - y_3}{r_{13}^3}\right) \\
&\frac{d^2y_3}{dt^2} + 2\omega\frac{dx_3}{dt} = \omega^2y_3 - \frac{H}{m}\left(\frac{y_3 - y_1}{r_{31}^3}\right) + \frac{J}{m}\left(\frac{y_3 - y_2}{r_{32}^3}\right) \\
&\qquad \frac{d^2z_1}{dt^2} = -\frac{H}{m_1}\left(\frac{z_1 - \zeta_2^{(0)} - z_2}{r_{12}^3} + \frac{z_1 - \zeta_3^{(0)} - z_3}{r_{13}^3}\right) \\
&\qquad \frac{d^2z_3}{dt^2} = -\frac{H}{m}\left(\frac{\zeta_3^{(0)} + z_3 - z_1}{r_{31}^3}\right) + \frac{J}{m}\left(\frac{\zeta_3^{(0)} + z_3 - \zeta_2^{(0)} - z_2}{r_{32}^3}\right)
\end{aligned}$$

where

$$H = 2k^2\epsilon^2 + m_1mK^2, \qquad J = k^2\epsilon^2 - m^2K^2.$$

We develop the right-hand side of (9) into power series in x_i, y_i, z_i, $i=1, 3$, by means of Taylor's Formula. Considering only the linear terms in these series we have, on expanding about the triangular points,

$$
(10)\quad
\begin{aligned}
\frac{d^2x_1}{dt^2} - 2\omega\frac{dy_1}{dt} &= A_{11}x_1 + C_{11}z_1 + C_{13}z_3\\
\frac{d^2x_3}{dt^2} - 2\omega\frac{dy_3}{dt} &= A_{21}x_1 + A_{23}x_3 + C_{21}z_1 + C_{23}z_3\\
\frac{d^2y_1}{dt^2} + 2\omega\frac{dx_1}{dt} &= 0\\
\frac{d^2y_3}{dt^2} + 2\omega\frac{dx_3}{dt} &= B_{41}y_1 + B_{43}y_3\\
\frac{d^2z_1}{dt^2} &= A_{51}x_1 + A_{53}x_3 + C_{51}z_1\\
\frac{d^2z_3}{dt^2} &= A_{61}x_1 + A_{63}x_3 + C_{61}z_1 + C_{63}z_3,
\end{aligned}
$$

where

$$A_{11} = \omega^2\left(3 - \frac{3a^2}{4}\right),\quad C_{11} = -\frac{H}{m}\,\frac{3a\sqrt{4-a^2}}{4},\quad C_{13} = -\frac{H}{m_1}\,\frac{3a\sqrt{4-a^2}}{2},$$

$$A_{21} = -\frac{H}{m}\left(2 - \frac{3a^2}{4}\right) - J\frac{m_1}{a^3m^2},\quad A_{23} = \omega^2 + \frac{H}{m}\left(\frac{8-3a^2}{4}\right) + \frac{2J}{ma^3},$$

$$C_{21} = -\frac{H}{m}\,\frac{3a\sqrt{4-a^2}}{4},\quad C_{23} = \frac{H}{m}\,\frac{3a\sqrt{4-a^2}}{4},\quad B_{41} = \frac{H}{m} + \frac{m_1J}{m^2a^3},$$

$$B_{43} = \omega_2 - \frac{H}{m} + \frac{2J}{ma^3},\quad A_{51} = -H\frac{3a}{4m}\sqrt{4-a^2},\quad A_{53} = -\frac{H}{m_1}\,\frac{3a}{2}\sqrt{4-a^2},$$

$$C_{51} = -\frac{H}{m_1}\,\frac{4-3a^2}{4}\left(\frac{2m+m_1}{m}\right),\quad A_{61} = -\frac{H}{m}\,\frac{3a}{4}\sqrt{4-a^2},$$

$$A_{63} = \frac{H}{m}\,\frac{3a}{4}\sqrt{4-a^2},\quad C_{61} = \frac{H}{m}\left(1 - \frac{3a^2}{4}\right) - \frac{2m_1J}{a^3m^2},$$

$$C_{63} = -\frac{H}{m}\left(1 - \frac{3a^2}{4}\right) - \frac{4J}{a^3m}.$$

The characteristic equations. In (10) let

$$x_i = K_ie^{\lambda t},\qquad y_i = L_ie^{\lambda t},\qquad z_i = M_ie^{\lambda t},\qquad i = 1, 3,$$

to obtain the solution of the system for the values of ω given in (7) and (8). This substitution gives six linear equations in K_i, L_i, M_i with the common factor $e^{\lambda t}$, which may be divided out. In order that there be a solution other than $K_i = L_i = M_i = 0$, $i = 1, 3$, the determinant formed of the coefficients of K_i, L_i,

M_i in these equations should equal zero. This determinant is of the sixth order in λ^2. It may be simplified by dividing out λ from the third row, and making the following substitutions

$$\omega^2 \text{ as given in (8)}$$

$$\lambda = \sqrt{\frac{H}{2}}\, x, \qquad s = \frac{2J}{mH}.$$

After some reductions x may be divided out, leaving a fifth order determinant in x^2. The value of a is set equal to 1, the unit of mass is chosen so that $m=1$, and $1/m_1$ set equal to y. The determinant is then expanded to give the following equation in x

$$\begin{aligned}
&x^{10} + 8(2y+1)x^8 + \left(6y^2 + \frac{105}{2}y + 6 - 108y^3 + 48ys + 21s - 12s^2\right)x^6 \\
&+ \left(604y^3 + 324y^2 - 8y + \frac{19}{2} + 432y^4 + 610sy^2 + 386sy + \frac{957s}{2} - 10s^2\right. \\
&\left. + 16s^3 + 216y^3s - 32s^2y\right)x^4 + \left(112y^4 - 48y^3 - 424y^2 - 298y - 56\right. \\
&\left. + 904sy^3 + 541sy^2 + \frac{413}{2}sy + 81s + 51s^2 + 64s^3 + 102s^2y + 128s^3y\right)x^2 \\
&+ (28s^3 + 242s^2 - 36s - 160y^4 - 988y^3 - 868y^2 - 207y - 428sy \\
&+ 1080s^2y + 112s^3y + 1416s^2y^2 + 2256y^2s + 2736sy^3 + 448sy^4 \\
&+ 448s^2y^3 + 112y^2s^3) = 0.
\end{aligned} \tag{11}$$

If now we put $K=0$, the parameter s reduces to 1 and the above equation reduces to a much simpler equation of the tenth degree, agreeing with the one found by Professor Buchanan. He found two roots of this equation to be $x = \pm iu$, so dividing thru by the factor x^2+u^2, he secured the following eighth degree equation, where y has the value .00014,

$$\begin{aligned}
&x^8 + 6(2y+1)x^6 + 3(13y^2 + 22y + 1)x^4 + (2y+1)^2(7y+62)x^2 \\
&\quad + 18(2y+1)^2(y+5) = 0.
\end{aligned}$$

He found the following for roots of this equation

$$x^2 = .99965\cdots - i3.00063\cdots, \qquad x^2 = .99965\cdots + i3.00063\cdots,$$
$$x^2 = -1.35382\cdots, \qquad x^2 = -6.64717\cdots.$$

The computed value of $s = -.00027$. We can get a solution of (11) in the form $x^2 = \phi + \psi s + \cdots$, and since s is very small we need take only the linear terms. Substituting this value of x^2 in equation (11) and collecting the first degree terms in s we find

$$\psi = \frac{-21\phi^3 - \frac{957}{2}\phi + 36}{5\phi^4 + 64y\phi^2 + 32\phi^3 + \frac{315}{2}y\phi^2 + 18\phi^2 - 8y + \frac{19}{2}}$$

where ϕ is the value of x^2 when $s=0$, already found above.

Therefore the approximate roots of (11) are as follows:

$$x^2 = s3.79\cdots, \qquad x^2 = -2.00056 - s7.19\cdots,$$
$$x^2 = .99965\cdots - i3.00063\cdots + (.759\cdots - i.251\cdots)s,$$
$$x^2 = .99965\cdots + i3.00063\cdots + (.759\cdots + i.251\cdots)s,$$
$$x^2 = -1.35382\cdots + s36.6\cdots, \qquad x^2 = -6.64717\cdots + s8.03\cdots.$$

The characteristic exponents. Since s is very small, these values differ but little from the ones found by Professor Buchanan. When the numerical value of s is substituted and the square root taken, the form of the roots is as follows:

$$x = \pm a, \qquad x = \pm ib, \qquad x = \pm(c+id), \qquad x = \pm(c-id)$$
$$x = \pm gi, \qquad x = \pm hi$$

where the numerical values of a, b, c, d, g, and h can be easily computed. Hence the values of λ which satisfy the original equation are not completely known. Writing ρ for $\sqrt{H/2}$, they are

$$\pm a\rho, \qquad \pm ib\rho, \qquad \pm(c+id)\rho,$$
$$\pm(c-id)\rho, \qquad \pm ig\rho, \qquad \pm ih\rho.$$

The solutions of equations (10). Substitute the above values of λ in the characteristic equations

$$x_i = K_i e^{\lambda t}, \qquad y_i = L_i e^{\lambda t}, \qquad z_i = M_i e^{\lambda t}, \qquad i = 1, 3$$

to obtain the general solutions of equations (10) as follows:

$$\begin{aligned}
x_i &= K_{i1}e^{a\rho t} + K_{i2}e^{-a\rho t} + K_{i3}e^{ib\rho t} + K_{i4}e^{-ib\rho t} + K_{i5}e^{ig\rho t} + K_{i6}e^{-ig\rho t}\\
&\quad + K_{i7}e^{ih\rho t} + K_{i8}e^{-ih\rho t} + e^{c\rho t}(K_{i9}e^{id\rho t} + K_{i10}e^{-id\rho t})\\
&\qquad + e^{-c\rho t}(K_{i11}e^{id\rho t} + K_{i12}e^{-id\rho t})\\
y_i &= L_{i1}e^{a\rho t} + L_{i2}e^{-a\rho t} + L_{i3}e^{ib\rho t} + L_{i4}e^{-ib\rho t} + L_{i5}e^{ig\rho t} + L_{i6}e^{-ig\rho t}\\
&\quad + L_{i7}e^{ih\rho t} + L_{i8}e^{-ih\rho t} + e^{c\rho t}(L_{i9}e^{id\rho t} + L_{i10}e^{-id\rho t})\\
&\qquad + e^{-c\rho t}(L_{i11}e^{id\rho t} + L_{i12}e^{-id\rho t})\\
z_i &= M_{i1}e^{a\rho t} + M_{i2}e^{-a\rho t} + M_{i3}e^{ib\rho t} + M_{i4}e^{-ib\rho t} + M_{i5}e^{ig\rho t} + M_{i6}e^{-ig\rho t}\\
&\quad + M_{i7}e^{ih\rho t} + M_{i8}e^{-ih\rho t} + e^{c\rho t}(M_{i9}e^{id\rho t} + M_{i10}e^{-id\rho t})\\
&\qquad + e^{-c\rho t}(M_{i11}e^{id\rho t} + M_{i12}e^{-id\rho t})
\end{aligned} \tag{12}$$

where the subscript $i=1, 3$.

These solutions consist of the non-periodic terms $e^{\pm apt}$, the non-periodic vibrations $e^{(c\pm id)pt}$, the non-periodic damped vibrations $e^{-(c\pm id)pt}$ and the periodic terms $e^{\pm ibpt}$, $e^{\pm igpt}$, $e^{\pm ihpt}$. So that if the system is disturbed in any way it may have periodic oscillations, but most likely will break up, because the coefficients of the non-periodic terms are such that these terms fail to cancel out.

These solutions give the variations from those of Professor Buchanan when the gravitational forces are taken into consideration. The periods of the oscillating terms are $2\pi/bp$, $2\pi/gp$, $2\pi/hp$.

The solutions (12) contain 72 arbitrary constants. Twelve of them can be chosen arbitrarily as constants of integration, and the others found as linear functions of these twelve.

Summary. The problem of the motion of the electrons of the helium atom as considered by Professor Buchanan has been extended to take care of gravitational forces as well as electrical forces. With some numerical changes, all of his conclusions are true in this case. An isosceles triangle solution could be found by taking $a\neq 1$, and the limitations on this solution could be considered, taking into account either gravitational forces or electrical forces, or both simultaneously.

ON ISOGONAL POINTS

By J. H. WEAVER, Ohio State University

Let there be a triangle $A_1A_2A_3$ and a point P in its plane. The reflections of the lines A_iP in the bisectors of the angles A_i also meet in a point P' called the isogonal conjugate of P.* It is the object of this note to determine certain properties of these points when they are collinear with the circumcenter, O, of the triangle $A_1A_2A_3$.

Let the vertices A_i be represented by the complex numbers t_i, $|t_i|=1$. Let the point P be represented by the complex number z. Then without difficulty it may be shown that the complex number y, representing P' is given by

$$y=\frac{\bar{z}^2\sigma_3-\bar{z}\sigma_2+\sigma_1-z}{1-z\bar{z}}.$$

where the σ's are the elementary symmetric functions of the t_i, and $\bar{z}$ is the conjugate of z.

Since P, P' and O are to be collinear we must have

$$\begin{vmatrix} z & \bar{z} & 1 \\ y & \bar{y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0. \tag{1}$$

* Gallatley, *Modern Geometry of the Triangle*, p. 57, calls these points counterpoints; Morley, *Inversive Geometry*, p. 196, calls them a focal pair. In this connection see also Encyklopädie der Math. Wissenschaften, Band III, Heft 7, p. 1207.

If we substitute in (1) the values of y and $\bar{y}$ in terms of z and $\bar{z}$ we obtain

$$z^3 - z^2\sigma_1 + z\sigma_2 = \bar{z}^3\sigma_3^2 - \bar{z}^2\sigma_2\sigma_3 + \bar{z}\sigma_1\sigma_3. \tag{2}$$

Hence the locus of points such that a point and its isogonal conjugate are collinear with the circumcenter of the given triangle is the cubic (2).*

A solution of (2) and the equation $z\bar{z}=1$ shows that the cubic (2) cuts the circumcircle of the triangle in the six points,

$$z = t_1,\ t_2,\ t_3,\ -\sqrt[3]{\sigma_3},\ -\omega\sqrt[3]{\sigma_3},\ -\omega^2\sqrt[3]{\sigma_3},$$

(ω a complex cube root of unity); that is in the three vertices A_i and in three other points Q_i which determine an equilateral triangle.

The complex number representing
(a) the circumcenter is $z=0$,
(b) the orthocenter, $z=\sigma_1$,
(c) the incenter and excenters, $z=\pm\sqrt{t_1t_2}\pm\sqrt{t_1t_3}\pm\sqrt{t_2t_3}$, with proper choice of square roots in each case,
(d) the points† where the lines A_iO cut the sides A_jA_k,

$$z = \frac{t_i^2(t_j+t_k)}{t_i^2+t_jt_k}.$$

A substitution of the above values of z together with their conjugates in (2) shows that the corresponding points also lie on the cubic.

If we write the equation

$$x = m\bar{x} + n \tag{3}$$

and determine m and n so that this line will be an asymptote of (2) we obtain

$$x = \omega_i\sigma_3^{2/3}\bar{x} + 1/3\left(\sigma_1 - \frac{\omega_i\sigma_2}{\sigma_3^{1/3}}\right), \tag{4}$$

where ω_i is any one of the cube roots of unity.

The three lines (4) all meet in the point given by $x=\sigma_1/3$. Hence *the three asymptotes of the cubic* (2) *meet at the centroid of the triangle.* Since the coefficient of $\bar{x}$ in (4) is the clinant of the line and determines its direction, we see that the three lines (4) make angles of 60° with each other.‡

Now in (2)

$$\frac{dz}{d\bar{z}} = \frac{3\sigma_3^2\bar{z}^2 - 2\sigma_2\sigma_3\bar{z} + \sigma_1\sigma_3}{3z^2 - 2\sigma_1 z + \sigma_2}, \tag{5}$$

which is the clinant of the tangent to the cubic (2) at any point z on the curve. The equation of the Euler line is $z=\sigma_1\sigma_3\bar{z}/\sigma_2$. Hence its clinant is $\sigma_1\sigma_3/\sigma_2$, which

* This cubic is called M'Cay's Cubic. See Gallatley, loc. cit., p. 80.

† J. H. Weaver, *Curves Determined by a One-parameter Family of Triangles*, American Mathematical Monthly, vol. 40, 1933, p. 85. Also Morley, *Inversive Geometry*, p. 193.

‡ See Morley, *Inversive Geometry*, p. 153.

is the value of the clinant (5) when $z=0$. Therefore *the cubic* (2) *is tangent to the Euler line at the circumcenter of the triangle.*

At the points t_i the clinant (5) becomes $dz/d\bar{z}=t_jt_k$, which is the clinant of the altitude on the side A_jA_k. Hence *the altitudes of the triangle are tangent to the cubic* (2) *at the vertices.*

Any line through the circumcenter has the equation

$$z = t^2\bar{z}. \tag{6}$$

A substitution of the value of z from (6) in (2) gives

$$\bar{z}^3t^6 - \bar{z}^2t^4\sigma_1 + \bar{z}t^2\sigma_2 = \bar{z}^3\sigma_3^2 - \bar{z}^2\sigma_2\sigma_3 + \bar{z}\sigma_1\sigma_3. \tag{7}$$

If the two roots of (7) other than $\bar{z}=0$ are equal we must have

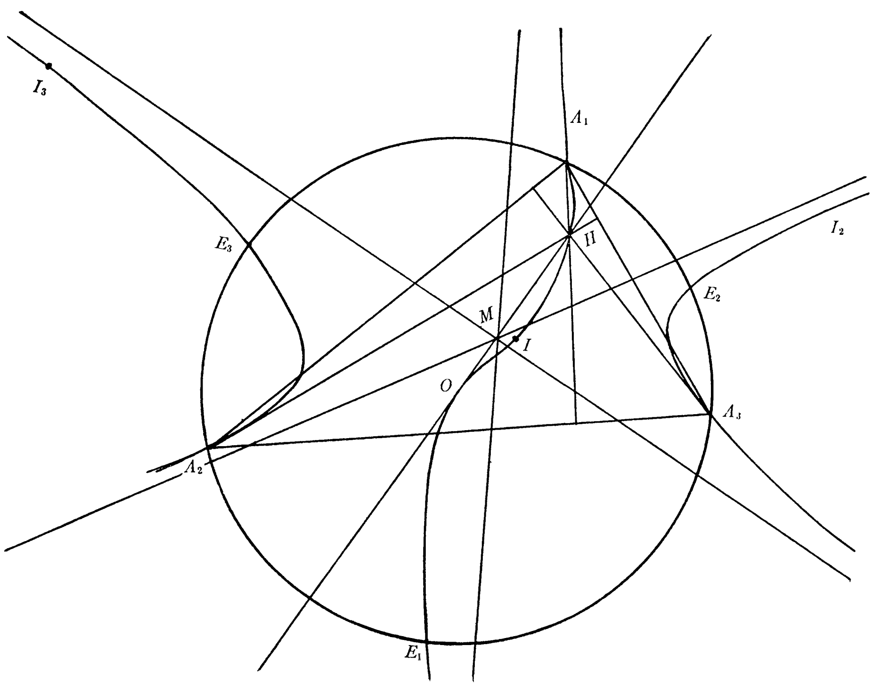
$$(t^4\sigma_1 - \sigma_2\sigma_3)^2 - 4(t^6 - \sigma_3^2)(t^2\sigma_2 - \sigma_1\sigma_3) = 0.$$

Hence there are four values for t^2 such that a point and its isogonal conjugate coincide. These values of t^2 determine the incenter and the excenters. We therefore conclude that *the four tangents to the cubic* (2) *at the incenter and the excenters are concurrent at the circumcircle of the triangle.*

The above contact properties are believed to be new.

If the triangle $A_1A_2A_3$ is isosceles, the cubic (2) degenerates into a hyperbola and its conjugate axis, and if equilateral, into three straight lines.

The figure shows the graph of the cubic (2).



Since isogonal conjugate points have a common pedal circle, whose center is midway between the points, the complex number representing this center is given by the half sum of the two roots of (7) which are different from zero, that is, by

$$\bar{z} = \frac{t^4\sigma_1 - \sigma_2\sigma_3}{2(t^6 - \sigma_3^2)}. \tag{8}$$

If we write the conjugate of (8) and eliminate t we obtain

$$2(z^3 - \sigma_3^2\bar{z}^3) = z^2\sigma_1 - \bar{z}^2\sigma_2\sigma_3. \tag{9}$$

Hence the locus of the centers of the common pedal circles of isogonal conjugate points which are collinear with the circumcenter is the cubic (9).

If we solve (6) and (9) simultaneously for z we obtain

$$z = 0 \text{ twice}, \quad \text{and } z = \frac{t^2(t^4\sigma_1 - \sigma_2\sigma_3)}{2(t^6 - \sigma_3^2)}.$$

If the line (6) cuts the cubic (9) three times at the circumcenter we must have $t^4\sigma_1 = \sigma_2\sigma_3$, or $t^2 = \pm\sqrt{\sigma_2\sigma_3/\sigma_1}$. This shows that the cubic (9) has a double point at the circumcenter and that the two branches intersect at right angles

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SERIES USEFUL IN THE COMPUTATION OF π

By J. S. FRAME, Brown University

One of the standards ways of computing π is based on Machin's formula:

$$\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \tag{1}$$

and the series expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{2}$$

W. Shanks used precisely this in computing π to 707 decimal places. In applying this series to the case $x = 1/5$, the individual terms are easily computed as decimals, and the series converges rapidly enough so that 35 terms suffice for 50-place accuracy. When we set $x = 1/239$, however, the individual terms, involving powers of $1/239$, are not easily expressed as decimals, so that compu-

tation beyond 15 decimals is laborious despite the rapid convergence. If, however, we expand the terms in powers of $1/240$, we obtain a new series which converges rapidly, and whose terms are easier to compute as decimals. The result is expressed by the formula:

$$\begin{aligned}\tan^{-1}\frac{t}{1-t} &= \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{4}\right)\frac{(t\sqrt{2})^n}{n} \\ &= \frac{t}{1}+\frac{2t^2}{2}+\frac{2t^3}{3}+0-\frac{4t^5}{5}-\frac{8t^6}{6}-\frac{8t^7}{7}-0+\cdots.\end{aligned} \tag{3}$$

The terms are alternately positive and negative in groups of three, so the error in breaking off the series is less in absolute value than the first group omitted. The series converges for $|t| < 1/\sqrt{2}$. Setting $t = 1/240$, we obtain the series

$$\begin{aligned}\tan^{-1}\frac{1}{239} = \frac{1}{240} &+ \frac{2}{2}\left(\frac{1}{240}\right)^2+\frac{2}{3}\left(\frac{1}{240}\right)^3+0 \\ &-\frac{4}{5}\left(\frac{1}{240}\right)^5-\frac{8}{6}\left(\frac{1}{240}\right)^6-\frac{8}{7}\left(\frac{1}{240}\right)^7+\cdots.\end{aligned}$$

The computation is conveniently arranged as follows: Divide 1 by 240, this by 120, this in turn by 240, and so on alternately. This takes care of the numerators automatically. It remains only to divide each term by the corresponding exponent, and add and subtract appropriate terms. Sixteen terms give 50-place accuracy.

The proof of formula (3) is a special case of the following: Let

$$x = \frac{at+b}{ct+d}; \qquad z = re^{i\theta} = \frac{ia-c}{-ib+d}.$$

Then

$$\begin{aligned}2i\tan^{-1}x &= \log\frac{1+ix}{1-ix} = \log\frac{(ct+d)+i(at+b)}{(ct+d)-i(at+b)} \\ &= \log\frac{(c+ia)t+(d+ib)}{(c-ia)t+(d-ib)} = \log\frac{1+\dfrac{c+ia}{d+ib}t}{1+\dfrac{c-ia}{d-ib}t}+\log\frac{d+ib}{d-ib},\end{aligned}$$

$$\tan^{-1}\frac{at+b}{ct+d}-\tan^{-1}\frac{b}{d} = \frac{1}{2i}\log\frac{1-\bar{z}t}{1-zt} = \sum_{n=1}^{\infty}\frac{z^n-\bar{z}^n}{2i}\,\frac{t^n}{n} = \sum_{n=1}^{\infty}(r^n\sin n\theta)\frac{t^n}{n}.$$

This series converges for $|t| < 1/r$, but it is useful for computation only when the values of $r^n \sin n\theta$ are convenient rational quantities. If $z = i$, we have $x = t$,

and obtain the series (2). The other case of interest is $z=1+i$, $x=t/(1-t)$, which leads to formula (3), and can be applied to the computation of π as discussed above. This same series (3) can be used to advantage in computing $\tan^{-1} 1/239$ by means of the formula

$$(4) \qquad \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{41} - 2\tan^{-1}\frac{1}{99}.$$

A Note on the Conics

By L. S. Johnston, University of Detroit

For each of the three conics $x^2/a^2+y^2/b^2=1$, $x^2/a^2-y^2/b^2=1$, and $y^2=4px$ consider the following set of points:

F, the focus
A, the corresponding intersection of the conic with the X axis
P, a corresponding intersection of the conic with its latus rectum
T, the foot of the tangent to the conic at P
N, the foot of the normal to the conic at P
C, the center of curvature of the conic for the point A, taking, for definiteness, the focus with positive abscissa. It is quite easily shown, though not generally noticed, apparently, that for the central conics the abscissas of the points T, A, F, C, and N respectively are in geometric progression with the eccentricity of the conic as the common ratio, and that for the parabola the abscissas of the same five points are in arithmetic progression with p as the common difference.

A Proof of the Fundamental Theorem of Algebra

By R. P. Boas, Jr., Harvard University

This note gives a proof, believed to be new, of the fundamental theorem of algebra; it is obtained by the use of the classical theorem of Picard: If there are two distinct values which a given entire function never assumes, the function is a constant. The proof is extremely simple and may be of interest as an application of Picard's theorem.

The fundamental theorem of algebra may be formulated as follows: An arbitrary polynomial,

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n,$$

where n is an integer >0, and the a_i are constants, has at least one zero (in the complex plane). We shall use in addition to Picard's theorem only the facts that $f(z)$ is an entire function—hence, in particular, continuous—and that $f(z)$ has a pole at infinity.

The proof is indirect. Suppose that $f(z)$ is never zero. I say then that $f(z)$ also fails to take on one of the values $1/k$ $(k=1, 2, \cdots)$. In fact, suppose that there are points z_k such that $f(z_k)=1/k$ $(k=1, 2, \cdots)$. Since $f(z)$ has a pole at

infinity, $|f(z)|>1$ uniformly outside some circle C. The points z_k all lie within C, and hence have at least one limit point Z within C. Since $f(z)$ is continuous,

$$f(Z) = \lim_{z_k \to Z} f(z_k) = 0.$$

This contradiction allows us to conclude that for some integer k, $f(z)$ fails to take on the value $1/k$. By Picard's theorem, $f(z)$, never assuming the distinct values 0 and $1/k$, must be constant, contrary to the hypothesis that the degree of $f(z)$ was at least 1. This contradiction shows that $f(z)$ must have at least one zero, and the proof is complete.

Note on a Quadratic Diophantine Equation

By E. T. Browne, University of North Carolina

In his *Introduction to the Theory of Numbers*, pp. 44–48, Dickson gives a treatment of the Diophantine equation

$$ax^2 + bxy + cy^2 = ez^2. \tag{1}$$

He first derives formulas for all *rational* solutions and expresses his results in the following theorem:

If a, b, c, e are integers such that $e \neq 0$ and $d = b^2 - 4ac$ is not the square of an integer, and if ξ, η, ζ are given rational solutions of (1), then *all* rational solutions of (1) are given by

$$x = \rho r,\ y = \rho s,\ z = \rho t, \tag{2}$$

$$\begin{cases} r = -(a\xi + b\eta)u^2 - 2c\eta uv + c\xi v^2 \\ s = a\eta u^2 - 2a\xi uv - (b\xi + c\eta)v^2 \\ t = \zeta T,\ T = au^2 + buv + cv^2, \end{cases} \tag{3}$$

where u and v are relatively prime integers and ρ is rational.

Dickson then supposes that ξ, η, ζ are *integers* satisfying (1) and he proceeds to find what restrictions must be placed on the choice of the parameters u, v, ρ so that the formulas (2), (3) may give only *integral* solutions and *all* integral solutions of (1).

It is the purpose of this note to derive, by a method somewhat different from Dickson's, the conditions on u, v and ρ.

Let $\rho = N/k$ $(k>0)$ be a fraction in its lowest terms. Obviously, then, for integral values of u, v and k, the numbers x, y and z given by (2) and (3) will be integers if, and only if, u, v and k satisfy the congruences:

$$\left.\begin{aligned} &-(a\xi + b\eta)u^2 - 2c\eta uv + c\xi v^2 \equiv 0 &(4)\\ &a\eta u^2 - 2a\xi uv - (b\xi + c\eta)v^2 \equiv 0 &(5)\\ &\zeta(au^2 + buv + cv^2) \equiv 0 &(6)\end{aligned}\right\} \pmod{k}.$$

Let us denote by Δ the determinant of the coefficients of u^2, uv, v^2 in (4), (5) and (6). If then A_1, A_2, A_3 denote the cofactors of the elements in the first column of Δ, we have on multiplying the congruences through by the A's and adding,

$$\Delta u^2 \equiv 0 \pmod{k}.$$

Similarly, if C_1, C_2, C_3 denote the cofactors of the elements in the last column of Δ, we have on multiplying through by the C's and adding,

$$\Delta v^2 \equiv 0 \pmod{k}.$$

Since u and v are relatively prime, it follows at once that

$$\Delta \equiv 0 \pmod{k}.$$

But on expanding we find that

$$\Delta = -\zeta(b^2 - 4ac)(a\xi^2 + b\xi\eta + c\eta^2) = -de\zeta^3, \tag{7}$$

whence

$$de\zeta^3 \equiv 0 \pmod{k}. \tag{8}$$

If now g is the greatest common divisor of $k = gD$ and $\zeta = gZ$, so that D is relatively prime to Z, it follows that D divides deg^2.

If b is even, say $b = 2B$, the second column of Δ consists entirely of even integers. Hence the A's and the C's are all even integers. We may therefore replace the multipliers above by their halves so that instead of (7) and (8), we have

$$\tfrac{1}{2}(b^2 - 4ac)e\zeta^3 = 2(B^2 - ac)e\zeta^3 \equiv 0 \pmod{k}. \tag{9}$$

More generally, it should be obvious that any common factor of $2c\eta$, $2a\xi$ and $b\zeta$ can be removed from the left-hand member of (8).

These are the results arrived at by Dickson by a somewhat different method.

It is clear then that only a finite number of integers k need be considered. If k is an integer satisfying (8), or (9), we have now only to follow the method outlined by Dickson for finding relatively prime integers u, v satisfying (4), (5) and (6). Thus we finally obtain a finite number of formulas which together yield all integral solutions of (1).

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

Differential and Integral Calculus. By C. E. Love. Third edition. New York, The Macmillan Company, 1934. xv+383 pages. $2.75.

The author states in the preface: "The aim is still, as in the past, to present the subject in such a way as to reward the student who strives for real understanding as distinguished from the one who works by rote." In this revision

evidently emphasis has also been placed on making the text itself more complete and more helpful to the student.

In the early part of the book, a more forceful introduction of the derivative is achieved by replacing the somewhat scattered treatment of the former text by a single chapter which includes the definition of dy/dx and a detailed discussion of its interpretations. A chapter on applications, immediately following differentiation of algebraic functions, deals not only with geometric topics, as formerly, but also with time rates, so that early in the course the student is able to solve a variety of exercises, requiring a knowledge of significance of results as well as of algebraic methods. The text has been amplified for other topics largely through the use of illustrative problems. These have been revised and increased in number and are presented so as to form a valuable supplement to the discussion. It may be said in criticism, however, that the explanatory material could have been extended to greater advantage in some cases. The introductory chapter on functions, limits and continuity still seems inadequate and, in later work, efforts to economize time and space result frequently in statements of necessary theorems with partial proofs or without any proof, a fact which is apt to leave the critical student dissatisfied with the treatment. Another criticism which the reviewer found in teaching from the earlier text, and one which remains, is that all definitions and rules for procedure are not given in general form. As an example, the derivative is defined only in symbolic form and only for $y=f(x)$.

The book seems particularly commendable for its problem material. The large number of exercises has been arranged so that, in general, even and odd numbered problems are alike in type. Throughout the text, each set applies to allied topics already studied as well as to the subject under consideration, making reviews unnecessary.

Changes in content are minor ones. Some topics of physics, namely, work, kinetic energy, impulse, momentum and the simple pendulum have been omitted. In the study of tangents to curves, the subtangent, subnormal, and tangent and normal lengths have been introduced, without, however, developing the general formula for each. Other additions include approximate solutions of equations by means of the differential and the tracing of a curve by means of its intersections with lines through the origin. In the later work, a major change of order is the postponement of chapters on indeterminate forms and on curve tracing. These are placed after integration. The opportunity for added applications of the definite integral is well used, but it is a question whether this arrangement offers sufficient time for a topic which requires practice over an extended period, and whether the results should not be available earlier in the course.

The new text seems a good one for a first course and should require less supplementing than did the former one. It does give the student an opportunity to acquire ability in the use of the subject.

HELEN K. KUTMAN

A First Course in Calculus. By E. S. Crawley and P. A. Caris. New York, F. S. Crofts and Company, 1933. x+342 pages. \$3.25.

This text is properly named a first course. Two thirds of the book is devoted to the differential calculus, and no introduction to differential equations is given. In each part of the work, the authors have developed all formulae first, these being accompanied by appropriate drill exercises. The applications, making up about half of the text, follow this formal work in each part. The problems are abundant and varied. The uses of series expansion are lightly treated, and the last chapter is devoted to mechanical integration by rules and instruments.

A. L. UNDERHILL

Analytic Geometry. By R. W. Brink. New York, Appleton-Century Company, 1935. xiv+350 pages. \$2.90.

This text is a very thorough revision and expansion of an earlier edition which was reviewed in this MONTHLY.* New figures, better paper and typography, and a large increase in the number of problems constitute the most noticeable changes. There has been added a well written chapter on *Curve Fitting* in which the methods of Average Points and Least Squares are used on curves of the linear, parabolic, exponential, or power types.

Some idea of the increase in the number of problems may be obtained by comparing the first four chapters in the two editions. In these chapters, which deal with coordinates and loci, the straight line, the locus of an equation, and locus problems, the number of problems has been increased from 286 to 806.

Topics such as limiting and degenerate forms of the conic, lines parallel to the y-axis, and others which are so frequently muddled are clearly and carefully treated by the author. For example it is correctly stated that, "Lines parallel to the y-axis are the only lines for which no slope is defined, for their inclination is 90°, and tan 90° is not defined." Too frequently such lines are said to have "infinite slope" and then follows some apologetic explanation of the anomalous term.

L. A. DYE

Trend Analysis of Statistics and Technique. By Max Sasuly. The Brookings Institution, Washington, D.C., 1934. xiii+421 pages. \$6.00.

This book marks a radical change in the policy of the Brookings Institution, which heretofore has confined its publications to materials which lie strictly within the fields of the social sciences. The present work lies on the borderline between the social sciences and mathematics. The social scientist will be interested primarily in the methods which are outlined here, and the mathematician will find his major interest in the mathematical treatment of the theory underlying these methods. The Brookings Institution is to be congratulated on its

* Volume 33, pp. 332 and 428 (1926).

decision to define the scope of its work broadly enough to include the present volume.

The aim of this book is, according to the preface, "to present the basis and the technique of trend analysis in such complete and dependable form as will serve the objective of ascertaining relevant facts and presenting them effectively." As a matter of fact, little space is devoted to the underlying philosophy of trend fitting. The relative merits of the free-hand trend as compared with trends of the more formal mathematical types is barely mentioned, and hardly enough attention is given to the difficult problem of determining the type of function which should be fitted to a given series. The student who is interested in these matters must look elsewhere for a satisfactory treatment.

But when we come to the actual techniques of curve fitting, and to the formal mathematical basis underlying them, the treatment is remarkable in its completeness, in its clarity, and in its organization. Dr. Sasuly has evidently given careful study to the literature of the field, and his work is copiously documented. He draws freely on the work of his predecessors and succeeds admirably in showing the relationships between seemingly diverse methods. Fortunately the discussion of methods is accompanied by a generous number of numerical illustrations, for the treatment is highly mathematical, and the lay reader would experience considerable difficulty in following the argument were the illustrations less numerous or the explanations less lucid.

The treatment of curve fitting is confined to the problem of fitting curves to time series. No attempt is made to apply the methods to the important problems of correlation. Yet the techniques are described in such general terms that the application of most of the methods to correlation problems will be obvious to the reader who is already familiar with the correlation concept.

The book begins with a treatment of the problem of fitting polynomials. Moving polynomial arcs are fitted, having the characteristics of both polynomial trends and of moving averages. The method is simple and flexible, and admirably adapted to the requirements of the social sciences. The author explains the less familiar methods of trend fitting by means of factorial moments, the parameters being discovered by means of successive summations. The power polynomials and other functions are resolved into orthogonal components, giving a simple yet general solution for the constants in the trend equations. The methods suggested appear to offer real advantages to the practicing statistician, especially in those cases in which he is dealing with equidistant data in time series analysis.

The book will get no small part of its value from the materials which have been summarized and tabulated both in the appendix and at various points throughout the text. The author has collected the more important formulae and also presents tables which should reduce considerably the routine arithmetical work involved in trend-fitting.

No social scientist can read this book without taking away a better under-

standing of the meaning and limitations of the methods which he is using, and no mathematician can read the book without comprehending to a greater degree the peculiar problems faced in the analysis of social data. The author has managed to combine the viewpoints of the mathematician and of the practicing statistician to an unusual degree.

A. E. WAUGH

Differential Geometry. By W. C. Graustein. New York, The Macmillan Company, 1935. xii+230 pages. $3.00.

This book treats of the fundamentals of the metric differential geometry of curves and surfaces in ordinary three-dimensional space. It is an excellent introduction to the classical theory, and will probably be found very useful as a text for beginners in the subject. The reader is supposed to be familiar with the calculus and solid analytics, and to have some knowledge of differential equations, but the treatment is elementary throughout, as it should be in a book designed for students devoting their first quarter or half-year to the subject.

As to contents, the first chapter is devoted to an introduction to basic geometric ideas, and is designed to familarize the reader with the elements of vector algebra employing Study's notation for products of vectors. The next two chapters are concerned with space curves and the surfaces naturally associated with them. The theory of surfaces is then developed in the next six chapters, the last of which presents a discussion of the absolute geometry of a surface and concludes with a section on Riemannian geometry. The book really reaches its climax here. However, one more chapter is added on special classes of surfaces, namely, surfaces of revolution, ruled surfaces, surfaces of translation, and minimal surfaces.

Opinions may differ as to whether it is wise to use vector methods in an exposition of this kind. In case this question is answered in the affirmative, moreover, a decision must still be made as to which one of the many existing vector notations will be employed. The author of the book before us, for reasons which seem to him adequate, favors the notation of Study. The reviewer is not altogether convinced that the usefulness of the book may not have been somewhat impaired thereby, since Study's notation for products of vectors is probably not the most widely used of the various notations which have been invented. Moreover, according to the reviewer's taste, the appearance of some of the pages is rendered aesthetically less pleasing because of the spreading of the lines necessitated by the exigencies adherent to printing formulas in Study's notation.

The book is plentifully supplied with exercises scattered throughout. The exposition is painstaking, accurate, and scholarly. It promises to be influential in the training of succeeding generations of scholars. When an expert in a branch of mathematics writes such a book as this he does his science a great

service, inasmuch as he provides a highway over which others may make the journey to knowledge with greater facility and comfort than did the pioneers.

E. P. LANE

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 171. *Proposed by J. E. Trevor, Cornell University.*

A chord of constant length slides around in a circle with fixed diameter. The midpoint of the chord and the projections of its ends upon the fixed diameter form the vertices of a triangle. Prove that this triangle is isosceles and never changes its shape.

E 172. *Proposed by V. Thébault, Le Mans, France.*

Determine a perfect square containing each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 just once, and having its four digits at the right form an arithmetic progression. Show that the solution is unique.

E 173. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

An exact division was made, and then the even digits were all replaced by x's and the odd digits were all replaced by y's, getting this pattern:

$$\begin{array}{lccccccccl}
y\,y\,y\;) & y & x & x & x & y & x & x & (\;x\,y\,y\,x\\
 & \underline{y} & \underline{x} & \underline{x} & \underline{x} & & & & \\
 & & & x & x & y & & & \\
 & & & \underline{y} & \underline{x} & \underline{y} & & & \\
 & & & & y & x & x & & \\
 & & & & \underline{x} & \underline{y} & \underline{y} & & \\
 & & & & y & x & y & x & \\
 & & & & \underline{y} & \underline{x} & \underline{y} & \underline{x} &
\end{array}$$

Reconstruct the original long division problem, and show that the solution is unique.

E 174. *Proposed by Roy MacKay, Eastern New Mexico Junior College.*

Defining a median triangle as the triangle determined by an edge of a tetrahedron and the midpoint of the opposite edge, show that the sum of the areas of the median triangles is less than three-halves of, but greater than the total area of the tetrahedron.

E 175. *Proposed by C. E. Sharp, Jr., Clayton, Mo.*

If a conic of eccentricity e is revolved about its major axis, and the resulting quadric surface cut by a plane which makes an angle A with the axis of revolution, prove that the eccentricity of the resulting conic section is $e \cos A$.

SOLUTIONS

E 4 [1932, 489]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

What is the simplest way to cut a wooden block 1ft. ×1 ft. ×2 ft. into pieces which may be reassembled into a cube?

Solution by A. H. Wheeler, Worcester, Massachusetts

A solution of this problem by W. R. Ransom of Tufts College [1933, 113]

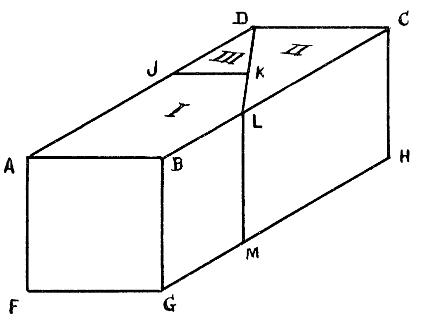

FIG. 1

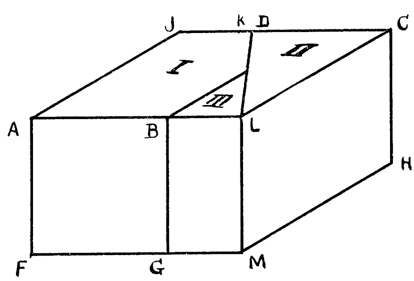

FIG. 2

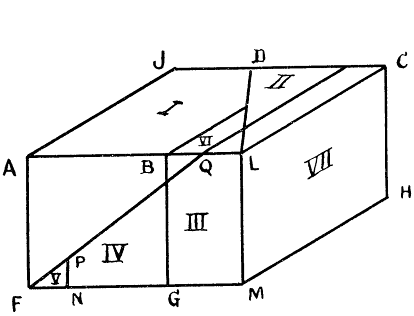

FIG. 3

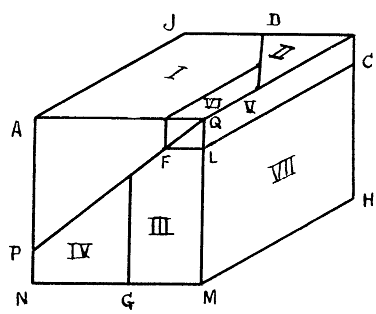

FIG. 4

made use of eight pieces, and with the solution was a note that the problem had also been solved by A. H. Wheeler, of Worcester, Massachusetts, using only

seven pieces. In response to repeated requests, the seven-piece solution is given herewith.

We will letter the block as shown in Fig. 1, in which AB is one of the shorter edges. On one of the longer edges, as AD, set off a length AJ equal to $2^{1/3}$ ft., which is the edge of the equivalent cube. Also take CL along a parallel edge and equal to AJ. Draw JK perpendicular to AD, to meet DL at K. Cut the block along the line DL perpendicular to the face $ABCD$, and then along JK, perpendicular to the same face.

We now have three pieces, I, II and III, as in Fig. 1, which we reassemble by pure translation into the relative positions shown in Fig. 2. Now we perform on face $FMLA$ of Fig. 2 the same operations that we performed on face $ABCD$ of Fig. 1, laying off MN and AQ each equal to $2^{1/3}$ ft., connecting FQ, erecting NP perpendicular to FM at N, and cutting the block with planes through FQ and NP perpendicular to the face $FMLA$. This results in the setup shown in Fig. 3.

The original block is now divided into seven pieces, numbered I, II, III, IV, V, VI and VII, as shown in Fig. 3, and these may be reassembled, again by pure translation, into the cube shown in Fig. 4.

If instead of setting off the lengths equal to the edge of the equivalent cube from the vertices A and C of the face $ABCD$ in Fig. 1, we had followed the same procedure from the vertices D and B, a dissection into seven parts would have been obtained such that the final figure would have had the parts arranged symmetrically with reference to those of the first figure.

The method described can easily be applied to solving the more general problem of dissecting a given right rectangular prism into parts which can be reassembled to form an equivalent right rectangular prism having properly chosen specified dimensions, or as a special case, into an equivalent cube.

Furthermore, since an oblique parallelopiped can be separated by two cuts into four parts which can be reassembled to form a right rectangular prism, it follows that an oblique parallelopiped can be separated into parts which can be reassembled to form an equivalent cube, and the minimum number of parts will be sixteen.

E 142 [1935, 173]. *Proposed by W. B. Clarke, San Jose, California.*

From the midpoint of a side of a triangle a line is drawn through the incenter of the triangle, intersecting the altitude to that side at the point P. Prove that the distance from P to the vertex opposite the side already referred to, is equal to the radius of the inscribed circle.

Solution by D. L. MacKay, Evander Childs High School, New York City

In the given triangle ABC, let E be the midpoint of BC, I the incenter and H the intersection of EI with the altitude AD. Draw the circumcircle with center O. Now G, the midpoint of arc BC of the circumcircle, lies on AI, and OG is the perpendicular bisector of BC. Draw IF perpendicular to AC.

Since angle ICG equals angle CIG (because of equality of intercepted arcs), IG equals CG. Triangles AIF and AIH are respectively similar to triangles CGE and GIE. Thus we have $CG:EG::AI:FI$ and $CG:EG::IG:EG::IA:HA$. Hence FI equals HA, as was to be proved.

The property stated in this problem was given by Wm. Walker in the proof of a theorem in the *Gentleman's Mathematical Companion* for 1803, page 50.

Also solved by W. E. Buker, W. Douglas, R. A. Johnson, L. M. Kelly, J. Balasundara Rao, Leon Recht, E. P. Starke, Simon Vatriquant and the proposer.

E 143 [1935, 173]. *Proposed by H. T. R. Aude, Colgate University.*

If x, y and z are restricted to positive integers, find how many solutions exist for the equation $6x+3y+2z=49$. Also find the maximum and the minimum values of $(x+y+z)$ and of (xyz).

Solution by E. L. Harp, Jr., Junior High School, Roswell, New Mexico

Excluding zero as a positive integer, we find $x<8$, $y<14$, and $z<21$. There are then seven possible values for x. Since the coefficients of x and z are even and the constant term is odd, y must always be odd. Also, the coefficients of x and y being divisible by 3 restrict the values of z to those which are one less than a multiple of three. Consequently,

When x is 1, there are 7 pairs of values for y and z;
when x is 2, there are 6 pairs of values for y and z;
when x is 3, there are 5 pairs of values for y and z;
when x is 4, there are 4 pairs of values for y and z;
when x is 5, there are 3 pairs of values for y and z;
when x is 6, there are 2 pairs of values for y and z, and
when x is 7, there is 1 pair of values for y and z.

Hence there are 28 possible solutions.

By inspection, the minimum value of $(x+y+z)$ is 10, and the maximum value of this function is 22. For the function xyz, the minimum value is 14 and the maximum value is 120.

The following table may be made to show all of the 28 solutions:

x:	1	2	3	4	5	6	7
y:	1	3	5	7	9	11	13
z:	20	17	14	11	8	5	2

Choose a value for x, slide the y-row to the right until the 1 is under the chosen x-value. Then each opposing pair of values of y and z will fit with the chosen value of x. For example, picking 4 as a value for x, the table appears:

x:	1	2	3	4	5	6	7		
y:				1	3	5	7	9	11	13
z:	20	17	14	11	8	5	2		

giving the four solutions: (4, 1, 11), (4, 3, 8), (4, 5, 5) and (4, 7, 2).

Also solved by W. E. Buker, Richard Fowler, Theodore Lindquist, F. L.

Manning, E. P. Starke, Dorothy Stephenson, J. E. Trevor, C. W. Trigg, Simon Vatriquant and the proposer.

E 144 [1935, 173]. *Proposed by W. P. Udinski, University of Texas.*

Show that a triangle must be equilateral if any pair of the following centers coincide: Incenter, Circumcenter, and Centroid.

Solution by Theodore Lindquist, Michigan State Normal College

Let ABC be the triangle with sides a, b and c, and common point O.

Case I. Incenter and Circumcenter at O. The perpendicular bisector of c and the bisectors of angles A and B meet at O. AOB is then an isosceles triangle, and so $\frac{1}{2}A$ equals $\frac{1}{2}B$, whence by symmetry the angles A, B and C are all equal and the triangle ABC is equilateral.

Case II. Circumcenter and Centroid at O. The median from A and the perpendicular bisector of a have two points in common and hence coincide. If the perpendicular from the vertex of a triangle to the opposite side bisects that side, the triangle is isosceles. Consequently angles B and C are equal. Similarly, angle A equals angle B, and the triangle ABC is equilateral.

Case III. Centroid and Incenter at O. The bisector of angle A and the median from A both go through O. Hence the angle bisector bisects the opposite side. Therefore sides b and c are equal. Similarly sides a and b are equal, and the triangle ABC is equilateral.

Simon Vatriquant extends the problem to include the Orthocenter with the three given points.

Also solved by W. E. Buker, R. A. Johnson, Sidney Kaplan, L. M. Kelly, J. Balasundara Rao, Maxwell Reade, Leon Recht, E. P. Starke, C. W. Trigg and the proposer.

E 145 [1935, 173]. "*Proposed humbly and anonymously by one, presumably able, but actually unable, to do it himself,*" *West Lafayette, Indiana.*

A cube is circumscribed about a sphere of radius unity. At the eight points where the sphere is pierced by the radial lines from the center to the vertices, tangent planes are drawn cutting off the corners, thus forming a polyhedron with fourteen faces and twenty-four vertices. The process is then repeated; that is, at the twenty-four points where the sphere is pierced by the radial lines from the center to these new vertices, tangent planes are drawn cutting off the corners and forming a polyhedron with thirty-eight faces. Find its volume in simplest form in terms of radicals.

Solution by E. P. Starke, Rutgers University

Let the sphere have its center at the origin, and the cube have its vertices at $(\pm 1, \pm 1, \pm 1)$. The tangent planes which cut off the corners corresponding to these vertices have the equations $\pm x \pm y \pm z = \sqrt{3}$. These planes intersect each other and the original faces to determine the polyhedron with twenty-four vertices, $(0, \pm 1, \pm[1-\sqrt{3}])$ and others obtained from these by permuting the x, y and z. (This polyhedron can be shown to have the volume $40-20\sqrt{3}$.)

The tangent plane which cuts off the corner of the above polyhedron corresponding to $(\sqrt{3}-1, 1, 0)$ has the equation $(\sqrt{3}-1)x+y=(5-2\sqrt{3})^{1/2}$. The calculations seem simpler if we put $s^2=3$ and $r^2=5-2s$. Our required polyhedron now has faces of three classes, of which the following are typical: (A) $x=1$, (B) $x+y+z=s$, and (C) $(s-1)x+y=r$. The equations of all six planes of class (A) and eight of class (B) and twenty-four of class (C) can be written immediately from considerations of symmetry. These thirty-eight planes intersect each other to form the seventy-two vertices of the required polyhedron. These vertices appear in three classes of which the following are typical:

(D) $[(1+s)(r-1)/2,\ 1,\ (1+s)(r-1)/2]$;

(E) $[(1+s)(s-r)/2s,\ (rs-s+r)/s,\ (1+s)(s-r)/2s]$;

(F) $[(3-2r)/s,\ r/s,\ r/s]$.

There are twenty-four of each class which can be written immediately from considerations of symmetry. The face of the polyhedron in the plane $x=1$ has the vertices: $[1,\ \pm(1+s)(r-1)/2,\ \pm(1+s)(r-1)/2]$. It is a square whose area is easily computed to be $4(3+s)-4r(2+s)$. The face in the plane $x+y+z=s$ has six vertices, the three of each of the classes (E) and (F) whose coordinates are all positive. This hexagon is easily shown to have each side of length $s(r-1)$ with angles alternately 90° and 150°. By trigonometry, its area is computed to be $18-3r(3+s)$. The face in the plane $(s-1)x+y=r$ has six vertices, two of each of the classes (D), (E) and (F) for which x and y are both positive. This plane and the planes $y=1$ and $x+(s-1)y=r$ which it intersects are all parallel to the z-axis. Hence two opposite edges of the face are parallel lines, and its area may be easily computed as the sum of two trapezoids. It is $13-(22+5s)r/3$.

If we take six, eight and twenty-four faces in the planes of classes (A), (B) and (C) respectively, the sum of their areas is

$$24(22+s)-8r(37+11s),$$

the total surface of the polyhedron. Since each face is tangent to the unit sphere, the volume of the pyramid having that face as base and the origin as apex is one-third the area of its base. Hence the total required volume is one-third of the above total area, or

$$8(22+s)-8r(37+11s)/3, \quad \text{or}$$

$$8(22+\sqrt{3})-(8/3)(37+11\sqrt{3})(5-2\sqrt{3})^{1/2}.$$

Also solved by W. B. Carver.

E 146 [1935, 174]. *Proposed by C. W. Foard, Youngstown College, Ohio.*

A cow is tethered by a seventy foot rope which passes over a long straight fence seven feet high, to a stake in the ground, twenty-four feet back from the fence. The ground is level. It is required to find the area over which the cow can graze.

Note by W. E. Buker, Leetsdale High School, Pennsylvania

This problem, with length of rope l, height of wall h, and stake a feet from the wall, was proposed by W. F. Harlow in this MONTHLY, April, 1918, and solved by C. F. Gummer, April, 1919, pages 174–175. He sets up the problem in rectangular coordinates, getting

$$A = \int_0^{y_1} \sqrt{a^2 + l^2 - y^2 - 2\sqrt{a^2 + h^2}\sqrt{l^2 - y^2}}dy,$$

where

$$y_1 = \sqrt{l^2 - (\sqrt{a^2 + h^2} + h)^2}.$$

The same problem in numerical form was proposed as problem 1303 in School Science and Mathematics. Instead of a solution, reference was made to the solution mentioned above.

E 147 [1935, 174]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a certain college of under five thousand total enrollment, a third of the students were freshmen, two-sevenths of the students were sophomores, a fifth of them were juniors and the rest seniors. The history department offered a popular course in which there were registered a fortieth of all the freshmen in college, a sixteenth of all the sophcmores and a ninth of all the juniors, while the remaining third of this history class were all seniors. How many students were there in the history class?

Solution by Mary L. Constable, Philadelphia High School for Girls

Let x represent the number of students in the college. Then $x/3$, $2x/7$, $x/5$ and $19x/105$ are the numbers of students in the freshman, sophomore, junior and senior classes respectively. Correspondingly, $(x/3)/40$, $(2x/7)/16$, $(x/5)/9$ and $\frac{1}{2}(x/3\cdot 40+2x/7\cdot 16+x/5\cdot 9)$ are the numbers of freshman, sophomores, juniors and seniors in the history class respectively. Since each of these eight numbers must be an integer, x must be a multiple of $5\cdot 7\cdot 8\cdot 9$ or 2520. But the problem specifies that the total enrollment is under 5000, so x must be 2520. Consequently, the history class contains 21 freshmen, 45 sophomores, 56 juniors and 61 seniors, or 183 students in all.

Also solved by John Bimmerle, W. B. Brown, W. E. Buker, Daniel Finkel, E. L. Harp, Jr., L. M. Kelly, Theodore Lindquist, F. L. Manning, Maxwell Reade, Leon Recht, F. C. Schmidt, E. P. Starke, Dorothy Stephenson, J. E. Trevor, C. W. Trigg, Simon Vatriquant, R. C. Yates and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially

sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department, In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3749. *Proposed by H. D. Grossman, New York.*

Construct, similar to a given triangle, a triangle whose vertices lie each on a corresponding curve in a set of three given curves in a plane.

3750. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

A circle with a fixed tangent rolls along a straight line AB. Show that the envelope of the tangent is the locus of the point of intersection of the tangent with the perpendicular to it drawn through the point of contact of the circle with AB.

3751. *Proposed by V. Thébault, Le Mans, France.*

Let Q_b, Q_c, Q_b', Q_c' be the intersections of the bisectors of the angles B and C with the opposite sides of the triangle ABC; D' and D_a', the points on the inscribed and escribed circles in the angle A diametrically opposite to their points of contact with BC; P and P', the intersections of Q_bQ_c and $Q_b'Q_c'$ with the interior bisector of angle A. Show that the lines $D'P$ and $D_a'P'$ meet in the foot of the altitude AA' and are symmetric with respect to BC.

3752. *Proposed by V. Thébault, Le Mans, France.*

Let S_1, S_2, S_3, S_4 be the centers of four spheres. The symmetric of the center of the sphere passing through S_1, S_2, S_3, S_4 with respect to the radical center of the four spheres coincides with the radical center of the spheres orthoptic to the first four spheres. Generalize.

Editorial Note. The following definitions were taken from the large Larousse dictionary. Given a quadric surface the locus of the vertex of a trirectangular trihedral angle whose faces are tangent to the quadric is a sphere called the orthoptic sphere, or the Monge sphere, of the given quadric surface. For plane curves there is a similar definition. The locus of the point from which the two tangents to a given conic are perpendicular is called the orthoptic circle, or the Monge circle, of the given conic. The circle is imaginary for certain cases of the hyperbola, and it is the directrix for a parabola.

3753. *Proposed by Frank Ayres, Jr., Dickinson College.*

Let P_4 be the orthocenter of the triangle $P_1P_2P_3$; and O_i, the circumcenter of the triangle $P_jP_kP_4$, where i, j, k is a permutation of 1, 2, 3. Prove that the sum of the squares of the radii of the circles having O_i as centers and orthogonal to the nine-point circle of the given triangle is one-fourth of the sum of the squares of the edges of the given triangle.

SOLUTIONS

3673 [1934, 269]. *Proposed by E. B. Escott, Oak Park, Ill.*

Factor $x^8+98x^4y^4+y^8$ into two polynomial factors with integral coefficients.

I. *Solution by F. D. Rigby, Student, Reed College*

Assume the factors

$$(x^4 + ax^3y + bx^2y^2 + cxy^3 + y^4)(x^4 + dx^3y + ex^2y^2 + fxy^3 + y^4),$$

a, b, c, d, e, f, being integers. Multiplying, and equating coefficients,

$$a + d = 0, \quad b + e + ad = 0, \quad c + f + ae + bd = 0, \quad af + be + cd = 96,$$

$$c + f = 0, \quad b + e + cf = 0, \quad a + d + bf + ce = 0.$$

Since $d = -a, f = -c$, the third equation gives $a = 0$ or $b = e$.

If $a=0$, it follows that $b = -e$, $c=d=f=0$, which would require $-b^2=96$. This case must be discarded. Hence $b=e$; and the second and sixth equations give: $2b = -ad = -cf = a^2 = c^2$. Since $2b$ is a perfect square, b must have one of the values, 2, 8, 18, 32, $\cdots$.

But from the fourth equation, $b^2-2ac=96$; and this is not satisfied by $b=2$ (with the corresponding values, $a=\pm 2$, $c=\pm 2$). If $b=8$, $a=\pm 4$, $c=\pm 4$, and since then $2ac=-32$, $a=-c=\pm 4$. With either sign, $a=-c=-d=f=\pm 4$, $b=e=8$, and the factors are

$$x^4 + 4x^3y + 8x^2y^2 - 4xy^3 + y^4,$$

$$x^4 - 4x^3y + 8x^2y^2 + 4xy^3 + y^4.$$

II. *Solution by Hansraj Gupta, Govt. College, Hoshiarpur (India)*

Solving $x^8+98x^4y^4+y^8=0$ as a quadratic in x^4, we get

$$x^4 = -(49 \pm 20\sqrt{6})y^4, \qquad x^2 = \pm(5 \pm 2\sqrt{6})iy^2,$$

and finally

$$x = \pm(\sqrt{3} \pm \sqrt{2})\left(\frac{1 \pm i}{\sqrt{2}}\right)y = \pm(1 \pm i)\left(1 \pm \sqrt{\frac{3}{2}}\right)y.$$

The four roots $(1 \pm i)(1 \pm \sqrt{3/2})y$, lead to the factor:

$$x^4 - 4x^3y + 8x^2y^2 + 4xy^3 + y^4;$$

and the four roots $-(1 \pm i)(1 \pm \sqrt{3/2})y$, give the factor:

$$x^4 + 4x^3y + 8x^2y^2 - 4xy^3 + y^4.$$

III. *Solution by E. E. Strock, Yale University*

$$
\begin{aligned}
x^8+98x^4y^4+y^8 &= (x^4+y^4)^2+96x^4y^4\\
&= (x^4+y^4)^2+16x^2y^2(x^4+y^4)+64x^4y^4-16x^2y^2(x^4+y^4)+32x^4y^4\\
&= (x^4+y^4+8x^2y^2)^2-16x^2y^2(x^4-2x^2y^2+y^4)\\
&= (x^4+y^4+8x^2y^2)^2-(4x^3y-4xy^3)^2\\
&= (x^4-4x^3y+8x^2y^2+4xy^3+y^4)(x^4+4x^3y+8x^2y^2-4xy^3+y^4).
\end{aligned}
$$

IV. *Solution by Morgan Ward, California Institute*

It is no more difficult to treat the general problem of factoring $x^8+Nx^4y^4+y^8$ in the ring of rational integers for N any rational integer, and it suffices to perform this factorization with $y=1$. Denote the resulting polynomial in x, x^8+Nx^4+1, by $F(x)$.

Linear factors of $F(x)$ can obviously occur only when $N=-2$, giving $F(x)=(x-1)^2(x+1)^2(x^2+1)^2$. We assume henceforth that $N\neq-2$.

Consider next irreducible quadratic factors of $F(x)$. Such a factor is readily seen to be of the form $Q(x)=x^2+Mx\pm1$, M an integer $\neq 0$. Since $F(x)$ is an even function of x, if $Q(x)$ is a factor of $F(x)$, so is $Q(-x)$, and accordingly $P(x^2)=Q(x)\,Q(-x)=x^4-(M^2\mp2)x^2+1$ is a factor of $F(x)$. Similarly, $P(-x^2)$ must be a factor of $F(x)$. Therefore, if $F(x)$ has an irreducible quadratic factor, it is necessarily of the form

$$F(x)=(x^2+Mx\pm1)(x^2-Mx\pm1)(x^4+(M^2\mp2)x^2+1),$$

so that N must be of the form $2-(M^2\mp2)^2$. (If $M=\pm2$, the lower sign must be taken in the formulas). Conversely, if N is of the form $2-(M^2\mp2)^2$, $F(x)$ possesses a quadratic factor. Finally, it is easily seen that the quartic $x^4+(M^2\mp2)x^2+1$ is irreducible when $N\neq-2$.

$F(x)$ can have no irreducible cubic factor $K(x)=x^3+\cdots\pm1$. For if such a factor existed, $-K(-x)$ would be still another one, distinct from $K(x)$. Therefore $F(x)$ would have either a quadratic or a linear factor, and in either eventuality, we have seen that no irreducible cubic factors exist.

The only remaining possibility is that $F(x)$ has two irreducible quartic factors, $G(x)=x^4+\cdots\pm1$, $H(x)=x^4+\cdots\pm1$, so that

$$F(x)=G(x)H(x). \tag{1}$$

On replacing x by $-x$ in (1), we see that either $G(x)=G(-x)$, $H(x)=H(-x)$ or $G(x)=H(-x)$. In the first case, $G(x)$ and $H(x)$ are of the forms

$$G(x)=x^4-Mx^2\pm1,\qquad H(x)=x^4+Mx^2\pm1$$

and N is of the form $\pm2-M^2$. We must also assume that the integer $\pm M$ is not of the form $\pm2-L^2$, as otherwise either $G(x)$ or $H(x)$ would be reducible. For the same reason, if $M=0$, the positive sign must be taken with the constant terms of $G(x)$ and $H(x)$.

There remains then only the case when $G(x)=H(-x)$. If we replace x by x^{-1} in (1) and then multiply by x^8, we see that either $G(x)=\pm x^4 G(x^{-1})$ or $G(x) = \pm x^4 H(x^{-1})$. We find then that either

$$G(x) = x^4 + Ax^3 + Bx^2 - Ax + 1, \text{ or } G(x) = x^4 + Ax^3 + Bx^2 + Ax + 1,$$

where A and B are integers and not zero, and $H(x)=G(-x)$.

On substituting these expressions in (1), we find in the first case that $2B - A^2=0$, $N=B^2+2A^2+2$, and in the second case that $2B-A^2=0$, $N=B^2 - 2A^2+2$. Therefore if we set $A=2M$, N must be of one or the other of the forms $4M^4+8M^2+2, 4M^4-8M^2+2$. The sufficiency of this condition to insure a factorization (1) is again obvious. To summarize: A necessary and sufficient condition that $F(x)=x^8+Nx^4+1$ be the product of two irreducible quartic factors containing odd powers of x is that N be of the form $4M^4\pm 8M^2+2$ and $\neq \pm 2$. The factorization of $F(x)$ is then

$$F(x) = (x^4 + 2Mx^3 + 2M^2x^2 \mp 2Mx + 1)(x^4 - 2Mx^3 + 2M^2x^2 \pm 2Mx + 1).$$

The first few admissible values of N are 14; 98, 34; 398, 254; 1154, 898. In particular, for $M=2$, $N=4M^4+8M^2+2=98$, we have

$$x^8 + 98x^4 + 1 = (x^4 + 4x^3 + 8x^2 - 4x + 1)(x^4 - 4x^3 + 8x^2 + 4x + 1).$$

Solved also by Richmond Albert, W. N. Birchby, W. E. Buker, J. E. Burnam, J. W. Clawson, F. L. Griffin, Ivar Highberg, Sidney Kaplan, Harry Langman, F. L. Manning, A. S. Merrill, Otto J. Ramler, E. P. Starke, A. L. Starrett, Elijah Swift, C. W. Trigg, F. Underwood, S. Vatriquant, and the proposer.

Editorial Note. The proposer used the method in solution III for the general case

$$x^8 + Nx^4y^4 + y^8 = (x^4 + ax^2y^2 + y^4)^2 - (bx^3y + cxy^3)^2$$

and, after equating coefficients, it was found that

$$a = 2m^2, \quad b = 2m, \quad c = \pm 2m, \quad N = 4(m^2 \pm 2)^2 - 2.$$

This device was used by several solvers for the given polynomial. A number of solvers gave two solutions. Swift gave two solutions with a complete discussion, and he referred to a more difficult problem of this kind on page 449 in Weber's *Lehrbuch der Algebra*, 2nd ed. vol. 1 (1898).

3674 [1934, 269]. *Proposed by Garrett Birkhoff, Harvard University.*

For any positive integer k show that

$$\phi_k = \frac{(2k-2)!}{k!(k-1)!} = \binom{2k-1}{k}\Big/(2k-1)$$

is an integer; and prove the recurrence formula

$$\phi_n = \sum_{i=1}^{n-1} \phi_i \phi_{n-i}.$$

I. *Solution by E. D. Rainville, Junior Engineer, U. S. Bureau of Reclamation, Denver, Colorado*

By actual expansion

$$\psi(x) = \tfrac{1}{2} - \tfrac{1}{2}(1 - 4x)^{1/2} = \sum_{k=1}^{\infty} \frac{(2k-2)!}{k!(k-1)!} x^k = \sum_{k=1}^{\infty} \phi_k x^k, \qquad |x| < \tfrac{1}{4},$$

and also

$$[\psi(x)]^2 + x = \psi(x).$$

Now, using the Cauchy product,

$$[\psi(x)]^2 = \sum_{k=2}^{\infty} \left(\sum_{s=1}^{k-1} \phi_s \phi_{k-s} \right) x^k.$$

Hence we have the identity

$$x + \sum_{k=2}^{\infty} \left(\sum_{s=1}^{k-1} \phi_s \phi_{k-s} \right) x^k \equiv \sum_{k=1}^{\infty} \phi_k x^k,$$

from which $\phi_1 = 1$; and for $k > 1$

$$\phi_k = \sum_{s=1}^{k-1} \phi_s \phi_{k-s}.$$

Since $\phi_1 = 1$ and $\phi_2 = 1$, ϕ_k is evidently integral for all integral k.

II. *Solution by Morgan Ward, California Institute*

We observe that

$$\phi_k = {}_{2k-2}C_{k-1}/k = {}_{2k-1}C_k/(2k-1).$$

From the last two equalities, $(2k-1) \cdot {}_{2k-2}C_{k-1} = k \cdot {}_{2k-1}C_k$. Therefore k divides ${}_{2k-2}C_{k-1}$ and ϕ_k is an integer.

We may readily express the summation in the problem as a hypergeometric series. On writing $(\alpha)_i$ for $\alpha(\alpha+1) \cdots (\alpha+i-1)$, we find that

$$\phi_i = (-)^{i+1} 2^{2i-1}(-1/2)_i/i!, \quad \phi_{n-i} = (-)^i \phi_n (-n)_i/(-n+3/2)_i 2^{2i}.$$

Therefore, with the customary notation for the hypergeometric series,

$$\sum_{i=1}^{n-1} \phi_i \phi_{n-i} = -\frac{\phi_n}{2} \sum_{i=1}^{n-1} \frac{(-1/2)_i (-n)_i}{(-n+3/2)_i i!}$$

$$= \phi_n - \frac{\phi_n}{2} F(-1/2, -n; -n+3/2; 1) = \phi_n,$$

for by the classical formula of Gauss

$$F(-1/2, -n; -n+3/2; 1) = \frac{\Gamma(-n+3/2)\Gamma(1)}{\Gamma(-n+2)\Gamma(3/2)} = 0,$$

since n is an integer greater than one.

Solved also by Hansraj Gupta, Harry Langman, and E. P. Starke.

Editorial Note. The remaining three solutions were similar to I in the proof of the recurrence formula. Gupta noted that $\phi_k = {}_{2k-1}C_k - 2_{2k-2}C_k$, while Starke observed that $\phi_k = {}_{2k-2}C_{k-1} - {}_{2k-2}C_k$. Either form shows that ϕ_k is an integer. Langman used induction to prove this part.

Solution II suggests a similar proof which may appear simpler to those not familiar with the functions used in that proof. The theory involved will be developed in the proof. We define $x^{(k)} = x(x-1) \cdots (x-k+1)$, $x^{(0)} = 1$; and it is easily verified that $\Delta x^{(k)} = kx^{(k-1)}$. It may also be verified that

$$\phi_k = -\frac{(-4)^k}{2}\frac{(1/2)^{(k)}}{k!}. \tag{1}$$

Hence we are to prove that

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{(1/2)^{(k)}(1/2)^{(n-k)}}{k!(n-k)!} = -2\frac{(1/2)^{(n)}}{n!}, \quad \text{or} \\ &\sum_{k=0}^{n} \frac{(1/2)^{(k)}(1/2)^{(n-k)}}{k!(n-k)!} = 0, \end{aligned} \tag{2}$$

where a factor $(-4)^n/4$ has been removed.

Define the operator U so that $U^k f(x) = f(x+k)$. Then, if n is a positive integer or zero,

$$f(x+n) = U^n f(x) = (1+\Delta)^n f(x) = \sum_{k=0}^{n} {}_nC_k \Delta^k f(x). \tag{3}$$

Suppose now that $f(x)$ is a polynomial of degree n in x and that h is any kind of number, then

$$f(x+h) = \sum_{k=0}^{n} \frac{(h)^{(k)}}{k!} \Delta^k f(x). \tag{4}$$

For, if x is fixed, each side is a polynomial of degree n in h; and (3) shows that (4) is true when $h = 0, 1, \cdots, n$. Therefore (4) is an identity in h. Set $f(x) = x^{(n)}$ and $x = h = 1/2$; then, for $n \geqq 2$,

$$0 = (1/2 + 1/2)^{(n)} = \sum_{k=0}^{n} \frac{(1/2)^{(k)}}{k!} \frac{n!}{(n-k)!} (1/2)^{(n-k)},$$

and the proof is complete.

The above proof has established the binomial theorem for the function $x^{(n)}$,

$$(x+y)^{(n)} = \sum_{k=0}^{n} {}_nC_k x^{(k)} y^{(n-k)}.$$

3675 [1934, 269]. *Proposed by H. Grossman, New York.*

Denote by $P^{-1}(ABC)$ the triangle which contains the given triangle ABC in its interior as its pedal triangle, by $P^{-2}(ABC)$ the triangle determined by $P^{-1}(ABC)$ in the same manner, and so on. Show that as n becomes infinite the angles of $P^{-n}(ABC)$ approach 60°.

Solution by Harry Langman, Brooklyn, N. Y.

Let $A_nB_nC_n$ be the nth circumscribed triangle of the specified type. Then

$$A_n = \frac{\pi}{2} - \frac{A_{n-1}}{2}.$$

Applying this formula to itself, we have in sequence

$$\begin{aligned} A_n &= \frac{\pi}{2} - \frac{\pi}{2^2} + \frac{A_{n-2}}{2^2} \\ &= \frac{\pi}{2} - \frac{\pi}{2^2} + \frac{\pi}{2^3} - \frac{A_{n-3}}{2^3} \\ &= \frac{\pi}{2} - \frac{\pi}{2^2} + \frac{\pi}{2^3} - \frac{\pi}{2^4} + \frac{A_{n-4}}{2^4} = \cdots \\ &= \frac{\pi}{2} - \frac{\pi}{2^2} + \cdots + (-1)^{n-1}\frac{\pi}{2^n} + (-1)^n \frac{A}{2^n}. \end{aligned}$$

Obviously, then,

$$\lim_{n\to\infty} A_n = \frac{\pi}{2} - \frac{\pi}{2^2} + \frac{\pi}{2^3} - \frac{\pi}{2^4} + \cdots = \frac{\pi}{3}.$$

Similarly for the other angles.

Solved also by J. W. Clawson, J. M. Feld, Otto J. Ramler, and F. Underwood.

Editorial Note. The other solutions were similar to the one given above. It is obvious that the angles for any one of the triangles are the arithmetic means of each pair of angles for the preceding triangle; and the desired result follows from a general theorem.

Given an initial set of $n>2$ positive quantities $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$, we obtain a second set by taking in n ways the arithmetic means of all omitting one; a third set is obtained from the second in the same way; and so on. The members of the sets converge each to the arithmetic mean of the initial set.

Proof. Let the members of each set be arranged in descending order of magnitude, so that

$$x_1^{(i)} \geqq x_2^{(i)} \geqq \cdots \geqq x_n^{(i)}.$$

Then $x_1^{(i+1)}$ is the arithmetic mean of the first $n-1$ of the above; $x_2^{(i+1)}$, that of all omitting $x_{n-1}^{(i)}$, etc. We have then

$$x_1^{(0)} \geqq \cdots \geqq x_1^{(i)} \geqq x_1^{(i+1)} \geqq x_n^{(i+1)} \geqq x_n^{(i)} \geqq \cdots \geqq x_n^{(0)}.$$

Hence as n becomes infinite $x_1^{(i)}$ approaches a limit a_1, while $x_n^{(i)}$ approaches a limit a_n, where $a_1 \geqq a_n$. Denote the sum of the initial set by na; then it easily follows that na is the sum of every set. Thus $(n-1)x_1^{(i+1)} = na - x_n^{(i)}$, $(n-1)x_n^{(i+1)} = na - x_1^{(i)}$. By taking the limit of each side, we find that

$$(n-1)a_1 = na - a_n, \qquad (n-1)a_n = na - a_1;$$

and, since $n>2$, we have $a_1 = a_n = a$. Moreover, since $x_1^{(i)} \geqq x_k^{(i)} \geqq x_n^{(i)}$, it follows that the limit of $x_k^{(i)}$ is also a, and the proof is complete.

The theorem may be varied by taking, for example, the geometric mean of all n terms omitting one; the proof is quite similar.

3676 [1934, 269]. *Proposed by R. E. Gaines, University of Richmond.*

Let $P_1, P_2, P_3, \cdots$ be a series of points on a parabola such that the successive chords $P_1P_2, P_2P_3, \cdots$ are the bases of segments of the parabola whose areas are in decreasing geometric progression. Then P_n approaches a limit P as n becomes infinite; and the series of areas of the segments on the chords $P_1P, P_2P, P_3P, \cdots$ are in geometric progression with the same ratio as the former. Also any set of P_s whose subscripts are in arithmetic progression determines a series of areas in geometric progression, and the areas of the triangles formed by successive sets of three such points are also in geometric progression.

Solution by C. L. Weaver, New England Mutual Life Ins. Co., Boston, Mass.

Without loss of generality we may take the equation of the parabola in the form $y^2 = a^2x$. Let the coordinates of P_n be x_n, y_n; we take $y_n > y_{n-1}$, where negative as well as positive y's are admitted. Denote by ${}_1A_n$ the area of the segment on the chord P_nP_{n+1}, then by a simple integration we find that

$$(1) \qquad {}_1A_n = \frac{(y_{n+1} - y_n)^3}{6a^2} = \frac{c^3r^{3n}}{6a^2}, \qquad r < 1,$$

where the last equation is given by the problem. This result is true for all the cases mentioned above. Hence, by a summation of the differences of the y's, we find that

$$(2) \qquad y_n = y_1 + \frac{cr(1 - r^{n-1})}{1 - r}, \qquad y = y_1 + \frac{cr}{1 - r},$$

where y is the ordinate of P.

Let A_n be the area of the segment on P_nP, then

$$(3) \qquad A_n = \frac{1}{6a^2}(y - y_n)^3 = \frac{1}{6a^2}\left(\frac{cr^n}{1 - r}\right)^3 = r^3 A_{n-1};$$

and the second proposition is proved.

Let ${}_sA_n$ be the area of the segment on P_nP_{n+s}, then

$$(4) \qquad {}_sA_n = \frac{1}{6a^2}\left(\frac{cr^n(1 - r^s)}{1 - r}\right)^3 = r^{3t}{}_sA_{n-t}.$$

This proves a general proposition including the third.

Let ${}_sT_n$ be the area of the triangle $P_nP_{n+s}\ P_{n+2s}$, then ${}_sT_n = {}_{2s}A_n - {}_sA_n - {}_sA_{n+s}$, or

$$(5) \qquad {}_sT_n = \frac{r^s + r^{2s}}{2a^2}\left(\frac{cr^n(1 - r^s)}{1 - r}\right)^3 = r^{3t}{}_sT_{n-t}.$$

This is a general proposition that includes the last one to be proved.

Solved also by J. W. Clawson, Geraldine Coon, Harry Langman, Otto J. Ramler, and F. Underwood.

Editorial Note. All parabolas are similar, and a proof for $y^2 = x$ is quite general. The solution by Ramler used $y^2 = 4x$ in parametric form.

Certain simple transformations of the plane give the theorems of the parabola essential for this problem. Let AB be a chord of a given parabola, and let the parallel to the axis through M, the mid-point of AB, cut the curve in V. We shall call VM the breadth of the parabolic segment, and the distance between the parallels to the axis through A and B the altitude of the segment. If V is the vertex of the parabola, the altitude is the length of the chord of the segment. In this latter case let t be any chosen straight line through the vertex V not parallel to the axis. By a linear shear of the plane parallel to the axis the tangent at the vertex can be carried into t, which is now obviously the tangent to the transformed curve at V. This shear does not alter the form of the parabola but merely translates it so that the axis remains parallel to its former position. Thus by altering the position of t the parabola is made to slide through V. The proof is simple: if the equations of the parabola and t are, respectively, $x = cy^2$ and $x = my$, the new curve has the equation $x = cy^2 + my$. But the addition of the linear term does not alter the shape of the parabola and does not change the direction of the axis. The shear does not alter the area, the breadth, or the altitude of the segment. Hence all segments of a given parabola having the same breadth, or the same altitude, have the same area, and conversely. Equidistant parallels to the axis determine on a given parabola segments, bounded by the intercepted arc and corresponding chord, having the same breadth and area. This recalls the other similar Archimedean theorem in regard to the areas

of zones of a sphere cut off by equidistant parallel planes; but the analogy is not complete.

The tangents at A and B meet in T on the axis VM, where here V is the vertex, and $TV = VM$. After the shear this is again true, but now V is the intersection with the curve of the parallel to the axis through M, the mid-point of the chord.

We now consider the triangle ABV inscribed in the segment; its area T is also unaltered by the shear. If S is the area of the segment, $S = kT$, where k does not depend upon the form of the parabola or upon the position of the segment. For, the transformation $x = r^2 \cdot X$, $y = r \cdot Y$, multiplies areas by r^3, and it carries the segment and its inscribed triangle into another segment and inscribed triangle of the same parabola. But the ratio $S:T$ remains unaltered. Denote by S' and T' the areas of the segments and triangles for the chords AV and BV. Then $T = S - 2S' = k(T - 2T')$. The altitude of S' in each case is half that of S, and the breadth is one-fourth of that of S. Hence $T = 8T'$, and thus $k = 4/3$ or

$$S = \frac{4}{3} T.$$

If x and y denote the breadth and semi-altitude, then $T = xy = cy^3$, and

$$S = \frac{4}{3} cy^3.$$

If we assign to S a set of values forming a decreasing geometric progression, then the corresponding altitudes form also a decreasing geometric progression. Select any point P_0 of the given parabola and draw through it an initial parallel to the axis. This initial parallel determines in two ways a sequence of parallels whose successive distances apart are the determined altitudes in geometric progression; and these determine a limit parallel through P on the curve. There is also determined a sequence of chords and corresponding segments upon the chords. If the set of parallels is moved as a rigid system, remaining parallel to the axis, the corresponding sequence of segments slide along the parabola preserving their individual areas. The area of the segment on P_0P, as well as that of a segment on the chord joining any two selected points of the set, is also unaltered. This is obviously true also for the inscribed triangles. Let $AVBM$ be any one of these segments; and translate the parabola by the vector VM. The parabola in its new position will be tangent to AB in its first position at M. This gives the theorem: If two parabolas are such that the second is a translation of the first in the direction defined by that of the vertex to the focus, the tangent at any point M of the second determines a chord AB of the first which is bisected at M, and the segment of the first on this chord has an altitude and area which is independent of the position of M. Thus the sequence of areas S determine a sequence of equal parabolas which approach an equal limit parabola. As the system of parallels move the chord of each segment slides on its corresponding

parabola in the manner described. The chord P_0P, as also any other selected chord of the system, has its corresponding parabola for sliding.

Another theorem in regard to areas results from the above, or in the following manner. Let P_i and P_j be any two points of the parabola on the same side of any chosen diameter l; let M_{ij} be the mid-point of the chord P_iP_j; let the two tangents at P_i and P_j intersect in T_{ij}, and cut l in T_i and T_j; and finally let M_i and M_j be the intersections with l of its conjugate chords through P_i and P_j. If V is the intersection of l with the curve, $T_jV=VM_j$, $T_iV=VM_i$, and therefore, $T_jT_i=M_iM_j$. Also $T_{ij}M_{ij}$ is parallel to l. Hence the area of triangle $T_{ij}T_jT_i$ is precisely one-half of that of the trapezoid $M_iM_jP_jP_i$, since they have equal bases and the altitude of the triangle is the altitude of the mid-point of P_iP_j. Thus as the segment of the tangent, P_jT_j moves from the position AT to $P'T'$ it sweeps out the area $ATT'P'$ which is one-half of the area swept out simultaneously by P_jM_j as it moves from AM to $P'M'$, the two areas being separated by the arc AP' of the parabola.

Archimedes derived the area of a parabolic segment in two ways, in one of which he used properties of the lever; the other proof is given in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, page 134. In this proof there is a summation of a geometric series, which may be avoided. If the segment is $AVBM$ with the tangents at A and B meeting in T, where M is the mid-point of chord AB, then the tangent at V with AT and BT form an outer triangle whose area is one-half of that of the inner triangle AVB. On the arcs VA and VB we have similar pairs of inner and outer triangles. We may continue in this way to introduce inner and outer triangles on the new arcs without end. Hence the parabola divides the area of the triangle ATB into two parts the ratio of whose areas is 1:2. Hence the area of the segment AB of the parabola is 2/3 the area of triangle ABT or 4/3 the area of triangle ABV.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Mathematical Association of America will hold its 1936 summer meeting August 31, in connection with the Tercentenary Celebration of Harvard University. The American Mathematical Society will meet September 1–5. Harvard University is inviting distinguished mathematicians from each of four countries, and is generously scheduling the lectures of these guests to take place before the Society and the Association. Professor E. W. Chittenden will deliver the Colloquium Lectures on *Topics in General Analysis*. There will be a joint meeting of the Society with the American Astronomical Society. Arrangements will be made to house a considerable number of the visiting mathematicians in the Harvard dormitories and there will be numerous excursions to points of historic interest in Cambridge and the neighborhood. This meeting

should prove one of the most interesting and important that the organizations have held.

The International Mathematical Congress, under the presidency of Professors Alf Guldberg and Carl Størmer, will be held at Oslo July 13–18, 1936. This date early in the summer has been set in order to avoid conflict with the Summer Meeting of the American Mathematical Society. The Congress will be organized in the following sections: (1) Algebra and Theory of Numbers; (2) Analysis; (3) Geometry and Topology; (4) Theory of Probability, Insurance, Statistics; (5) Astronomy; (6) Mechanics and Mathematical Physics; and (7) Philosophy, History, Education. Nearly a score of leading mathematicians have been invited to give general lectures before the Congress, including the following representatives from American universities: Professors G. D. Birkhoff, Oystein Ore, Oswald Veblen and Norbert Wiener. Titles of papers intended for presentation at the Congress should be sent to the Sécrétaire Général Edgar B. Schieldrop, Universitetet, Oslo, Norvège. After the Congress, trips will be arranged to some of the picturesque spots in Norway.

At the June Commencement of the University of Wisconsin, the degree of Doctor of Science was conferred upon Professor G. A. Bliss, head of the department of mathematics of the University of Chicago.

Solomon Lefschetz, professor of mathematics at Princeton University, has been elected a member of the Royal Society of Letters and Sciences of Bohemia.

R. C. Tolman, professor of mathematical physics at the California Institute of Technology, has been elected president of the Pacific division of the American Association for the Advancement of Science.

R. H. Fowler, Plummer professor of physics at Trinity College, Cambridge, has been appointed visiting lecturer in mathematics at Princeton for the second term of the year 1935–36.

R. L. Jeffery, professor of mathematics at Acadia University, has been appointed visiting lecturer at the University of Wisconsin for the year 1935–36.

At the University of Michigan, G. Y. Rainich and R. L. Wilder have been promoted to professorships.

At Harvard University, J. H. Van Vleck has been appointed to a professorship of mathematical physics.

H. F. Archibald has been appointed temporary instructor at Acadia University for the year 1935–36.

J. G. Estes, professor of mathematics at North Carolina State College, was killed on June 1, when his plane crashed at the Raleigh airport.

Volume I of Runkle's Mathematical Monthly (October 1858–September 1859) and Volume II except No. 11 (August 1860) may be obtained at a reasonable price through the office of the Secretary of the Mathematical Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4; Beaver Falls, Pa., Oct. 26.
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11; College Park, Md., Dec. 7.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLII, 1935

NUMBER 9, NOVEMBER

PART I

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THE NINETEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The nineteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of Michigan, Ann Arbor, Michigan, on Monday and Tuesday, September 9 and 10, 1935, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Three hundred ten were present at the meetings, including the following one hundred sixty-nine members of the Association:

C. R. Adams, Brown University
R. P. Agnew, Cornell University
Nola Lee Anderson, Sophie Newcomb College
R. C. Archibald, Brown University
W. L. Ayres, University of Michigan

Grace M. Bareis, Ohio State University
W. D. Baten, University of Michigan
H. M. Beatty, Ohio State University
Samuel Beatty, University of Toronto
A. A. Bennett, Brown University
Felix Bernstein, Columbia University
H. L. Black, Westminster College
G. A. Bliss, University of Chicago
Henry Blumberg, Ohio State University
J. W. Bradshaw, University of Michigan
J. B. Brandeberry, University of Toledo
Richard Brauer, University of Toronto
R. W. Brink, University of Minnesota
Myrtle C. Brown, North Texas State Teachers College
H. E. Buchanan, Tulane University
W. H. Bussey, University of Minnesota

W. D. Cairns, Oberlin College
B. H. Camp, Wesleyan University
C. C. Camp, University of Nebraska
J. W. Campbell, University of Alberta
W. B. Carver, Cornell University
R. V. Churchill, University of Michigan
C. J. Coe, University of Michigan
L. W. Cohen, University of Kentucky
Julia T. Colpitts, Iowa State College
A. T. Craig, University of Iowa
C. C. Craig, University of Michigan
A. R. Crathorne, University of Illinois
S. E. Crowe, Michigan State College
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University

Wayne Dancer, University of Toledo
H. T. Davis, Indiana University
F. F. Decker, Syracuse University
W. W. Denton, Detroit, Michigan
L. L. Dines, Carnegie Institute of Technology
Olive May Draper, Taylor University
F. G. Dressel, Duke University
P. S. Dwyer, Antioch College

C. M. Erikson, Michigan State College
H. P. Evans, University of Wisconsin
H. S. Everett, University of Chicago
J. P. Everett, State Teachers College, Kalamazoo, Mich.

Rev. F. J. Feinler, Loudonville, Ohio
Peter Field, University of Michigan
C. H. Fischer, Wayne University
N. C. Fisk, U. S. Rubber Products, Inc., Detroit
C. W. Foard, Youngstown College
W. B. Ford, University of Michigan
T. C. Fry, Bell Telephone Laboratories

D. C. Gillespie, Cornell University
J. W. Glover, University of Michigan
Cornelius Gouwens, Iowa State College
L. M. Graves, University of Chicago
V. G. Grove, Michigan State College

J. G. Hardy, Williams College
W. L. Hart, University of Minnesota
M. L. Hartung, University of Wisconsin
Olive C. Hazlett, University of Illinois
E. R. Hedrick, University of California at Los Angeles
H. C. Hicks, Carnegie Institute of Technology
T. H. Hildebrandt, University of Michigan
L. A. Hopkins, University of Michigan
W. A. Hurwitz, Cornell University

M. H. Ingraham, University of Wisconsin

Dunham Jackson, University of Minnesota
R. L. Jeffery, Acadia University
Marie M. Johnson, Oberlin College
E. H. Johnson, University of Detroit
L. S. Johnston, University of Detroit
F. C. Jonah, Western Reserve University
B. W. Jones, Cornell University
Margaret E. Jones, Ohio State University

H. S. Kaltenborn, University of Michigan
L. C. Karpinski, University of Michigan
A. J. Kempner, University of Colorado
W. J. Kirkham, Oregon State College
Elizabeth E. Knight, State Teachers College, Milwaukee, Wis.
F. W. Kokomoor, University of Florida
H. W. Kuhn, Ohio State University

E. P. Lane, University of Chicago
R. E. Langer, University of Wisconsin
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Theodore Lindquist, Michigan State College
C. E. Love, University of Michigan
C. I. Lubin, University of Cincinnati

Roy MacKay, Junior College, Portales, N.M.
H. F. Mac Neish, Brooklyn College
E. D. McCarthy, University of Detroit
W. H. McEwen, Mount Allison University
H. W. March, University of Wisconsin
R. H. Marquis, Ohio University
Ruth G. Mason, Hood College
L. C. Mathewson, Dartmouth College
H. R. Mathias, Bowling Green State University
Sister M. Mercedes, Toledo Teachers College
C. A. Messick, Florida Topographic Mapping Survey, Gainesville, Fla.
A. D. Michal, California Institute of Technology
Norman Miller, Queen's University
E. C. Molina, Bell Telephone Laboratories
Max Morris, Case School of Applied Science
Richard Morris, Rutgers University
E. J. Moulton, Northwestern University
Paul Muehlman, University of Detroit
J. R. Musselman, Western Reserve University

A. L. Nelson, Wayne University
Marie M. Ness, University of Minnesota
J. A. Nyswander, University of Michigan

H. J. O'Connor, Niagara University
Rufus Oldenburger, Armour Institute of Technology
E. G. Olds, Carnegie Institute of Technology
A. L. O'Toole, College of St. Catherine
F. W. Owens, Pennsylvania State College
Mrs. F. W. Owens, State College, Pa.

T. S. Peterson, Shipley School
A. E. Pitcher, Institute for Advanced Study
H. H. Pixley, Wayne University
J. E. Powell, Michigan State College

Tibor Radó, Ohio State University
T. E. Raiford, University of Michigan
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Max Sasuly, Brookings Institution
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R. S. Shaw, College of the City of New York
W. A. Shewhart, Bell Telephone Laboratories
C. J. Shires, Detroit, Mich.
C. Grace Shover, Shipley School
T. M. Simpson, University of Florida
Mary E. Sinclair, Oberlin College
E. R. Sleight, Albion College
G. W. Snedecor, Iowa State College
Virgil Snyder, Cornell University
G. G Speeker, Michigan State College
Vivian E. Spencer, University of Pennsylvania
C. C. Spooner, State Teachers College, Marquette, Mich.

J. D. Tamarkin, Brown University
Mildred E. Taylor, Mary Baldwin College
C. C. Torrance, Case School of Applied Science
P. L. Trump, University of Wisconsin

H. S. Vandiver, University of Texas
H. E. Vaughan, Ogdensburg, N. Y.

C. C. WAGNER, Pennsylvania State College
L. E. WARD, University of Iowa
J. F. WARDWELL, Johns Hopkins University
J. H. WEAVER, Ohio State University
K. W. WEGNER, Whitworth College
E. T. WELMERS, University of Michigan
E. A. WHITMAN, Carnegie Institute of Technology
G. T. WHYBURN, University of Virginia
R. L. WILDER, University of Michigan
K. P. WILLIAMS, Indiana University
C. R. WILSON, Rutgers University
J. B. WINSLOW, University of Toledo
MARGARETE C. WOLF, University of Wisconsin
F. E. WOOD, Northwestern University
F. L. WREN, George Peabody College

The meetings were held in Hutchins Hall of the Law School of the University, the newest building on the campus. The mathematicians stayed in comfortable quarters in the Michigan League and in Betsy Barbour and Helen Newberry dormitories. Social rooms available in each place furnished many opportunities for the delightful group meetings which have so commonly been attractive features of our summer meetings. Professors Ayres and Anning with their assistants on the local committee planned fully and effectively to an extent that greatly enhanced the enjoyment of the guests.

On Monday evening a reception was given the mathematicians and their guests by the University; in the receiving line were Professor and Mrs. Hildebrandt, Dean Kraus, Professor and Mrs. Glover, Secretary Richardson of the Society, President Curtiss of the Association and Professor Rietz representing the statisticians. Tuesday noon a luncheon for ladies was held at Huron Hills Country Club; the visiting ladies were furnished transportation by the ladies of the local group. Thursday afternoon tea was served in Helen Newberry dormitory by the ladies of the department of mathematics of the University. A group photograph was taken Friday afternoon on the steps of Angell Hall.

Wednesday afternoon was devoted to excursions. Some members went to Greenfield Village where Mr. Henry Ford has assembled many exhibits showing materials of the past which are of interest to the present and to posterity. Others chose to make a visit and tour of the Ford factory.

The joint dinner of the two organizations was held Thursday evening in the beautiful ballroom of the Michigan League, under the able toastmastership of Professor Buchanan. Almost two hundred were present at the dinner. Dean Kraus voiced the pleasure of the University authorities in having the mathematicians as their guests for the week, with the inspiration that the lectures and discussions have brought. Professor Rietz explained the newly organized Institute of Statisticians by saying that for a number of years many valuable papers on mathematical statistics could be placed only with difficulty for publication in existing mathematical journals, that the new Institute is enabled now to have the Annals of Mathematical Statistics as a useful organ, that the analysis of statistical data is only slowly developing in the United States and that the Institute should be the agent for devising experiments better adapted to statistical treatment. President Curtiss described the division of activities which have become established between the Society and the Association; many activities are necessary which are not strictly matters of research, and the Association

serves in a complementary way to the Society, serving, for example, as an indirect means for stimulating membership in the Society. Professor Tamarkin held that mathematicians on the one hand have an inferiority complex, are too apologetic and feel too often that they do not deserve recognition, that this need not be true is shown by the ability with which the recent campaign for funds for the Society has been carried through; on the other hand, they sometimes have an unfortunate superiority complex, for it is one thing to develop a correct mathematical paper, and quite another thing to prepare it in a form that is clear and readable, some mathematicians not deigning to descend to the level of those who seek to understand these papers.

The American Mathematical Society held its forty-first summer meeting and eighteenth colloquium from Tuesday to Friday. The colloquium lectures were given by Professor H. S. Vandiver of the University of Texas on "Fermat's last theorem and related topics in number theory." By invitation, Professor G. Y. Rainich of the University of Michigan spoke Wednesday morning on "Remarks on product integrals and their applications to geometry" and Professor G. T. Whyburn gave a lecture Friday morning "On the structure of continua." Sessions of the Society for the reading of papers were held on Tuesday, Thursday and Friday afternoons and on Wednesday, Thursday and Friday mornings. A joint session of the Society with the newly-formed Institute of Statisticians was arranged for Thursday morning, the session considering papers on probability and statistics.

The Mathematical Association held sessions on Monday afternoon, Tuesday morning and Tuesday evening under the presidency of Professor D. R. Curtiss, the chair being occupied by Vice-President Dines for most of the Monday afternoon session and by Vice-President Kempner for the Tuesday morning session. The Program Committee consisting of Professors T. H. Hildebrandt, chairman, R. W. Brink, Tomlinson Fort, A. L. Nelson and P. R. Rider, arranged an admirable program, for which the Association owes them sincere thanks.

First Session of the Association

1. "The confusion in secondary education and its influence on the teaching the mathematics" by Professor K. P. Williams, Indiana University.

2. "Geometric constructions without the classical restriction to ruler and compasses" by Professor W. H. Bussey, University of Minnesota.

3. "On an introduction to higher geometry" by Professor Tibor Radó, Ohio State University.

1. Professor Williams's paper began with a discussion of the development of American secondary education, and set forth the ideas that have governed it from time to time, beginning with the first high school organized in Massachusetts in 1821. It was shown that the early high schools were not primarily devoted to preparing for college, but sought to be truly higher schools for the community with quite a wide offering of studies, contrary to statements fre-

quently made. The work of the Committee of Ten of 1890 was described, followed by that of the Committee of Nine of 1910, the alterations being largely due to the increase in enrollment which had taken place in the meantime. The seven Cardinal Principles of Secondary Education set forth by the Committee of Twenty-Six in 1918 were considered, and it was shown how their very general character differed from previous conceptions, and also made possible quite weak programs of instruction.

The report of the National Committee entitled *Reorganization of Mathematics in Secondary Education* of 1923 was discussed in so far as it set forth the precept that one year of mathematics should be required of all pupils in the high school. This thought was out of harmony with the ideas that were in control of secondary mathematics, and could hardly be expected to win favor.

Certain general aims for mathematical study were set forth and it was argued that they would not be advanced by having mathematics a universally required study at a time when the high schools have come to enroll so large a percentage of youths of high school age. It should have been possible to expand the schools and extend the benefits of education to a wider group without a sacrifice of some of the older values and distinctions, which have been quite eliminated. All persons interested in education should help create a sounder and more substantial philosophy than that frequently announced at the present time. At the same time, however, it is necessary to take account of actual controlling conceptions and the wide group of youths being dealt with, when the position of mathematics in the curriculum is considered.

2, 3. The papers by Professors Bussey and Radó will appear in forthcoming issues of the MONTHLY.

SECOND SESSION OF THE ASSOCIATION

1. "Some recent applications of the theory of elasticity" by Professor H. W. MARCH, University of Wisconsin.

2. "Operational calculus, its applications and foundations" by Dr. HILLEL PORITSKY, The General Electric Company.

3. "Some unsolved problems of topology" by Professor R. L. WILDER, University of Michigan.

1. The question of the elastic stability of a structure has become of increasing importance in the last twenty-five or thirty years because of the desire to secure the greatest economy of material that is consistent with adequate performance. Elastic instability manifests itself in the buckling, wrinkling, or twisting of the members of a structure. In the case of a thin plate under loads acting in the plane of the plate the critical load may be found from a consideration of the differential equation, with appropriate boundary conditions, for the deflection of such a plate, viz.

$$D^4\nabla\omega+P_1\frac{\partial^2\omega}{2x^2}+P_2\frac{\partial^2\omega}{2y^2}+2S\frac{\partial^2\omega}{2x\,2y}=0$$

where D is the flexural rigidity of the plate as determined by its thickness and the elastic constants of its material. P_1 and P_2 are compressive forces acting parallel to the x and y axes, respectively, and S is a shearing force, all these forces being measured per unit length of edge. For example, the thin outstanding flanges of a column subjected to a compressive load may buckle as thin plates.

In the case of each flange of a column whose length is in the direction of the axis of x, the quantities P_2 and S in the foregoing equation are zero while P_1 is the load per unit length of the upper and lower edges of the flange. Considered as a plate the upper and lower edges are simply supported; the outer edge is free, while the remaining inner edge is considered to be clamped, or simply supported, or subject to an intermediate condition depending upon circumstances. These conditions, expressed mathematically, form the boundary conditions of the problem. The critical load is the smallest value of P_1 for which a solution of the differential equation exists that satisfies the boundary conditions and that does not vanish identically. At this critical load the plate will buckle from its plane and the column will fail by the wrinkling of its flanges or by twisting about its vertical axis.

An almost limitless number of problems connected with the fourth order equation written above arise in the determination of the stability of thin plates as they occur in structures in various ways. Any one or all of the quantities P_1, P_2 and S may be different from zero. Further, the boundary conditions may specify that all or some of the edges are simply supported or clamped, or that some of them are free. Of the numerous problems that have thus far been treated many have been solved with satisfactory mathematical rigor. Of the approximate solutions some are based upon reasoning that is mathematically sound or that appears to be capable of being made so while others are much less satisfactory in this respect.

The design of a column with thin outstanding flanges referred to above depends upon a combination of the information obtained from the theory of thin plates with a knowledge of the torsional rigidity of the column, i.e., its resistance to a couple tending to twist it about its vertical axis. The torsional rigidity of a column, when it cannot be readily calculated, may be obtained with the aid of the soap film analogy or with the aid of empirical formulas that have been set up as the result of studies based on the soap film analogy.

2, 3. The papers by Dr. Poritsky and Professor Wilder will appear in later issues of the MONTHLY.

THIRD SESSION OF THE ASSOCIATION

"Early mathematical books in the library of the University of Michigan" by Professor L. C. KARPINSKI, University of Michigan.

This lecture was held Tuesday evening in the General Library of the University. In addition to the many books which were shown during the lecture, numerous rare mathematical books were exhibited in the corridors of the Library and in the Clements Library nearby. Many of the guests of the week

visited this smaller library which contains one of the leading collections in America on American history. Following the lecture, refreshments were served by the ladies of the Library staff.

Professor Karpinski gave the evening lecture on what he called the strange topic, "Logs and tables and why and how people collect books." It was explained by the lecturer that the title was purposely put somewhat vaguely in order to induce a larger attendance. The purpose of the talk was to call attention, through the medium of the specimen topic, mathematical tables and logarithms, to the magnificent collection of early works on mathematics in the University Library, quite certainly the greatest collection in America of mathematical works covering the period from the beginning of printing to 1850.

The earliest document exhibited was a tax-collector's table in Greek but emanating from excavations in Egypt. This document is definitely Graeco-Egyptian of about 100 A.D. This table gives the fractional parts of 1/2, 1/3, 1/4, up to 1/19 of 2 drachmas to 10, then by tens to 100, and by hundreds to 1000, and then by thousands to 6000 drachmas which is the talent. The lecturer stressed that this document corresponds precisely to the tables made necessary by the most recent income and inheritance taxes in the United States. In all mathematical work, even of the most ancient documents of Babylon and Egypt, the connection with modern mathematics and modern problems is direct and intimate. This constitutes the justification for the renewed interest in mathematical studies. This justifies a great university like Michigan for a program devoting funds to the purchase of historical works.

Five rolling trucks of the kind commonly used in libraries were filled with some four hundred volumes, representing only a part of the University's great collection of arithmetical, astronomical, trigonometrical, actuarial and life, and logarithmic tables. This was the material used by the lecturer to illustrate his talk.

Among the tables exhibited were practically all the fundamental trigonometrical tables published before the invention of logarithms, including the Rheticus-Otho tables to ten places of 1596 in the so-called Opus Palatinum, and the 1613 Pitiscus revision of the sines to 15 places.

In 1614, just one year later, John Napier, Baron of Merchiston, published in Edinburgh his work on logarithms, the greatest mathematical labor-saving device published to that time. It is estimated that for any astronomer his effective life was more than doubled since he could by logarithms easily do five times to ten times the computation in the same amount of time. The University of Michigan has some of the approximately forty or more volumes on logarithms published before 1660. The only library in America which approximates this number is Brown University where Dr. Archibald has made a specialty of tables.

The University of Michigan has similarly complete collections in Arithmetic, Algebra, Geometry, Trigonometry, Analytic Geometry and the Calculus.

It was announced that Professor Emeritus W. H. Butts of Michigan, for

37 years a member of the staff, had given 150 rare historical volumes to the University Library. The late Professor W. W. Beman and the late Professor Alexander Ziwet, both widely known, gave large collections to the University. Professor Ziwet gave not only some 15,000 volumes to the library but he left twenty thousand dollars to the University for scientific purposes.

The lecturer stated that the collecting of these books involved his personal visits to more than 500 antiquarian bookshops in Pennsylvania, New York, Maine, Massachusetts, Vermont, in Washington, D.C., and in bookshops in England, Denmark, Germany, France, Czecho-Slovakia, Austria, Italy to Sicily, Spain, Hungary, Switzerland, Belgium, Holland and, in his sabbatical half-year 1933–1934, in Greece, Egypt, Palestine, Syria and Turkey. The collecting involved also searching some 25,000 book catalogues during the lecturer's thirty-one years of service at the University of Michigan.

The "why people collect books" was, in conclusion, given as in effect a kind of mental aberration to which human beings are subject. The lecturer told of one professor who collected hymn-books, another collected prayer-books, another collects dead Indians or their bones, and that the lecturer himself had at various times collected works on Martin Luther and the Protestant Reformation (now in the Rochester Theological Seminary), works on the topography of the Holy Land (now in the Hebrew Union College of Cincinnati), and a large collection (60 boxes, three tons of material) of atlases, geographies, gazetteers and separate maps as well as 600 works on Christopher Columbus, both collections now at Yale University.

Meeting of the Board of Trustees

Eleven members of the Board were present at the Ann Arbor meeting. The following seventeen persons and one institution were elected to membership on applications duly certified:

To Individual Membership

F. A. Alfieri, Audit Clerk, Metropolitan Life Ins. Co., New York, N.Y.; Student, College of the City of New York

H. M. Bacon, Ph.D.(Stanford) Instr., Stanford Univ., Stanford University, Calif.

Kathryn P. Bartlett (Mrs. J. H., Jr.), B.S. (Chicago) 415 Algoma Blvd., Oshkosh, Wis.

H. L. Black, Ph.D.(Illinois) Prof., Westminster Coll., New Wilmington, Pa.

J. F. Calvert, B.S. in E.E. (Missouri); M.S. (Pittsburgh) Electrical Engineer, Westinghouse Elec. and Mfg. Co., Pittsburgh, Pa.

B. G. Clark, Ph.D.(Illinois) Asst. Prof., Univ. of Alabama, University, Ala.

H. W. Emmons, M.S.(Stevens Inst.) Research Asst., Harvard Univ., Cambridge, Mass.

Sister Mary C. Garvin, Ph.D.(St. Louis Univ.) Head of Dept. of Math. and Science, Notre Dame Coll., South Euclid, Ohio

Mrs. Joy Gilder-Claude, B.S.(Columbia) Grad. Student, Columbia Univ., New York, N.Y.

R. H. Marquis, Ph.D.(Chicago) Asso. Prof., Ohio University, Athens, Ohio

Christian Oehler, A.M.(Columbia) Cert. Pub. Accountant with Haskins & Sells, New York, N.Y.; Instr., Math. of Accounting, Fordham Univ.

Walter Penney, Actuarial Clerk, Metropolitan Life Ins. Co., New York, N.Y.

C. J. Shires, 13573 Monica, Aev. Detroit, Mich.

AUGUSTUS SISK, Ph.D.(Cornell) Asst. Prof., Univ. of Tennessee, Knoxville, Tenn.

JOHN TODD, B.S. (Queen's Univ.) Asst. Lecturer, Queen's Univ., Belfast, Northern Ireland

P. L. TRUMP, Ph.D.(Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis.

HENRY VAN ENGEN, Ph.D.(Michigan) Instr., School of Educ., Western Reserve Univ., Cleveland, Ohio.

To Institutional Membership

WILSON COLLEGE, Bombay, India (John McKenzie, Principal)

The Trustees voted to accept the invitation of George Washington University to meet there at the time of the meetings of the American Association for the Advancement of Science in December 1936.

A report of further progress in the work of the Commission on the Training and Utilization of Advanced Students of Mathematics was made by Professor Moulton; and plans for future meetings were considered and will be announced as they are formulated.

W. D. CAIRNS, *Secretary-Treasurer*

ERNEST BROWN SKINNER

By R. E. LANGER, University of Wisconsin

"Bless'd as a man, and as a craftsman bless'd,
He died; . . . "

We who are members in this Mathematical Association, are *ipso facto* united in the purposes and ideals for which the organization stands. Additions to our number strengthen and therefore gratify us, but by the same token we are weakened and own our loss in the departure of those who have filled out their allotted spans of life. This is an acknowledgment of such a loss, an uncommon one, for it concerns one who had endeared himself widely in the mathematical fraternity, and who also markedly personified in his life and in his work the things to which our Association is primarily dedicated;—the promotion of the interests of mathematics especially in the collegiate field;—the production of collegiate mathematical books;—the improvement of the teaching of mathematics. Professor E. B. Skinner, of the University of Wisconsin, died on April 3rd of this year. He was in his seventy-second year, and yet his end came with abruptness, and with only brief warning; perhaps not unfittingly so, for uninterrupted activity had been an outstanding characteristic of his life.

In the minds of those who knew him Professor Skinner is most intimately associated with the State of Wisconsin. This state was his, however, only by adoption, for he was born in Ohio in the town of Redfield, the son of Thomas Peter and Harriet Newell Brown Skinner, on December 12, 1863. Ohio University, at Athens, Ohio, was his Alma Mater. He graduated there in 1888, and after a period of teaching at Amity College he went to Clark University in

1891–92 as a scholar in mathematics. He was a young man of twenty-nine when he came thence to the University of Wisconsin, to begin a period of forty-two years of fine service to that institution, to the cause of the appreciation of mathematics, and generally to teaching as a noble profession. From an initial instructorship he rose by promotion successively to the ranks of assistant, associate and full professor, in the years 1895, 1910 and 1920. In 1910 he received the Ph.D. degree from the University of Chicago, and in 1932 Ohio University honored him and itself by bestowing upon him the degree of LL.D.

It was customary for Professor Skinner to teach a course in "advanced" mathematics, generally in the Theory of Groups or the Theory of Numbers, subjects which retained his keen and active interest to the end. His greatest service, however, was undoubtedly in the elementary collegiate field. In this field he taught, he made teachers, and he wrote texts. He was himself a fine teacher—one who knew how to impart facts with clarity and without confusion, but over and above that, one with the gift for transmitting unobtrusively his own enthusiasms and appreciations. He won from his students not only their unfailing respect for his own person, but their respect for his subject as well. To his colleagues it is a commonplace to hear "old grads" relate, and invariably with pleasure, their reminiscences of days spent in his classes.

For many years Professor Skinner served as the virtual director of the freshman mathematics courses in the College of Letters and Sciences at the University of Wisconsin. This laid upon him the task of supervising normally a considerable number of young instructors and graduate assistants, and of guiding them through their early and often their initial teaching experiences. For this work he was superlatively fitted both by temperament and personality, he seemed to know instinctively in every case the proportions in which sternness and strength needed to be admixed with kindliness and understanding. Many are those who will acknowledge with gratitude their having served in such apprenticeships with him, and through them he will yet teach this many a year. As a textbook author Professor Skinner is known in this field for: *A High School Course in Mathematics* (1910); *The Mathematical Theory of Investment* (1913, and revised in 1924); *College Algebra* (1917); and *Introduction to Trigonometry and Analytic Geometry* (1932). For the second one of these in particular he deserves recognition as a pioneer, for his book was widely used as a model by subsequent authors.

The sources of personal influence, and the subtle essences which distinguish the effective personality from the ineffective are scarcely in any case to be either isolated or enumerated. To account for Professor Skinner's state-wide influence among mathematics teachers it must suffice, therefore, though it be inadequate, to mention concrete facts. For many years he was a member of the University's so called "Committee on High School Relations." It was a part of his work in this connection to make tours throughout the state in the immediate capacity of an inspector of schools, and perhaps more covertly as a general ambassador from the University. To have known him, his integrity, his helpfulness and the

steadiness of his sense of humor, is to preclude in one any doubt of the manner in which he would have filled this rôle. During the years 1900 to 1905 he served as Secretary of the Wisconsin Academy of Sciences, Arts and Letters, and was at the same time editor of the Academy's transactions. In the University itself he was chosen with regularity, even unto the very end, to committee posts of the first importance.

In the year 1919 the people of Madison elected Professor Skinner to the Board of Education, and as chairman of that Board he served the city during the ensuing decade. It was a period in which the acquisition of school sites and the erection of school buildings were matters of urgency which required vision and a statesmanly perspective. No better man for the place could have been found. Finally Professor Skinner was a member—a former chairman—and a steadily active participant in the affairs of the Mathematics section of the Wisconsin Teachers Association. Of late, in particular, he was actively concerned in the matter of the criticism to which mathematics in the high school is currently subjected, and sought earnestly to help in fortifying the morale of its defenders. It was in fact while engaged in correspondence on this matter that he was stricken, late in the afternoon in his office at the University.

Professor Skinner was a member of this Association; of the American Mathematical Society; of the Wisconsin Academy of Sciences, Arts and Letters; of the American Association for the Advancement of Science; of the Wisconsin Teachers Association; of the American Association of University Professors; of Phi Beta Kappa; of Sigma Xi; of Beta Theta Pi; and of Phi Kappa Phi. He was a regular attendant at mathematical meetings, especially at those held in the Middle West. His interest in mathematics was alive, although he never himself devoted his energies to investigation. He was—to use again the simile given us by the late Professor J. W. Young—a member of the *line* in the mathematical football team. Let it be said, moreover, that as such a one he was ever anxious to give most generously his admiration and applause for those who could, and currently did, carry the ball. Many of our mathematical investigators would justly be gratified did they but know the homage he brought them.

Professor Skinner is survived by Mrs. Skinner (Adda C. Coe, of Rue, Ohio), by his son, Merrill Edmund (an engineer), and by his two daughters, Mrs. Helen Harriet McKenzie, and Miss Edith Virginia. His home life was one in which refinement and mutual devotion reigned; one in which he justly took the greatest pride.

Professor Skinner is gone. It is no more than appropriate that this Mathematical Association should acknowledge that he was an exemplary member whose death we mourn.

> "We are such stuff
> As dreams are made on, and our little life
> Is rounded with a sleep."

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the George Washington University, Washington, District of Columbia, on Saturday, May 11, 1935. The Chairman, Professor F. M. Weida of the George Washington University presided over both sessions, morning and afternoon. Four papers were presented at the morning session and one was read by title, while in the afternoon, at the invitation of the Section, Professor J. M. Thomas of Duke University delivered a lecture on "Existence theorems for differential equations."

The attendance was thirty-four, including the following twenty-seven members of the Association: O. S. Adams, C. C. Bramble, Abraham Cohen, Alexander Dillingham, J. A. Duerksen, Almeda J. Garland, Michael Goldberg, T. N. E. Greville, Harry Gwinner, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, C. M. Lennahan, Florence M. Mears, A. W. Richeson, R. E. Root, W. F. Shenton, J. H. Taylor, J. F. Wardwell, F. M. Weida, C. H. Wheeler, G. T. Whyburn, John Williamson, E. W. Woolard, R. T. Zoch.

The following officers were elected for the year 1935–1936: Chairman, Professor G. T. Whyburn, University of Virginia; Secretary, Dr. John Williamson, The Johns Hopkins University; Members of the Executive Committee, Professor C. H. Rawlins, U. S. Naval Academy, and Michael Goldberg, Navy Department.

The fall meeting will be held at College Park, Maryland, on Saturday, December 7, 1935.

The following six papers were read:

1. "A problem in installment finance" by Dr. T. N. E. Greville, Acacia Mutual Life Insurance Company, Washington, D.C.

2. "An elementary method of finding maxima and minima values of a function" by Professor John Tyler, U. S. Naval Academy. (Read by title.)

3. "On a Van der Corput Landau absolute constant" by Richard Kershner, The Johns Hopkins University, introduced by Professor Murnaghan.

4. "The extremals for a class of problems in the calculus of variations" by Almeda J. Garland, Randolph-Macon Woman's College.

5. "Continuous transformations preserving all topological properties" by J. F. Wardwell, The Johns Hopkins University.

6. "Existence theorems for differential equations" by Professor J. M. Thomas, Duke University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Greville presented a solution of the following problem which arises in the mathematical theory of installment financing: An investment of one unit is made in installment plan contracts under which repayments are made

monthly over a period of n months, the first installment being due at the end of one month from the date of the contract. The entire amount received in installment repayments at the end of each month is immediately reinvested in contracts of the same type. Assuming that n is greater than 1, (a) find the total amount of installment repayments received at the end of the p-th month from the date of the original investment; (b) show that the total monthly amount of repayments approaches a limit as p increases without bound, and find the limit. Solutions to both problems (a) and (b) were found by means of difference equations.

2. If two equal ordinates of a curve $y=f(x)$ are drawn near a maximum or minimum point, the ordinate midway between them will be near to the maximum or minimum ordinate. More accurately stated; with proper restrictions on $f(x)$, if x is the abscissa of the maximum or minimum point and x_1 and x_2 satisfy the conditions $x_1<x<x_2$, $f(x_1)=f(x_2)$, then

$$\frac{x-x_1}{x_2-x_1}\rightarrow\frac{1}{2}\quad\text{as}\quad x_2-x_1\rightarrow 0.$$

3. It is known that if $f(x)$ possesses on $[a, b]$ a second derivative $f''(x)\geqq r>0$, then

$$\left|\int_a^b \cos f(x)dx\right|\leqq\frac{\gamma}{\sqrt{r}},$$

where γ is an absolute constant. The best (lowest) possible value of γ is determined by showing that there is an attained maximum for the integral in question; specifically, under the above conditions on $f(x)$,

$$\begin{aligned}\left|\int_a^b \cos f(x)dx\right| &= \max_{-\pi/2\leqq\mu<\pi/2}\left|\int_{-g(\mu)}^{g(\mu)}\cos\left(\frac{rx^2}{2}+\mu\right)dx\right|\\ &= \int_{-g(\mu_0)}^{g(\mu_0)}\cos\left(\frac{rx^2}{2}+\mu_0\right)dx=\frac{\gamma_0}{\sqrt{r}},\end{aligned}$$

where $g(\mu)=\sqrt{\pi-2\mu/r}$, $\mu_0=-0.725\cdots$ and $\gamma_0=3.327\cdots$ is the desired best value of the constant γ above.

4. Miss Garland considered the extremals for the integral

$$I=\int_{x_1}^{x_2} y^{1/n}(1+y'^2)^m dx$$

with $n>0$ and $0<m<1/2$. Given a fixed point $P_0(x_0y_0)$ with $y_0>0$, the problem is to determine the set S of points $P_1(x_1y_1)$ which have $x_1>x_0$, and which can be joined to P_0 by an extremal of I. It is shown that the set S is divided into three distinct regions. The shape of the boundaries of these regions was investigated and the envelope for the extremals in one region was determined.

5. Let A and B be any compact metric spaces and $T(A)=B$ be a continuous transformation; let G_0 denote the collection of all non-degenerate sets of the collection $[T^{-1}(b)]$, for all points b of B. Let G_i denote the collection of all sets of G_{i-1} which intersect $L_{i-1}=\lim\sup G_{i-1}$, for $i=1, 2, 3, \cdots$. If the following conditions are satisfied: (1) for any $\epsilon>0$, any set g of G_0, and any point x of g, there exists a homeomorphism $W(A-x)=A-g$, which is the identity outside of the ϵ-neighborhood of g in A, (2) there exists some number α of the first or second number class such that $G_\alpha=0$, and (3) $\prod_0^\infty L_i$ is a zero-dimensional set, it was shown by Mr. Wardwell that there exists a topological transformation $S(A)=B$.

6. Professor Thomas discussed a method of generalizing a given existence theorem E which is taken as a postulate and which applies to a system of partial differential equations in a canonical form C. The processes employed are algebraic combination and differentiation. The resulting theorem E' applies to other canonical forms C'. The procedure was illustrated by classical examples, such as the deduction of the existence theorem for passive systems of total differential equations from Cauchy's existence theorem for partial differential equations.

JOHN WILLIAMSON, *Secretary*

THE SUMS OF POWERS OF INTEGERS

By E. E. WITMER, University of Pennsylvania

The problem of finding the sums of the powers of the integers from 1 to n has interested mathematicians for a long time. Expressions involving Bernoulli's numbers have been developed for $S_p(n)$ where

$$S_p(n) = 1^p + 2^p + 3^p + \cdots + n^p \tag{1}$$

with p a positive integer, as well as for sums of powers of the odd integers, from 1 to $2n-1$. For a review of the previous work in this field the reader is referred to Bachmann's *Niedere Zahlentheorie*, Second Part, pp. 16 ff., and to Nielsen, *Traité des Nombres de Bernoulli*, Chap. XVI.* As far as the writer is aware, the formulas for $S_p(n)$ and similar sums have always been derived by methods involving Bernoulli's numbers. In the present paper formulas are derived for $S_p(n)$ and related expressions by methods involving nothing more than the binomial theorem. A natural independent variable in terms of which to express $S_p(n)$ is the triangular number $n(n+1)/2\equiv m=S_1(n)$. When p is odd, $S_p(n)=f_p(m)$ and when p is even, $S_p(n)=(2n+1)g_p(m)$, where f_p and g_p are polynomials with rational coefficients of degrees $(p+1)/2$ and $p/2$ respectively.

It is shown, furthermore, that $S_p(n)=F_p(n+1/2)$ where F_p is a polynomial with rational coefficients of degree $p+1$. When p is odd only even powers of

* Cf. also Schwatt, *An Introduction to the Operations with Series*, Ch. 5, Philadelphia, 1924.

$(n+1/2)$ occur in F_p; when p is even only odd powers of $(n+1/2)$ occur, with the exception of $S_0(n)=(n+1/2)-1/2$.

Let

$$R_p(2n-1) = 1^p + 3^p + 5^p + \cdots + (2n-1)^p. \tag{2}$$

It will be shown that

$$R_p(2n-1) = G_p(n),$$

where G_p is a polynomial with rational coefficients of degree $p+1$. When p is odd, G_p contains only even powers of n; when p is even, G_p contains only odd powers of n.

It is easily shown that the following relation holds:

$$R_p(2n-1) = S_p(2n-1) - 2^p S_p(n-1). \tag{2a}$$

We now proceed to establish*

THEOREM I:

$$S_{2p-1}(n) = \sum_{k=2}^{p} A_{pk} m^k, \tag{3}$$

where A_{pk} are rational numbers† *independent of n that satisfy*

$$A_{pp} = \frac{2^{p-1}}{p} \tag{4}$$

and the recursion formula

$$A_{pk} = -\frac{1}{p}\sum_{j=\mu}^{p-1}\binom{p}{2p-2j+1}A_{jk}, \qquad k < p. \tag{5}$$

Here μ is the least value which j can assume in (5) without making the binomial coefficient on the right side of the equation zero. This theorem is valid for all positive integral values of p except 1 in which case $S_1(n)=m$.

Proof.

$$\left[\frac{r(r+1)}{2}\right]^p - \left[\frac{r(r-1)}{2}\right]^p = \frac{2}{2^p}\sum_{k \text{ odd}}^{p}\binom{p}{k}r^{2p-k}$$
$$= \sum_{j=\mu}^{p}\frac{1}{2^{p-1}}\binom{p}{2p-2j+1}r^{2j-1}.$$

Summing for r from 1 to n, we have

* Theorems I, II, III and IV are obtained, with the aid of Bernoulli numbers, in Chap. XVI of Nielsen's *Nombres de Bernoulli*, and theorems V and VI in Chap. I of Bachmann, *Niedere Zahlentheorie*, Second Part, p. 26.

† Equation (3) of course implies that $A_{jk}=0$ for $k>j$. This is also true of the coefficients B_{jk}, C_{jk}, D_{jk}, E_{jk} and F_{jk} which occur in Theorems II, III, IV, V and VI respectively.

$$
(6)\qquad \begin{aligned} m^p &= \sum_{j=\mu}^{p} \frac{1}{2^{p-1}}\binom{p}{2p-2j+1} S_{2j-1}(n) \\ &= \frac{p}{2^{p-1}} S_{2p-1}(n) + \frac{1}{2^{p-1}} \sum_{j=\mu}^{p-1} \binom{p}{2p-2j+1} S_{2j-1}(n). \end{aligned}
$$

Henceforth, the proof rests on mathematical induction. Assume that (3) holds if p is replaced by j, for $j=2, 3, \cdots (p-1)$. Then we shall show that $S_{2p-1}(n)$ also has the form (3), and obtain recursion formulas for the coefficients.

Replacing p by j in (3), substituting the result in (6) and solving for $S_{2p-1}(n)$, we have

$$
(7)\qquad \begin{aligned} S_{2p-1}(n) &= \frac{2^{p-1}}{p} m^p - \frac{1}{p} \sum_{j=\mu}^{p-1} \binom{p}{2p-2j+1} \sum_{k=2}^{j} A_{jk} m^k \\ &= \frac{2^{p-1}}{p} m^p - \frac{1}{p} \sum_{k=2}^{p-1} \sum_{j=\mu}^{p-1} \binom{p}{2p-2j+1} A_{jk} m^k. \end{aligned}
$$

It is seen that $S_{2p-1}(n)$ has the form (3).

Since $S_1(n)$ contains the first power and only the first power of m, it is essential to the proof that $S_1(n)$ shall never occur in (6), for any value of p considered in the proof, i.e., for $p=3, 4 \cdots$. It is easily seen that this condition is fulfilled since even in the most unfavorable case, namely, when $p=3$, the value of μ is 2. Therefore, in equation (6), the sum of lowest order which occurs is $S_{2\mu-1}(n) = S_3(n)$.

Since, now, $S_3(n) = m^2$, a formula which is easily demonstrated, equation (7) permits us to conclude that $S_5(n)$ and hence in general, $S_{2p-1}(n)$, has the form given in equation (3). Theorem I therefore follows by mathematical induction. Formulas (4) and (5) are now obtained by comparison of equations (7) and (3).

THEOREM I(A): *The coefficients A_{pk} can be expressed in the following determinant form:*

$$
(8)\qquad A_{pk} = \frac{(-1)^{p-k} 2^{k-1} (k-1)!}{p!} \Delta_{pk},
$$

where

$$
(9)\qquad \Delta_{pk} = \left| a_{ij}^{pk} \right|,
$$

i and j take on the values $1, 2, 3, \cdots (p-k)$ *and*

$$
(9a)\qquad a_{ij}^{pk} = \binom{p-j+1}{2i-2j+3}.
$$

All of the elements of (9) are zero for $j > i+1$.

This is valid for $k < p$. For $k=p$ the equation (8) gives the correct result if the determinant (9) is arbitrarily assigned the value 1.

Proof. Equations (4) and (5) can be written in the following form:

$$\sum_{j=\mu}^{p}\binom{p}{2p-2j+1}A_{jk} = 2^{p-1}\delta_{pk}. \tag{10}$$

Here

$$\delta_{pk} = 1 \text{ if } p = k$$

$$\delta_{pk} = 0 \text{ if } p \neq k.$$

The A_{pk} can be obtained by solving the following $p-k+1$ equations from the set (10)

$$\begin{aligned}
\binom{p}{1}A_{p,k} + \binom{p}{3}A_{p-1,k} + \binom{p}{5}A_{p-2,k} + \cdots + \binom{p}{2p-2k+1}A_{kk} &= 0\\
\binom{p-1}{1}A_{p-1,k} + \binom{p-1}{3}A_{p-2,k} + \cdots + \binom{p-1}{2p-2k-1}A_{kk} &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\binom{k+2}{1}A_{k+2,k} + \binom{k+2}{3}A_{k+1,k} + \binom{k+2}{5}A_{kk} &= 0\\
\binom{k+1}{1}A_{k+1,k} + \binom{k+1}{3}A_{kk} &= 0\\
\binom{k}{1}A_{kk} &= 2^{k-1}.
\end{aligned} \tag{11}$$

Solving these equations by Cramer's rule and interchanging rows and columns in the determinants, we have the result given in equations (8) and (9).

THEOREM I(B): *The coefficients A_{pk} satisfy the following recursion formula*

$$A_{pk} = -\frac{1}{k}\sum_{i=k+1}^{2k-1} 2^{k-i}A_{pi}\binom{i}{2i-2k+1}. \tag{12}$$

This enables one to compute successively $A_{p,p-1}$, $A_{p,p-2}$, etc., from A_{pp} given by equation (4).

Proof. In formula (3) substitute $n-1$ for n and subtract the result from (3). This gives:

$$n^{2p-1} = \sum_{k=2}^{p}\frac{A_{pk}}{2^k}\left\{n^k\left[(n+1)^k-(n-1)^k\right]\right\}. \tag{13}$$

Expanding and rearranging terms on the right, and equating coefficients of powers of n, we obtain equations (4) and (12).

It is interesting to note that one can also obtain the determinant form (8) and (9) by starting with (12) instead of (10).

THEOREM II:

$$S_{2p}(n) = (2n+1)\sum_{k=1}^{p} B_{pk}m^k, \tag{14}$$

where B_{pk} are rational numbers independent of n that satisfy the relation

$$B_{pp} = \frac{2^{p-1}}{2p+1} \tag{15}$$

and the recursion formula

$$B_{pk} = -\frac{1}{2p+1}\sum_{j=\lambda}^{p-1}\left[2\binom{p}{2p-2j+1}+\binom{p}{2p-2j}\right]B_{jk}, \qquad k < p. \tag{16}$$

Here λ is the least value of j for which the bracket in (16) does not vanish.

In this case, p can have any integral value greater than or equal to 1.

Proof.

$$(2r+1)\left[\frac{r(r+1)}{2}\right]^p - (2r-1)\left[\frac{r(r-1)}{2}\right]^p$$
$$= \frac{1}{2^p}\sum_{j=\lambda}^{p}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]r^{2j}.$$

Summing r from 1 to n, we obtain

$$(2n+1)m^p = \frac{1}{2^p}\sum_{j=\lambda}^{p}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]S_{2j}(n).$$

Solving for $S_{2p}(n)$,

$$\begin{aligned} S_{2p}(n) &= \frac{2^p}{4p+2}(2n+1)m^p \\ &\quad - \frac{1}{4p+2}\sum_{j=\lambda}^{p-1}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]S_{2j}(n). \end{aligned} \tag{17}$$

Assuming (14) to hold if p is replaced by j, for $j=1, 2, 3, \cdots, p-1$, substituting in (17) and reversing the order of summation, we obtain

$$\begin{aligned} S_{2p}(n) &= \frac{2^{p-1}}{2p+1}(2n+1)m^p \\ &\quad - \frac{1}{2p+1}\sum_{k=1}^{p-1}\sum_{j=\lambda}^{p-1}(2n+1)\left[2\binom{p}{2p-2j+1}+\binom{p}{2p-2j}\right]B_{jk}m^k. \end{aligned} \tag{18}$$

It is seen that $S_{2p}(n)$ has the form (14), and by comparison of (18) with (14) formulas (15) and (16) follow. Theorem II, therefore, follows by mathematical induction.

THEOREM II(A): *The coefficients B_{pk} can be expressed in the following determinant form*

$$B_{pk} = \frac{(-1)^{p-k}2^{k-1}D_{pk}}{(2p+1)(2p-1)\cdots(2k+1)}, \tag{19}$$

where

$$D_{pk} = \left| b_{ij}^{pk} \right|, \tag{20}$$

i and j take on the values 1, 2, 3, $\cdots$, $(p-k)$, *and*

$$b_{ij}^{pk} = 2\binom{p-j+1}{2i-2j+3} + \binom{p-j+1}{2i-2j+2}. \tag{21}$$

As in the determinant (9), all of the elements of (20) are zero for $j>i+1$. This is valid for $k<p$. For $k=p$ equation (19) gives the correct result if D_{pk} is arbitrarily set equal to 1.

The proof of this theorem is similar to that of Theorem I(A).

It may be true that the determinants (9) and (20) are always positive, since for all values of p up to and including 5 the coefficients $A_{pp}, A_{p,p-1}, A_{p,p-2}, \cdots$, as well as $B_{pp}, B_{p,p-1}, B_{p,p-2}, \cdots$, alternate in sign as may be seen from Table I. Thus far the writer has not found a proof of this, however.

THEOREM II(B): *The coefficients B_{pk} satisfy the following recursion formula*

$$B_{pk} = -\frac{1}{2k+1}\sum_{i=k+1}^{2k} 2^{k-i}B_{pi}\left[2\binom{i}{2i-2k+1} + \binom{i}{2i-2k}\right]. \tag{22}$$

The proof is similar to that of Theorem I(B).

THEOREM III:

$$S_{2p}(n) = \sum_{k=0}^{p} C_{pk}(n+1/2)^{2k+1}, \tag{23}$$

where C_{pk} are rational numbers independent of n which satisfy the relation

$$C_{pp} = \frac{1}{2p+1}, \tag{24}$$

and the recursion formulas

$$C_{pk} = -\sum_{j=1}^{p-1} \frac{1}{2p+1}\cdot\frac{1}{2^{2p-2j}}\binom{2p+1}{2j}C_{jk}, \qquad 1 \leqq k < p, \tag{25}$$

and

$$C_{p0} = -\sum_{j=1}^{p-1} \frac{1}{2p+1}\cdot\frac{1}{2^{2p-2j}}\binom{2p+1}{2j}C_{j0} - \frac{1}{2p+1}\cdot\frac{1}{2^{2p}}. \tag{26}$$

The case $p=0$ is an exception since we have

$$S_0(n) = (n + 1/2) - 1/2.$$

Proof. We begin with the identity

$$(r + 1/2)^{2p+1} - (r - 1/2)^{2p+1} = \sum_{j=0}^{p} 2\binom{2p+1}{2j} r^{2j}(1/2)^{2p-2j+1}.$$

Proceeding as in Theorems I and II the proof is easily obtained.

THEOREM IV:

$$S_{2p-1}(n) = \sum_{k=0}^{p} D_{pk}(n + 1/2)^{2k}, \tag{27}$$

where D_{pk} are rational numbers independent of n that satisfy

$$D_{pp} = \frac{1}{2p}, \tag{28}$$

the recursion formulas

$$D_{pk} = -\frac{1}{2p}\sum_{j=1}^{p-1} (1/2)^{2p-2j}\binom{2p}{2j-1} D_{jk}, \qquad 1 \leqq k < p, \tag{29}$$

and

$$D_{p0} = -\frac{1}{2p}(1/2)^{2p} - \frac{1}{2p}\sum_{j=1}^{p-1} (1/2)^{2p-2j}\binom{2p}{2j-1} D_{j0}. \tag{30}$$

Proof. Starting with the identity

$$(r + 1/2)^{2p} - (r - 1/2)^{2p} = \sum_{j=1}^{p} 2\binom{2p}{2j-1} r^{2j-1}(1/2)^{2p-2j+1} \tag{31}$$

and proceeding as before, the proof is easily obtained.

THEOREM V:

$$R_{2p-1}(2n - 1) = \sum_{k=1}^{p} E_{pk}n^{2k}, \tag{32}$$

where E_{pk} are rational numbers independent of n that satisfy

$$E_{pp} = \frac{2^{2p-2}}{p}, \tag{33}$$

and the recursion formula

$$E_{pk} = -\frac{1}{2p}\sum_{j=1}^{p-1}\binom{2p}{2j-1} E_{jk}, \qquad k < p. \tag{34}$$

Proof. Starting with the identity

$$\frac{[(2r-1)+1]^{2p}}{2^{2p}} - \frac{[(2r-1)-1]^{2p}}{2^{2p}} = \sum_{j=1}^{p} \frac{1}{2^{2p-1}}\binom{2p}{2j-1}(2r-1)^{2j-1} \tag{35}$$

and proceeding as before, the proof follows by mathematical induction.

THEOREM VI:

$$R_{2p}(2n-1) = \sum_{k=0}^{p} F_{pk} n^{2k+1}, \tag{36}$$

where F_{pk} are rational numbers independent of n that satisfy

$$F_{pp} = \frac{2^{2p}}{2p+1}, \tag{37}$$

and the recursion formula

$$F_{pk} = -\frac{1}{2p+1}\sum_{j=0}^{p-1}\binom{2p+1}{2j}F_{jk}. \tag{38}$$

Proof. The starting point is the identity

$$\frac{[(2r-1)+1]^{2p+1}}{2^{2p+1}} - \frac{[(2r-1)-1]^{2p+1}}{2^{2p+1}} = \sum_{j=0}^{p} \frac{1}{2^{2p}}\binom{2p+1}{2j}(2r-1)^{2j} \tag{39}$$

and the proof is similar to that of the preceding theorems.

In the case of theorems III, IV, V and VI, determinant expressions similar to those obtained in Theorems I(A) and II(A) can be found by the same methods. We will not do this, however, because determinant expressions of that type can easily be obtained from the literature; for each coefficient in these theorems is expressible as the product of an algebraic factor and a Bernoulli number,* and every Bernoulli number can be written as a determinant expression.†

It is to be noted also that theorems analogous to I(B) and II(B) can be established in the case of Theorems III–VI inclusive, by the same methods as were used in proving I(B) and II(B).

N. Nielsen* gives a table of the formulas for $S_p(n)$ in powers of n from $p=1$ to 10 inclusive. We have put $S_p(n)$ into the forms indicated in Theorems I–IV inclusive for $p=0$ to $p=10$ inclusive. These formulas are given in Tables I and II. It will be observed from Table I that when we use the form given in Theorems I and II the coefficients are the ratios of fairly small integers.

In conclusion the writer wishes to express his thanks to the Faculty Research Committee of the University of Pennsylvania which provided funds for an assistant, Dr. A. V. Bushkovitch, who did most of the routine calculations connected with the paper, and to Prof. H. H. Mitchell for discussions and criticisms.

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* Cf. reference 2.

† Pascal, *Determinanten*, pp. 136–138.

TABLE I

$$m = n(n+1)/2$$

$$S_1(n) = m$$

$$S_2(n) = (2n+1)\frac{m}{3}$$

$$S_3(n) = m^2$$

$$S_4(n) = (2n+1)\left(\frac{2}{5}m^2 - \frac{1}{15}m\right)$$

$$S_5(n) = \frac{4}{3}m^3 - \frac{1}{3}m^2$$

$$S_6(n) = (2n+1)\left(\frac{4}{7}m^3 - \frac{2}{7}m^2 + \frac{1}{21}m\right)$$

$$S_7(n) = 2m^4 - \frac{4}{3}m^3 + \frac{1}{3}m^2$$

$$S_8(n) = (2n+1)\left(\frac{8}{9}m^4 - \frac{8}{9}m^3 + \frac{2}{5}m^2 - \frac{1}{15}m\right)$$

$$S_9(n) = \frac{16}{5}m^5 - 4m^4 + \frac{12}{5}m^3 - \frac{3}{5}m^2$$

$$S_{10}(n) = (2n+1)\left(\frac{16}{11}m^5 - \frac{80}{33}m^4 + \frac{68}{33}m^3 - \frac{10}{11}m^2 + \frac{5}{33}m\right)$$

TABLE II

$$S_0(n) = (n+1/2) - \frac{1}{2}$$

$$S_1(n) = \frac{1}{2}(n+1/2)^2 - \frac{1}{4}$$

$$S_2(n) = \frac{1}{3}(n+1/2)^3 - \frac{1}{12}(n+1/2)$$

$$S_3(n) = \frac{1}{4}(n+1/2)^4 - \frac{1}{8}(n+1/2)^2 + \frac{1}{64}$$

$$S_4(n) = \frac{1}{5}(n+1/2)^5 - \frac{1}{6}(n+1/2)^3 + \frac{7}{240}(n+1/2)$$

$$S_5(n) = \frac{1}{6}(n+1/2)^6 - \frac{5}{24}(n+1/2)^4 + \frac{7}{96}(n+1/2)^2 - \frac{1}{128}$$

$$S_6(n) = \frac{1}{7}(n+1/2)^7 - \frac{1}{4}(n+1/2)^5 + \frac{7}{48}(n+1/2)^3 - \frac{31}{1344}(n+1/2)$$

$$S_7(n) = \frac{1}{8}(n+1/2)^8 - \frac{7}{24}(n+1/2)^6 + \frac{49}{192}(n+1/2)^4 - \frac{31}{384}(n+1/2)^2 + \frac{51}{6144}$$

$$S_8(n) = \frac{1}{9}(n+1/2)^9 - \frac{1}{3}\cdot(n+1/2)^7 + \frac{49}{120}\cdot(n+1/2)^5 - \frac{31}{144}(n+1/2)^3 + \frac{127}{3840}(n+1/2)$$

$$S_9(n) = \frac{1}{10}(n+1/2)^{10} - \frac{15}{40}(n+1/2)^8 + \frac{49}{80}(n+1/2)^6 - \frac{155}{320}(n+1/2)^4 + \frac{381}{2560}(n+1/2)^2 - \frac{31}{2048}$$

$$S_{10}(n) = \frac{1}{11}(n+1/2)^{11} - \frac{5}{12}(n+1/2)^9 + \frac{7}{8}(n+1/2)^7 - \frac{31}{32}(n+1/2)^5 + \frac{127}{256}(n+1/2)^3 - \frac{2555}{33792}(n+1/2)$$

THE CONFIGURATION OF SIX POINTS OF THE PLANE

By B. G. CLARK, University of Illinois

I. *Introduction.* Based on a discussion of associated sets given in earlier papers by Professor Coble,* we obtain here a number of associated and projective sets of elements in the plane; and using these we derive many incidence properties of the configurations generated by six and also by five generic points of the plane. Then by specializing the results of the generic six-point we obtain some very interesting and apparently new properties of the Pascal figure.

II. *Associated and projective sets of elements.* Consider the two arrays

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \quad \text{and} \quad \begin{array}{ccc} -A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & -A \\ A_{11} & A_{12} & A_{13}, \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array}$$

where A denotes the determinant $|a_{ij}|$ and A_{ij} the cofactor of a_{ij} in A. Since each column of the one array has a zero product by each column of the other, we have here two associated sets of elements. But if we interpret the rows of the first array as coordinates of six points of the plane, the second array becomes the coordinates of six lines of the plane, whose relation to the points is expressed in†

THEOREM I. *If a set of six points of the plane is divided into two triangles, the coordinates of the six sides of the two triangles are associated with the coordinates of the six vertices.*

Thus, for six points a, b, c, d, e, f, of the plane we obtain the sets

$$\begin{array}{lcccccc} P: & a & b & c & d & e & f \\ L: & bc & ac & ab & ef & df & de \\ P_1: & ac,\, ef & bc,\, ef & d & c & de,\, ab & df,\, ab, \end{array}$$

where ac, ef denotes the point of intersection of the line ac with the line ef. The set of points P_1 makes up the vertices of two triangles whose sides are the same lines L as the sides of two triangles having the set P for vertices. Since the sets P and P_1 are each associated with the set L, they are projective to each other.

* A. B. Coble, *Point sets and allied Cremona groups I*, Transactions of the American Mathematical Society, vol. 16 (1915), pp. 155–198, §1, 2; *Associated sets of points*, Transactions of the American Mathematical Society, vol. 24 (1922), pp. 1–20, introduction; and *Hyperelliptic functions and irrational binary invariants, III*, American Journal of Mathematics, vol. 55, (1933), pp. 349–375, §11.

† Cf. Coble, Hyperelliptic functions, etc., loc. cit., p. 351, (5).

There are 90 sets of type P_1 since it is unaltered by the permutations (ab), (ef), and (ae) (bf) (cd). Hence

THEOREM II. *Let the six points a, b, c, d, e, f of a plane be divided into two triangles as (abc) (def); if from c we project a and b upon ef to get a', b' and from d project e and f upon ab to get e', f', then a, b, $\cdots$, f are projective in order to a' b', d, c, e', f'.*

By elementary principles of projection theorem II may be shown valid also for five points, or the case where the two centers of projection have been allowed to coalesce. That is, the sets

$$\begin{array}{llllll} Q: & a & b & c & d & e \\ Q_1: & a & ab,\, de & ac,\, de & ad,\, bc & ae,\, bc \end{array}$$

are projective. There are 15 sets of type Q_1 since the set is unaltered by the permutations (bc), (de), and (bd) (ce). Thus follows

THEOREM III. *If five points of a plane p_1, $\cdots$, p_5 are divided into two triangles having a common vertex p_i, a projectivity exists which leaves p_i unaltered and sends p_j into p_ip_j, p_kp_l, where p_j is a vertex of the triangle not containing p_k or p_l.*

III. *Some incidences in the configuration P_6^2.* In discussing six points of the plane we shall employ the symbol (a_0bc) (def_0) to denote that particular one of the 90 six-points projective to a, b, $\cdots$, f, which is the result of applying theorem II in such manner that a and f are interchanged, b and c projected from a upon de, and d and e projected from f upon bc. Thus we have such sets as

$$\begin{array}{lllllll} (a_0bc)(def_0) \equiv f & ab,\, de & ac,\, de & bc,\, df & bc,\, ef & a \\ (a_0bf)(cd_0e) \equiv d & ab,\, ce & bf,\, cd & a & bf,\, de & af,\, ce. \end{array}$$

These 90 sets of six points, being all projective to the original six points, are projective to each other. Isolating a pair of corresponding points in any two sets as centers of line pencils, we thus form two projective pencils of lines which generate a point conic. In this way are obtained 26 types of sets of six or seven points which lie on proper conics. Of these 26 conics 16 are on seven points and ten are on six points; 16 belong to the identity subgroup of $G_{6!}$ and 10 belong to subgroups of order 2. However, there is no relation between these two divisions of the 26 conics. Samples of a few of these 26 conics are given below. We use $(f; ab, de{:}ad, bf; ae, bc)$ to denote the point of intersection of the line joining f to (ab, de) with the line joining (ad, bf) to (ae, bc) etc.

[1]: $a, c, d, (af, ce), (bc, df), (ab: d; bf, ce), (de: a; bc, ef)$.

[2]: $d, e, f, (af, ce), (cd: f; ac, de), (f; ab, de: d; ab, ce)$.

[3]: $a, d, (ab, de), (ae, bc), (bc, df), (f; ab, de: d; ab, ce), (ab, de; bc, ef: ae, bc; bf, de)$.

. . . .

[26]: (ab, de), (bf, ce), $(ab:d; bf, ce)$, $(ce:f; ab, de)$, $(bf: ab, de; bc, df)$, $(de: ab, cd; bf, ce)$.

The sets [1] and [3] can easily be shown to lie on conics by using the converse of the Pascal theorem along with theorem IV, which is developed below. But it is by no means apparent that the other sets lie on conics.

The most interesting results, however, are obtained either by considering those projectivities which yield degenerate conics or by applying the Pascal theorem to [3] using various orderings of these points.

Here we shall call the six original points *fundamental points*, the 15 lines joining these six points *fundamental lines*, the 180 lines p_ip_j, p_kp_l; p_ip_m, p_kp_n *diagonal lines*, and the 45 points (p_ip_j, p_kp_l) *diagonal points*.

Using the same procedure as that by which the proper conics were obtained, we find from the sets (a_0bc) (def_0) and (a_0de) (bcf_0) that the line af contains the four points

$$(ab, de; bc, df: fb, de; bc, da),\ (ab, de; bc, ef: fb, de; bc, ea),$$
$$(ac, de; bc, df: fc, de; bc, da),\ (ac, de; bc, ef: fc, de; bc, ea);$$

this set is symmetric in b and c, and the points are individually symmetric in a and f. Hence there must be twelve points of this sort on the line af. The other two sets of four points on af may be given by applying the permutations (bd) and (cd), respectively, to the four points given above. That is,

THEOREM IV. *The* 180 *diagonal lines arising from six points of the plane meet in pairs in* 180 *points r which lie by* 12*'s on the* 15 *fundamental lines.*

Or, *the* 24 *possible diagonal lines which join diagonal points of opposite sides of a complete quadrangle, whose vertices are four of the given six points of the plane, meet in pairs in* 12 *points on the line joining the two excluded points of the original six.*

If the six fundamental points are on a conic these 12 points come together in pairs to give the six Pascal points which lie on each of the 15 fundamental lines.*

Similarly using (a_0bc) (def_0) with (a_0cd) (bef_0), we have as points of a line

$$f,\ (de: ac, be; bf, cd),\ (be: ac, de; bc, df),\ (ac, de; bc, ef: ac, be; cd, ef).$$

By theorem IV these points may be written

$$f,\ (ac, be; bf, cd: ac, bd; bf, ce),\ (ac, de; bc, df; ac, bd; ce, df),\ (ac, de; bc, ef: ac, be; cd, ef).$$

Since the set is symmetric in b, c, d there are 120 lines of this type. We write then

THEOREM V. *The* 180 *r-points lie by* 3*'s on* 120 *lines R which pass by* 20*'s through the six fundamental points. The r-points and R-lines thus form a* 180_2, 120_3 *configuration.*

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* Friedrich Levi, *Geometrische Konfigurationen*, S. Hirzel, Leipzig, 1929; p. 181, theorem 1.

Again, in case the six original points are on a conic these 120 lines coincide in sets of eight to give merely the 15 fundamental lines.

We now apply Pascal's theorem to the hexagon 147352 of conic [3], where the numbers refer to the seven points in the order first given in [3]; that is, 1 denotes a, 3 denotes (ab, de) etc. From this we obtain as collinear points

$$S_1 \equiv (ae\colon ab, de; bc, df), (ad\colon ab, de; bc, ef), (df\colon ae, bc; bf, de), (ef\colon ad, bc; bf, de),$$

symmetry adding the last point to the first three. The first point is unaltered by the permutation (ae) (bd) (cf), and by no other. There are then 360 points of the type of these four points. Also the line S_1 is unaltered by the permutations (af) and (de); the number of lines S is then 180. Thus follows

THEOREM VI. *The* 180 *diagonal lines meet the* 15 *fundamental lines in* 360 *points s which lie by* 4's *on* 180 *lines S.*

For the Pascal figure these 360 points coincide in pairs; for example, the point $(ae{:}ab, de; bc, df)$ is the same as $(ae{:}af, ce; bc, df)$. The line S_1 may then be written

$$V_1 \equiv (ae\colon ab, de; af, ce), (ad\colon ab, de; af, cd), (df\colon bf, de; af, cd), (ef\colon bf, de; af, ce),$$

and another line of the same type would be

$$V_2 \equiv (ae\colon ac, de; af, be), (ad\colon ac, de; af, bd), (df\colon cf, de; af, bd), (ef\colon cf, de; af, be).$$

Since V_1 belongs to the group of order 8 generated by $(adfe)$ (bc) and (af), there are 90 lines V. The points of V_1 and V_2 are the intersections of the eight possible Pascal lines which contain Pascal points of both af and de with the other sides of the complete quadrangle $adef$. One easily verifies analytically that V_1 and V_2 intersect on the line $ad, ef; ae, df$, a diagonal line of the complete quadrangle $adef$. We thus have

THEOREM VII. *The eight possible Pascal lines which join Pascal points of two opposite sides of a complete quadrangle, whose vertices are four of the six points on a conic, meet the other four sides in eight points v which lie by* 4's *on two V-lines. There are* 90 *such V-lines, meeting in pairs on the* 45 *diagonal lines of the* 15 *complete quadrangles, and* 180 *v-points.*

Using the points of [3] in the order 137452 we show in a similar way that a line T_1 contains the points

$$T_1 \equiv b, (ad\colon ae, bc; bf, de), (df\colon ab, de; bc, ef), (ac\colon ae, bd; bf, ce), (cf\colon ab, ce; bd; ef),$$

symmetry requiring the last two to be added to the first three obtained by Pascal's theorem. Since T_1 is unaltered by the permutations (af) and $(adfc)$, there are 90 T-lines. Then follows

THEOREM VIII. *The* 180 *diagonal lines meet the* 15 *fundamental lines in* 360 *points t lying by* 4*'s on* 90 *lines T, which pass by* 15*'s through the six original points.*

If the six fundamental points are on a conic the four lines $(ae, bc; bf, de)$, $(ab, de; bc, ef)$, $(ae, bd; bf, ce)$, $(ab, ce; bd, ef)$, mentioned above become the four Pascal lines which pass through the point (af, cd). This interesting and apparently hitherto unnoticed property of the Pascal figure is embodied in

THEOREM IX. *The four Pascal lines on any diagonal point of a complete quadrangle, formed by taking four of the six points of a conic, intersect the other four sides in eight w-points, lying by* 4*'s on two W-lines, each W-line passing through one of the two remaining points of the original six.*

These 90 W-lines divide into 45 pairs, each pair arising from using a particular Pascal point, called therefore their *related point.*

Since the 180 v-points and the 360 w-points are distinct, and since $540=9\cdot 60$ is the total number of such intersections, aside from the Pascal points themselves, we have established incidence properties for all such points of the Pascal figure.

By use of Desargue's theorem on perspective triangles and other elementary principles we readily establish

THEOREM X. *Two W-lines meet on the side of the diagonal triangle which is opposite their related point. Each of the* 45 *pairs of W-lines meet the conic in two further points, one lying on each of the two lines which join the two original points of the W-lines to their related point. And the junction of these further points meets the junction of the two original points of the W-lines in the side of the diagonal triangle opposite their related point.*

IV. *The configuration of* P_5^2. Part of the incidence properties of the configuration of five points of the plane are obtained in a manner similar to the treatment of six points, and part are taken as mere analogues of the properties for six points and then are verified analytically, or otherwise. We shall give here only the results.

Similar to the 26 proper conics in the case of six original points we obtain in the case of five points only 2 distinct types of proper conics. Each conic is on six points, and there are 60 of each type.

We shall, in discussing five points, speak of the 15 points p_ip_j, p_kp_l as *diagonal points* and the 60 lines p_ip_j, p_kp_l; p_ip_k, p_jp_m as *diagonal lines.*

Corresponding to theorem IV we have

THEOREM XI. *The* 60 *diagonal lines arising from five points of the plane meet in pairs in* 30 *points* r' *which lie by* 3*'s on the* 10 *lines of the complete five-point.*

Or, *if five points are divided to form the vertices of two complete quadrangles having three vertices in common, the six diagonal lines which join diagonal points of the*

one complete quadrangle to the diagonal points of the other meet in pairs on the line joining those two original points which are not common to the two quadrangles.

Similar to other theorems of section III we may add:

THEOREM XII. *The thirty points r' lie by 4's on 15 lines R', which pass by 3's through each of the original five points.*

THEOREM XIII. *The thirty points r' also lie by 3's on 20 lines S'.*

THEOREM XIV. *In addition to the 30 points r' the 60 diagonal lines meet the 10 fundamental lines in 60 points t' which lie by 4's on 15 lines T', one passing through each of the 15 diagonal points.*

But theorems XII and XIV may be combined into the more significant

THEOREM XV. *The four possible diagonal lines which join the diagonal points of any two opposite sides of a complete quadrangle, formed from four of a given set of five points, meet the other two pairs of opposite sides in eight points which lie by 4's on two lines R' and T', the first going also through the excluded point of the original five, and the other through a diagonal point of the complete quadrangle.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE COMPONENTS OF VELOCITY AND ACCELERATION

By B. C. PATTERSON, Hamilton College

1. *Introduction.* The following derivation of the formulas for the components of velocity and acceleration in the plane seems to possess certain advantages because of its brevity and simplicity. Frequently one wishes to make use of these formulas (in discussing certain applications of Differential Equations, for example) without taking the time from regular class work to derive them in the customary manner. The underlying idea is that of vector analysis but the notation is that of complex numbers and apparently offers no great difficulty to the undergraduate.

Let (x, y) be the rectangular and (r, θ) the polar coordinates of a point P in plane motion. The vector or complex number $z=x+iy$ is the complex coordinate of P, and

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = rt$$

is its polar form. In the latter r is the modulus or magnitude of z, and t is the direction factor of z. The direction factor t is a complex number of unit modulus. The coordinates of P are functions of time T, a real variable.

2. *Radial and Transverse Components.* Indicating differentiation with respect to time T by dots placed over the dependent variable, we differentiate

$$z = rt$$

with respect to T, which gives

$$\dot{z} = \dot{r}t + r\dot{t},$$

or, since $dt/dT = ie^{i\theta}\dot{\theta} = it\dot{\theta}$,

$$\dot{z} = \dot{r}t + itr\dot{\theta}. \tag{1}$$

This last equation gives the velocity vector $\dot{z}$ as the sum of the two vectors

$\dot{r}t$, with magnitude $\dot{r}$ and direction factor t, and
$itr\dot{\theta}$, with magnitude $r\dot{\theta}$ and direction factor it.

The direction factor t is the direction of the radius vector of P, and it is the direction perpendicular to that radius vector, as is evident from the equation

$$it = ie^{i\theta} = e^{i(\theta+\pi/2)}.$$

Hence the velocity vector $\dot{z}$ has

radial component, $\dot{r}$,
transverse component, $r\dot{\theta}$.

Differentiating (1) with respect to T, we have

$$\ddot{z} = (\ddot{r} - r\dot{\theta}^2)t + i(2\dot{r}\dot{\theta} + r\ddot{\theta})t,$$

thus exhibiting the acceleration vector $\ddot{z}$ as the sum of the

radial component, $\ddot{r} - r\dot{\theta}^2$,
and the transverse component, $2\dot{r}\dot{\theta} + r\ddot{\theta}$.

3. *Tangential and Normal Components.* As to the tangential and normal components of velocity and acceleration we first recall that

$$dx = \cos\phi ds$$
$$dy = \sin\phi ds,$$

where s represents the arc length and ϕ the inclination angle of the tangent at P. Moreover $ds/d\phi = \rho$, the curvature at P. We have then

$$\dot{z} = \dot{x} + i\dot{y}$$
$$\dot{z} = (\cos\phi + i\sin\phi)\dot{s} = \dot{s}e^{i\phi}, \tag{2}$$

i.e., the velocity vector $\dot{z}$ has the magnitude $\dot{s}$ and direction factor $e^{i\phi}$ and is directed along the tangent at P. Or in other words, the velocity vector $\dot{z}$ has the

tangential component, $\dot{s}$,
and the normal component is zero.

Differentiating (2) with respect to T, we have

$$\ddot{z} = \ddot{s}e^{i\phi} + i\dot{s}\dot{\phi}e^{i\phi}$$

or

$$\ddot{z} = \ddot{s}e^{i\phi} + ie^{i\phi}\dot{s}^2/\rho,$$

since

$$\dot{\phi} = \dot{s}d\phi/ds = \dot{s}/\rho.$$

The acceleration vector $\ddot{z}$ is here given as the sum of two vectors, the first directed along the tangent and the second along the normal; or, $\ddot{z}$ has the

tangential component, $\ddot{s}$,
and the normal component, $\dot{s}^2/\rho$.

4. *Other Relations.* Other relations between the various components may be deduced from the foregoing. For example, with regard to the velocity $\dot{z}$ we obtain from equations (1) and (2),

$$(\dot{r} + ir\dot{\theta})e^{i\theta} = \dot{s}e^{i\phi},$$

and therefore

$$(\dot{r} + ir\dot{\theta})e^{-i\psi} = \dot{s}, \tag{3}$$

where $\psi = \phi - \theta$ is the angle between the radius vector to P and the tangent at P. Hence, after writing

$$e^{-i\psi} = \cos\psi - i\sin\psi$$

and equating the real and imaginary parts of (3),

$$\dot{s} = \dot{r}\cos\psi + r\dot{\theta}\sin\psi$$
$$0 = -\dot{r}\sin\psi + r\dot{\theta}\cos\psi.$$

The first of these equations gives the tangential velocity component in terms of the radial and transverse velocity components and the angle ψ; and the second equations leads us at once to the familiar formula

$$\tan\psi = r\dot{\theta}/\dot{r},$$

or

$$\tan\psi = rd\theta/dr.$$

With regard to the acceleration vector $\ddot{z}$ we may begin, for example, with

$$\ddot{x} + i\ddot{y} = \ddot{s}e^{i\phi} + ie^{i\phi}\dot{s}^2/\rho,$$

whence

$$(\ddot{x} + i\ddot{y})(\cos\phi - i\sin\phi) = \ddot{s} + i\dot{s}^2/\rho.$$

Equating real and imaginary parts of this equation,

$$\ddot{s} = \ddot{x} \cos \phi + \ddot{y} \sin \phi$$
$$\dot{s}^2/\rho = - \ddot{x} \sin \phi + \ddot{y} \cos \phi,$$

we have the tangential and normal components of acceleration in terms of the horizontal and vertical components and the inclination angle ϕ of the tangent. Continuing in this way, relations between the various types of components can easily be found.

Concerning Sequences of Integral Right Triangles

By E. C. Kennedy, The Rice Institute

There is no integral right triangle similar to the $1:2:\sqrt{5}$ right triangle. It is possible however to find sets of integers which approach this triangle to within any desired degree of accuracy. To do this we consider the right triangle with sides X, Y, Z (Z the hypotenuse). Let

$$X = 2mn$$
$$Y = m^2 - n^2$$
$$Z = m^2 + n^2.$$

Since $\lim Y/X = 1/2$ we write

$$\frac{m^2 - n^2}{2mn} = \frac{1}{2}$$

and obtain

$$m = n(1 + \sqrt{5})/2 = n(1.61803399 \cdots).$$

We observe that if $n=5$, $m=8.09$—almost an integer. Using $n=8$ we find $m=12.94$, then using $n=13$ we get $m=21.03$ and so on. We obtain very quickly the sets of values:

$$(5, 8), (8, 13), (13, 21), \cdots, (10946, 17711) \cdots.$$

These values of n and m become increasingly accurate. Thus the last set tabulated (no. 18 in the series) gives the right triangle

253,772,064:507,544,127:567,451,585,

in which $Y/X = .5000000009$.

It is interesting to note that we could have started with $n=11$, $m=18$ and obtained an entirely different set of values: (11, 18), (18, 29), $\cdots$, (9349, 15127) $\cdots$. The last set tabulated yields the right triangle

370,248,447:740,496,904:827,900,705,

in which $Y/X = .499999993$. However, the first set of triangles given above is the "best" of all such sets.

The connection between our first sequence and continued fractions should be noted. If we expand $(1+\sqrt{5})/2$ into a continued fraction we see that the successive convergents,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots,$$

are precisely the successive values of m/n used, starting with the fifth fraction. The numerators and denominators of such successive convergents may be formed by simple recursion formulas; in this particular case each numerator (after the second) is the sum of the two immediately preceding numerators, and similarly for the denominators.

In a letter to the writer Professor Carver states that "It is of interest to note that the condition

$$\frac{m^2 - n^2}{2mn} = \frac{1}{2}$$

is equivalent to

$$\frac{m}{n} = \frac{m+n}{m}$$

which indicates that the ratio m/n is the so-called 'golden section' or 'mean and extreme ratio.' This means that n and m should be integers as nearly as possible in the ratio of the side of a regular decagon to the radius of its circumscribed circle."

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

The Poetry of Mathematics and Other Essays. By David Eugene Smith. *The Scripta Mathematica Library,* No. 1. New York, Scripta Mathematica, Yeshiva College, 1934. iv+90 pages. 75 cents.

This compact collection of charming essays by Professor Smith is the first volume of a series announced by *Scripta Mathematica* "designed to furnish, at a nominal price, material which will interest not only teachers of mathematics but all who recall their contact with the subject in their school or college days." If succeeding volumes maintain the high standard of interest and accuracy—without professorial pedantry—set by this, the initial volume, the success of the series is assured and will be well merited.

The five essays in the present volume are revisions and, in some instances, abridgements of articles previously published elsewhere. Their collection here is a welcome gift to all who believe mathematics to be more than the "mere mathematics" of some pedants who know little and care less about the evolution of their subject and its place in a modern civilized society.

Although it may not be evident at the first glance, these essays, addressed ostensibly to teachers, students, and enquiring laymen, have a very practical interest for those American mathematicians whose primary interest is in research. For it must now be obvious, even to a blind imbecile, that American mathematics and mathematicians are beginning to get their due share of those withering criticisms, motivated by a drastic revaluation of all our ideals and institutions, from the pursuit of truth for truth's sake to democratic government, which are only the first, mild zephyrs of the storm that is about to overwhelm us all. In the coming tempest only those things will be left standing that have something of demonstrable social importance to stand on. Mathematics, we as mathematicians believe, has so much of enduring worth to offer humanity on all sides from the severely practical to the ethereally cultural or spiritual, that we feel secure—until we stop to think.

The arresting thought that we as mathematicians have done next to nothing to inform and convince the sweating men and sweated women, whose hard labor makes possible our own leisurely pursuit of "the science divine," that mathematics does mean something in their lives and might mean much more, may well make us apprehensive of the future, for these too patient men and women in the storm ahead of us all will cast the deciding vote.

The harsh attrition has already begun. Are not mathematicians and teachers of mathematics in liberal America today facing the bitterest struggle for their continued existence in the history of our Republic? American mathematics is exactly where, by common social justice, it should be—in harassed retreat, fighting a desperate rear-guard action to ward off annihilation. Until something more substantial than has yet been exhibited, both practical and spiritual, is shown the non-mathematical public as a justification for its continued support of mathematics and mathematicians, both the subject and its cultivators will have only themselves to thank if our immediate successors exterminate both.

Taking a realistic view of the facts, anyone but an indurated bigot must admit that mathematics has not yet made out a compelling case for democratic support, so that the men and women who pay the bills which make mathematics possible can see clearly what they are asked to pay for. This must be done, and immediately, if mathematics is to survive in America.

In view of this desperate situation the publication of a volume of essays like Professor Smith's acquainting laymen with something of the potential worth of mathematics in any modern civilization is an event of social importance. These essays are at least a step in the right direction, and a long one, but, after all, only a step. However, as it is the first step which counts the most, our successors in America will probably look back on Professor Smith (and *Scripta*

Mathematica) as the pioneers in a vigorous movement to preserve American mathematics in the middle 30's from the savage assaults of a mob of influential haters of mathematics who, although high in our councils of education, have not the remotest conception of what even elementary mathematics accomplishes for a civilized society.

Were it not tragic it would be ludicrous that the public should accept as final the pronouncements of mathematical ignoramuses on the social value (or alleged lack of value) of mathematics. But cursing "the enemy" never did him one jot of harm, and it is our business as mathematicians to follow Professor Smith's lead, keep our tempers and our reason cool, and *convince* our detractors that they are grievously mistaken. They also, we believe, are honorable men who need only to be shown the justice of a position to admit that it is indeed just. We have not yet shown them. The fault is our own.

What might be a next step along the trail Professor Smith has so brilliantly blazed? The following suggestions are intended to be helpful, not carpingly critical. To the reviewer it seems obvious that *Scripta* will not accomplish its splendid purpose on any such road as it is now following. The material is excellent and admirably suited to its purpose; there can be no doubt about that. But *the price is prohibitive.* These are not essays for the delight of cultured men in comfortable studies—although any lover or appreciative student of mathematics will revel in them; they are messages to "the common man," in a language which he can understand, and by "common man" we mean that huge majority which hungers for knowledge (it *does*, in spite of superficial cynics—ask any astute publisher), but which has never been able to afford either a study or books of its own. The price is said to be "nominal," and it undoubtedly is: 75 cents for "not less than 96 pages, in handsome blue cloth binding, 50 cents in heavy grey paper cover." But what man or woman on a minimum wage scale can afford 75 cents, or even 50 cents, for a book which should be possessed and reread many times? The price should be five cents, or at most ten cents. If the parents don't read the pamphlets, their children will.

This is not as fantastic as it sounds. In the present attractive format it is of course out of the question to sell the books below their present price. But why maintain the format? Why not follow the lead of the ever-memorable Haldeman-Julius who put philosophy on the map of America—and also, somewhat ludicrously, into the American public consciousness—with his excellent five-cent reprints, on the cheapest paper available, of the classics? *Five* essays need not be put out under *one* cheap paper cover; *one* carefully worked-out essay is enough for five cents or even for ten cents. Authors need not be paid (unless their pamphlets run to the millions of copies, when they might be offered a bonus, which they should graciously decline): competent men surely would be happy to do a week's or a month's work gratis to sustain the sorely pressed science which gives them their means of livelihood and which they believe to be essential to the continued existence of civilization. To vary the mathematical monotony, essays on the mathematical significance of the sciences—or the other

way about, whichever it may be—might be interspersed in the ratio of about one to four. Consider, for example, what a fascinating story might be woven (without pedantry or technicalities) around Adams, Leverrier and the planet Neptune, or around Hamilton and conical refraction, or around Clerk Maxwell, Hertz, and wireless, or around Heisenberg and matrices. Finally, all this should be handled by some boldly progressive publishing firm, with *Scripta* in the all-important background as the selector, coordinator and mathematical expert guaranteeing reasonable scientific and mathematical accuracy.

One curse of "popular science" (which Professor Smith avoids, either deftly or by instinct, for he is that sort of a man) is the self-conscious pedantry of authors who imagine they must maintain their scientific punditry unimpaired when trying to tell a boy of sixteen what the quantum theory is all about. Meticulous accuracy is bad taste and is out of place in such work. A man who knows what he is about in this sort of thing scorns the stupid contempt of his ant-eyed colleagues. After all, almost any mathematical or scientific doctrine, even the most rigid, can be picked to pieces and shown up as fraudulent by any reasonably intelligent man with decent standards of rational skepticism. So those purists who insist upon the last epsilon and its penultimate delta in *popular* expositions are themselves in danger of hell fire when they call their humble popularizing brother "raca." Analysis, it may be recalled, has not at present a sound leg to stand on. Yet there is no bigot so narrow as a mathematical bigot.

Carried away by his enthusiasm for Professor Smith's book and the program which it suggests, the reviewer now notes with alarm that he has committed the unpardonable sin of all greenhorns at reviewing and has omitted to say what the book under review is about. To tell in detail what Professor Smith has done would be to kill the sale of his book, and as it is greatly to be desired that his essays find their way into the private library of every reader of this MONTHLY (and into the pocket of every busy practical man with a few moments for peaceful browsing on his way to and from work), we shall content ourselves with the briefest outline.

The first essay, *The Poetry of Mathematics* develops a thesis which may be succinctly stated by saying that mathematics and poetry are simply isomorphic. It is a relief to see that Professor Smith does not drag in all the frowsy references which poets have made to mathematics (those of Shakespeare are among the most trivial and worthless) to "prove" his point. In a word the point is this: both the mathematician and the poet are *creators*—indeed the word "poet" means, etymologically, one who creates. The creative imagination in mathematics, and the necessity for its presence before mathematical research can be profitably undertaken, is one of those mysteries regarding the true nature of mathematics which never fails to raise the eyebrows of the mathematically illiterate. They do not understand (why should they?—nobody has yet convinced them); the essay which opens this volume may serve to lift one corner of the curtain. To the reviewer one of the admirable features of this chapter—and indeed

of the whole book—is its freedom from ranting and oratory. Mathematics can not be made beautiful by spouting about its beauty. The third essay, *Religio Mathematici*, is closely allied to the first.

In *The Call of Mathematics* Professor Smith gives a brilliant and fascinating summary in brief compass of the reasons why mathematics should be studied, taught, and pursued. This essay might be prescribed reading for all professors of pedagogy.

The remaining essays on Thomas Jefferson and Gaspard Monge call for no comment—they are intensely interesting, as might be anticipated, but they raise no questions of the fundamental importance suggested by the other essays.

Professor Smith has scattered the seed broadcast with both hands. May it not have fallen on barren soil!

E. T. BELL

Actualités Scientifiques et Industrielles. Paris, Hermann et Cie. No. 72, *Les Espaces Métriques Fondés sur la Notion d'Aire.* By E. Cartan. 1933. 46 pages. 12 francs. No. 79, *Les Espaces de Finsler.* By E. Cartan. 1934. 41 pages. 12 francs. No. 194, *La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés.* By E. Cartan, 1935. 65 pages. 12 francs.

That certain concepts and relations of ordinary geometry find interesting generalizations in the Calculus of Variations is well known: transversality being an obvious though unsymmetrical generalization of orthogonality, and one of the envelope theorems expressing the "string" property of the generalized evolute. Recognizing these, it is natural to seek others, and in this connection are to be found papers by Bliss, Landsberg, and Underhill in the first decade of the century. Paul Finsler, however, in his Göttingen dissertation of 1918 was the first to develop systematically the geometry of the space thus generalized. By methods which were various and ingenious, Finsler succeeded in carrying over a great many of the ordinary theorems regarding curves and surfaces.

The spaces of Finsler differ from those of Riemann essentially in this, that while retaining the homogeneity property of the ds^2, Finsler releases the restriction that it be a quadratic form. This requires that the basic element of the space consist of point x and direction x', with both sets of variables affecting the measurements of length and angle. For this reason first attempts to define a parallel transport in the Finsler spaces met with virtual failure. Taylor and Synge, in 1925, defined such a transport, but only in the direction of the element itself. A little later Berwald adapted an invariant process developed by Noether in the year of Finsler's dissertation, but the Noether-Berwald process failed to retain the invariance of lengths and angles while involving the restrictive hypothesis that the x' of the element itself undergo simultaneous parallel displacement.

By a most natural device Cartan transcends the restrictions while retaining the essential properties for the parallel transport. Whereas his predecessors had thought to find the necessary coefficients right at hand, Cartan imposes the

desired postulates and calculates the coefficients compatible therewith. It is to be expected that they should be more complicated, but the geometric simplicity overweighs the analytical complications.

If the element of arc (as defined by a general Calculus of Variations integrand) is alone sufficient for the complete development of the geometry, may not the element of area of a hypersurface serve as well? More precisely, given a Calculus of Variations $(n-1)$-tuple integral in n-space, is this sufficient for developing the geometry of the space in which this integral defines the area? This is the problem attacked in the first monograph named above, and it turns out to be possible ordinarily, but not without exception, to establish a parallel transport here as in the Finsler spaces. However the basic element here consists of the point and the oriented hyperplane.

Given the equations of parallel transport, the rest is a matter of detail—a fact which Cartan explains in broader perspective in the third monograph. Consider a space of Klein to be studied from the standpoint of a finite continuous group G. In this space can be set up a system of *repères*, generalized trihedrals, each serving as a basis for a coordinate system, having the property that each repère can be transformed into any other of the system by one and just one transformation of G. The transformations sending a repère into neighboring repères are governed by a set of Pfaffians ω, equal in number to the number of essential parameters in G, which are identical with the coefficients of parallel transport in the case of the metric or affine group. In a space of Klein are satisfied the equations of structure, whose coefficients are the constants of structure c_{ijk} of G, and which govern the admissible variations of the ω's. From these equations it is possible to define for any subspace, a subset of repères, one associated with each point of the subspace in question, in a manner intrinsic to it and entirely analogous to the determination of the moving trihedral of Euclidean geometry. But if, with reference to the same group G, a set of ω's are given not satisfying the equations of structure without the addition of supplementary terms, these give rise to a generalized space whose curvature and torsion are defined by these supplementary terms. But the equations as so corrected still suffice to define, in the same way as before, the family of repères intrinsic to any subspace, and hence to provide the entire differential geometry of the space with reference to the basic group G. Indeed the geometry of curves is entirely the same in the generalized space as in the ordinary one, as is illustrated in the fact that in both the Riemann and the Finsler spaces the Frenet formulas are the same as in Euclidean space.

A. S. HOUSEHOLDER

Das Spiel der 30 Bunten Würfel—MacMahon's Problem. By F. Winter. Leipzig, B. G. Teubner, 1934. 128 pages. Rm. 3.60.

Using six colors it is possible to color 30 cubes so that each face of a cube has a different color and the arrangement of colors on no two cubes is the same. This may be seen easily as follows: Color one face of all the cubes with one

color. For the opposite face we have a choice of five colors. This being done, we may color any one of the four remaining faces. For the opposite face there is a choice of three colors. There remain two faces which may be colored in two ways. Hence the total possibilities are $5 \cdot 3 \cdot 2 = 30$ cubes.

This little book discusses in great detail some problems and puzzles in arrangements of these 30 cubes. The puzzles are very interesting in themselves, but the long discussion of the details with the use of page after page of tables of combinations is sure to try the patience of the reader. The problems yield to simple logical analysis and the reader will find it easier to solve the problems than to read the author's explanation. Perhaps the very length and tediousness of the author's discussion is really an advantage in forcing the reader to do this.

The book begins with MacMahon's problem: Given a cube of the 30 as model, to build a cube out of 8 cubes so that only like colored faces touch and the faces of the large cube (composed of faces of 4 small cubes) are colored as the model cube. This problem is discussed in minute detail and is shown to have two solutions. Following this the Kowalewski problem is proposed and solved: To build a cube out of 8 cubes so that only like colored faces touch and the large cube shall have the same color front and back, a second color on left and right faces, and third and fourth colors on upper and lower faces,—all colors being designated in advance. There is then discussed some relations between the cubes solving these two problems. The other sections of the book are devoted to various modifications of these problems with conditions on the colors inside the large cubes and arrangement puzzles of a similar nature. The book closes with a game using the cubes similar to dominoes and a magic square puzzle.

W. L. Ayres

Wie findet und zeichnet man Gradnetze von Land- und Sternkarten? By G. Scheffers. Mathematisch-Physikalische Bibliothek, Reihe I, 85/86. 1934. 98 pages.

Kristallprojektion. By W. Heintze. Mathematisch-Physikalische Bibliothek, Reihe I, 82. 1934. 31 pages.

Scheffers' excellent pamphlet contains an elementary account of the better known methods of map projection, with historical notes, and specific directions for the construction of some twenty types of map pictured herein. The author quotes with approval the dictum of Jacob Steiner to the effect that construction with the tongue is one thing, and construction with the pen quite another. This pamphlet is for those who use the pen.

There are two mathematical difficulties which must be met in any account based wholly on elementary methods. One is the solution of the transcendental equation, known as Kepler's equation, which is needed for Mollweide's projection. The other is the integration of the secant function which is needed for

Mercator's projection. The author handles the first by the intersection of graphs. For the second he gives first an approximate method which is not very accurate (the distance between the 70° and 80° parallels is over 40% in excess of what it should be), and then an exact one which requires a count of squares in various areas under the secant curve. The count is suspiciously good, and seems to be rather more accurate than the figure warrants. The reviewer is a personal exponent of Steiner's dictum, and from some personal experience wonders if the book would not be more useful to a wider group of constructors if it were not so self-consciously restricted to the "elementary."

In the pamphlet on Kristallprojektion, Heintze gives a brief account of the projections: stereographic, gnomonic, and orthographic, and explains and illustrates the applications of these to the study of crystals. To many mathematical readers the treatment of the Miller index-numbers for crystals will be the most interesting part of this pamphlet. The concluding section on the researches of Laue and others in crystal-structure, though very concise, is well written. The pamphlet should serve to give mineralogists some idea of the simple mathematical methods which have been found useful, and should whet the interest of many teachers and students of mathematics in finding out more about this fascinating subject. There is suitable material here for additional work in geometry classes, and for mathematics clubs.

B. H. Brown

MATHEMATICS CLUBS

This department, missing from the last two issues, will appear again in the December number under the editorship of Professor F. W. Owens of Pennsylvania State College and Dr. Helen B. Owens. Material for the department should be addressed to *F. W. Owens, 462 East Foster Avenue, State College, Pa.*

PROBLEMS AND SOLUTIONS

Elementary Problems

Edited by Otto Dunkel, H. L. Olson, and W. F. Cheney, Jr.

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics.

PROBLEMS FOR SOLUTION

E 176. *Proposed by V. Thébault, Le Mans, France.*

Find a number of five digits which are consecutive, though not in their natural order, and such that the square of this number contains the ten different digits. Show that there is just one solution.

E 177. *Proposed by F. A. Lewis, University of Alabama.*

If the cosines of the half-angles of a triangle be divided by the lengths of the corresponding bisectors, the sum of the three ratios thus formed equals the sum of the reciprocals of the sides of the triangle.

E 178. *Proposed by C. H. Forsyth, Dartmouth College.*

A, B and C play nine holes of golf, counting 5 points for the lowest score on a hole, 3 points for the next lowest, and 1 point for the highest. Their final total scores are; A 29, B 27, and C 25, although A makes the least number of low scores and C the greatest.

If there are no ties on any hole, determine the scores on each of the nine holes, without respect to the order of the holes. Show that the result is unique.

E 179. *Proposed by J. Rosenbaum, Hartford Federal College.*

In a triangle ABC the points D, E and F trisect the sides such that $BC = 3BD$, $CA = 3CE$, and $AB = 3AF$. Similarly the points G, H and I trisect the sides of triangle DEF such that $EF = 3EG$, $FD = 3FH$, and $DE = 3DI$. Prove that the sides of triangle GHI are parallel to the sides of triangle ABC and that each side of the smaller triangle is one-third as long as its parallel side in the larger triangle.

E 180. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find a number written in the scale of seven, with less than forty digits, such that if the digit "3" is moved from its extreme right to its extreme left, the result (still in the scale of seven) is four-fifths of the original number.

SOLUTIONS

E 148 [1935, 245]. *Proposed by V. Thébault, Le Mans, France.*

Form two numbers, one of which is twice the other, using the ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 just once each. Is more than one solution possible? If instead of one of the two numbers being twice the other, it must be k times the other, what is the smallest positive integer value of k not a power of ten, such that the problem will have no solution?

Solution by C. W. Trigg, Cumnock College, Los Angeles

If k be less than 10, the number, N, and its multiple, M, will each contain exactly five digits.

The sum of the ten digits is divisible by 9, regardless of their order. Hence, if $M = kN$, then $N(k+1)$ is a multiple of 9. Then N is a multiple of 9 if k is 3, 4, 6, 7 or 9. N is a multiple of 3 if k is 2 or 5, while if k is 8 this places no restriction on N.

If k is even, the units digit of M must be even. If k is 5, the units digit of M must be 0 or 5.

With these properties in mind, we proceed to discover the required numbers by constructing M from the left. Thus for $k = 2$, the first digit of M may be any

one of 1, 2, 3, $\cdots$, 9. If 1 be chosen, then the first digit of N is zero and the second digit of M may be any one of the remaining eight digits. Choice of 2 for the second digit of M leads to duplication of digits, but a 3 in this position makes 6 the second digit of N. Proceeding in this manner with the six remaining digits and recalling that twice the units digit of N must yield an available units digit for M, we find in succession the following sixty values for N, all corresponding to $k=2$.

06729	07692	13485	14853	20679	27069	30729	32907	45138	48351
06792	07923	13548	14865	20769	27093	30792	34851	45186	48513
06927	07932	13845	15486	20793	27309	30927	35148	45381	48516
07269	09267	14538	16485	23079	29067	31485	35481	46185	48531
07293	09273	14685	18546	26709	29073	32079	38145	46851	48615
07329	09327	14835	18645	26907	29307	32709	38451	48135	48651

If k is 3, we have the following eight values for N:

05823	05832	16794	17694	20583	23058	30582	32058

If k is 4, we have the following twelve values for N:

03942	05796	15237	17352	20439	23517
04392	07956	17235	20394	21735	23715

If k is 5, we have the following twenty-four values for N:

02697	02967	03729	09237	13458	14538	15384	18534
02769	02973	06297	09627	13584	14586	15846	18546
02937	03297	07629	09723	13854	14658	15864	18654

The corresponding numbers of values found for N when k takes the values 6, 7, 8 and 9 are 3, 8, 62 and 6 respectively.

It will be observed that the values of N corresponding to certain values of k group themselves into permutations of a few sets of digits. When k is 2, the sets 02379 and 13458 each occur 18 times and the sets 14568 and 02679 each occur 12 times, in different permutations. Note that these sets are mutually exclusive in pairs. When k is 5, these same four sets occur six times each.

Evidently if k is 10, all the digits of N will be duplicated in M, and if k is 11, the units digits of N and M will be the same. Hence 11 is the smallest value of k, not a power of 10, for which no solution exists.

Also solved by W. E. Buker, Richard Fowler, Dorothy Stephenson, and E. P. Starke.

E 149 [1935, 245]. *Proposed by J. A. Hurry, San Antonio Junior College, Texas.*

Show that the angle A has no value within the first quadrant which will satisfy the equation,

$$\sin\frac{A}{3}=\frac{\sin A}{2+\cos A}.$$

Solution by Maxwell Reade, Brooklyn, New York

Substituting the well-known formulas for the sine and cosine of thrice an angle in the given equation, it reduces readily to

$$(\sin A/3)(3 - 4 \cos^2 A/3)(1 - \cos A/3) = 0.$$

The roots of this equation are obviously $A = 0$ or any number of right angles, but no value "within" any quadrant.

Also solved by W. E. Buker, Richard Fowler, O. H. Hamilton, Sidney Kaplan, L. M. Kelly, J. Rosenbaum, E. P. Starke, C. W. Trigg, Simon Vatriquant, G. A. Williams, R. C. Yates and the proposer.

E 150 [1935, 246]. *Proposed by Maud Willey, Gulfport, Mississippi.*

Points M and N trisect side BC of triangle ABC, so that $BM = MN = NC$. A line parallel to AC meets lines AB, AM and AN in points D, E and F, respectively. Show that $EF = 3DE$.

Solution by W. B. Clarke, San Jose, California

Let DF cut BC at G. Through E and F draw lines parallel to BC cutting AB at H and J respectively.

Then EH is $CG/2$; FJ is $2CG$, and FJ is $4EH$.

Triangles DEH and DFJ are similar, so that $DF{:}DE{::}FJ{:}EH{::}1{:}4$. Consequently, EF is $3DE$.

Also solved by W. E. Buker, Mary L. Constable, D. H. Ewing, Daniel Finkel, Richard Fowler, D. P. Jones, L. M. Kelly, Maxwell Reade, Leon Recht, J. Rosenbaum, H. F. Schroeder, E. P. Starke, C. W. Trigg, Simon Vatriquant and R. C. Yates.

E 151 [1935, 246]. *Proposed by W. R. Ransom, Tufts College, Massachusetts.*

Under what circumstances may the cube of an integer equal the difference of the squares of two non-consecutive, relatively prime, positive integers?

Solution by Simon Vatriquant, Athénée Royale d'Ixelles, Brussels, Belgium

Let $N^3 = X^2 - Y^2 = (X+Y)(X-Y)$. We will determine the numbers X and Y by separating N^3 into two relatively prime factors and equating them to $X+Y$ and $X-Y$. During this process the following points must be observed:

(a) We must exclude the case which would make $X - Y = 1$, since the problem requires that X and Y be non-consecutive.

(b) If N is even, one of the factors $(X+Y)$ and $(X-Y)$ must be twice an odd number. For if one were even and the other odd, X and Y would be fractional. And if each contained the factor 4, then X and Y would have the common factor 2, contrary to hypothesis.

We have consequently to consider four cases.

CASE 1. $N = 2^k$. The unique decomposition is then $X + Y = 2^{3k-1}$, $X - Y = 2$. Here X and Y are consecutive odd numbers and $X = 2^{3k-2}+1$, $Y = 2^{3k-2}-1$. For example, $4^3 = 17^2 - 15^2$.

CASE 2. $N = p^r$. If N is a power of a single odd prime, then the decomposition is impossible, by (a) above.

CASE 3. $N = p^r \cdot q^s \cdots$. When N is a product of powers of odd primes, we may divide the factors into two groups, each prime being taken with its complete

exponent in N^3 in just one group. The larger product shall be equated to $X+Y$ and the smaller to $X-Y$. If the number of different prime factors in N is n, then the number of different decompositions is $2^{n-1}-1$.

CASE 4. $N=2^k p^r q^s \cdots$. When N is a product of powers of primes including a power of 2, the decomposition differs from that of case 3 only in that a single factor 2 must appear in just one of the two groups, in accordance with (b) above. If n denotes the number of different prime factors in N (including 2), the number of decompositions is in this case 2^{n-1}.

While the above four cases are desirable from the point of view of analysis, cases 1 and 2 are really limiting situations in cases 4 and 3 respectively, both as to method of solution and as to number of possible solutions.

Also solved by Richard Fowler, E. P. Starke, C. W. Trigg and the proposer.

E 152 [1935, 246]. *Proposed by J. Rosenbaum, Hartford Federal College, Connecticut.*

Simplify the product

$$(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)(2^{2^3}+1)\cdots(2^{2^n}+1).$$

Solution by Daniel Finkel, Brooklyn, New York

To simplify this product, multiply through by $(2^{2^0}-1)$ which is equal to 1, and hence does not change the value of the product. Now

$$(2^{2^0}-1)(2^{2^0}+1)=(2^{2^1}-1),\qquad (2^{2^1}-1)(2^{2^1}+1)=(2^{2^2}-1),$$
$$(2^{2^2}-1)(2^{2^2}+1)=(2^{2^3}-1),$$

et cetera. And finally

$$(2^{2^n}-1)(2^{2^n}+1)=(2^{2^{n+1}}-1),$$

which is the simplest expression for the given product.

Also solved by B. LeF. Brown, W. E. Buker, Richard Fowler, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 153 [1935, 246]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Last summer in Arizona I overheard Wild Bill and Pesky Pete discussing different incidents which had occurred during the past half century. It seems that on one occasion Pete had asked Wild Bill the date, and in typical western fashion the latter had spun around, drawn his revolver, and shot a bullet into the calendar hanging at the far end of the barn. "Thar's your date, Pesky," Bill had exclaimed, and a closer inspection had proved he was right.

Out of curiosity, the two had then added up the various numbers Bill's bullet had punctured on the successive sheets of the calendar. (It was the usual kind, with a sheet for each month, and on each sheet a square for each date in the month, arranged in seven columns according to the days of the week.) When I left them, the two were arguing over the total, Bill claiming it had been 317, and Pete, 319. Which was right, and just when had the incident occurred?

Solution by W. B. Campbell, Ithaca, New York

Since no month date exceeds 31, and 317/31 exceeds 10, there must have been either eleven or twelve month sheets on the calendar, and any case involving a blank for more than one month must be excluded. Also, since 317/12 exceeds 26, the "average" date of the month in the sheets punctured must be well into the last third of the month, so that dates in the early part of the month may be ignored.

Examining 1934, the then-current year, we find that the various months begin with the days indicated below, and we list for each month the date lying in the square corresponding to that of Jan. xth:

Month	1st Day of Month	$6 \leqq x \leqq 29$	$x=30$
Jan.	Monday	x	x
Oct.	Monday	x	x
May	Tuesday	$x-1$	$x-1$
Aug.	Wednesday	$x-2$	$x-2$
Feb.	Thursday	$x-3$	$x-3$
Mar.	Thursday	$x-3$	$x-3$
Nov.	Thursday	$x-3$	$x-3$
June	Friday	$x-4$	$x-4$
Sept.	Saturday	$x-5$	$x-5$
Dec.	Saturday	$x-5$	$x-5$
April	Sunday	$x+1$	None
July	Sunday	$x+1$	$x+1$
		$12x-24$	$11x-25$

For values of x from 6 to 29, each line indicates a correct date, but corresponding to $x=30$ we should have April 31, which must be excluded. Using 31 for x would introduce an empty square for July also and hence involve less than eleven months. There is no integral solution for $12x-24$, or $11x-25$, $=317$ or 319. If the January sheet is missing, cancellation of the first line gives the total in terms of October xth, as $11x-24$, which has an integral solution only for the excluded case of $x=31$. Hence the date was not in 1934.

To study an ordinary year with January first a Sunday, we change each symbol in the day column above to the preceding day, and notice that the last two lines become $x+1-7=x-6$, for values of x from 7 to 31. For an ordinary year with January first a Tuesday, we advance by one each day in the original table and add 7 to the lines opposite the new Sunday. This procedure may be continued to produce a table for each different day of the week to start the year.

The data for leap years may be built up from the case where January, April and July start on Sunday, October on Monday, May on Tuesday, February and August on Wednesday, March and November on Thursday, June on Friday and September and December on Saturday. The results for the 7 possible ordinary years and the 7 possible leap years are listed here:

Jan. 1 on	Ordinary Years Totals	x-limits	Leap Years Totals	x-limits
Sunday	$12x-38$	$x\leqq 31$	$12x-34$	$x\leqq 31$
Monday	$12x-24$	$x\leqq 29$	$12x-20$	$x\leqq 29$
	$11x-25$	$x=30$	$11x-21$	$x=30$
Tuesday	$12x-10$	$x\leqq 28$	$12x-13$	$x\leqq 28$
	$11x-12$	$x=29$	$11x-15$	$x=29$
Wednesday	$12x-3$	$x\leqq 27$	$12x+1$	$x\leqq 27$
	$11x-6$	$x=28$	$11x-2$	$x=28$
Thursday	$12x+18$	$x\leqq 24$	$12x+15$	$x\leqq 25$
	$11x+14$	$x=25$	$11x+11$	$x=26$
Friday	$12x+25$	$x\leqq 24$	$12x+22$	$x\leqq 25$
	$11x+21$	$x=25$	$11x+18$	$x=26$
Saturday	$12x+32$	$x\leqq 24$	$12x+29$	$x\leqq 25$
	$11x+28$	$x=25$	$11x+19$	$x=26$

Each case involving $12x$ must also be tested for $11x$.

The only case where a permissible integer x gives a total of 317 or 319 is the leap year beginning on Saturday, for which x must be 24, as then $12x+29=317$. Working backwards from 1934, dropping one day for each ordinary year and two for each leap year, and remembering that 1900 was an ordinary year, we find that 1916 was the only leap year during the given half century which began on a Saturday. Hence the date in question was January 24, 1916, and the total of the punctured dates was 317.

Also solved by Richard Fowler, E. P. Starke, Dorothy Stephenson, and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3754. *Proposed by Warren Jones, Maryville College, Tennessee.*

Find the minimum area of the segment cut from a parabola by a chord passing through a given point in its interior not on the curve or on its axis.

3755. *Proposed by J. Rosenbaum, Hartford Federal College, Connecticut.*

The volume V of an orthocentric tetrahedron $ABCD$ is given by

$$244V^2 = f(a, b, c) + f(a, y, z) + f(x, b, z) + f(x, y, c)$$

where a, b, c, x, y, z are the lengths of the edges BC, CA, AB, DA, DB, DC, and

$$f(a, b, c) = (a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2).$$

Show that the converse is not true.

3756. *Proposed by J. M. Feld, New York City.*

If

$$S_p = \sum_{k=1}^{n} (2k - 1)^p,$$

prove that

$$\sum_{i=0}^{k} \binom{2k+1}{2i} S_{2i} = 2^{2k} n^{2k+1}.$$

3757. *Proposed by V. Thébault, Le Mans, France.*

The sum of the powers of the vertices of a tetrahedron with respect to the sphere having a diameter with end points at the centroid and the Monge point is equal to the sum of the squares of the bimedians of the tetrahedron. The bimedians are the segments of straight lines joining the middle points of opposite edges of the tetrahedron.

SOLUTIONS

3677 [1934, 270]. *Proposed by Otto Dunkel, Washington University.*

Given the two equations

$$y^2 = (a^2 \pm 1)x^2 + 1,$$

in which a is a given positive integer greater than unity in the second equation; deduce for each equation, without use of the theory of continued fractions, iterative formulae which will give all the positive integral solutions in x and y.

Solution by E. P. Starke, Rutgers University

Consider first

$$y^2 = (a^2 - 1)x^2 + 1. \tag{1}$$

Put $a = \cosh\theta$ and $y = \cosh\phi$; then $x = \sinh\phi/\sinh\theta$.
If $\phi = n\theta$, $n = 0, 1, 2, 3, \cdots$, then

$$\begin{aligned} y &= \cosh n\theta = C_n(\cosh\theta) = C_n(a), \\ x &= \sinh n\theta/\sinh\theta = \frac{1}{n} C_n'(a), \end{aligned} \tag{2}$$

will be integers satisfying (1). Here $C_n(a)$ is the cosine polynomial of degree n, so that $C_n(\cos\alpha) = \cos n\alpha$ and hence $C_n(\cosh\alpha) = \cosh n\alpha$.

That (2) gives all the solutions of (1) follows from the following two facts: (i) $\cosh x > \sinh x$, and if $\beta > \alpha$ then $\cosh\beta > \cosh\alpha$ and $\sinh\beta > \sinh\alpha$; (ii) if ϕ and α correspond to two solutions (2) then $\cosh(\phi \pm \alpha)$ and $\sinh(\phi \pm \alpha)/\sinh\theta$ are integers satisfying (1). For then, let $Y = \cosh\alpha$ and $X = \sinh\alpha/\sinh\theta$ be any

integers satisfying (1). Put $\alpha = n\theta + g$ in which n is an integer and $0 \leqq g < \theta$. Then by (ii) $\cosh g$ and $\sinh g/\sinh\theta$ would be integers satisfying (1). But since $0 \leqq g < \theta$, (i) gives $0 \leqq \sinh g/\sinh\theta < 1$. So, if X is an integer, $g = 0$.

In the above, (i) is an immediate consequence of the definitions of $\cosh x$ and $\sinh x$. To prove (ii) note that $\cosh^2(\phi \pm \alpha) = \sinh^2(\phi \pm \alpha) + 1$, and that

$$
\begin{aligned}
&\cosh(\phi \pm \alpha) = \cosh\phi\cdot\cosh\alpha \pm \sinh\phi\cdot\sinh\alpha = yY \pm (a^2 - 1)xX,\\
&\sinh(\phi \pm \alpha)/\sinh\theta = \sinh\phi\cdot\cosh\alpha/\sinh\theta \pm \cosh\phi\cdot\sinh\alpha/\sinh\theta = xY \pm yX
\end{aligned} \tag{3}
$$

are integers.

The desired iterative formulae are given by (3) when we put $\phi = n\theta$ and $\pm\alpha = \theta$, obtaining

$$
x_{n+1} = ax_n + y_n, \qquad y_{n+1} = ay_n + (a^2 - 1)x_n, \tag{4}
$$

where $x_0 = 0$, $y_0 = 1$.

We may obtain solutions for

$$
y^2 = (a^2 + 1)x^2 + 1 \tag{5}
$$

by setting $a = \sinh\theta$, $y = \cosh\phi$, whence $x = \sinh\phi/\cosh\theta$. By a discussion entirely analogous to that given above, we find that x and y will provide integral solutions of (5) if and only if ϕ is an even multiple of θ. Thus

$$
\begin{aligned}
y &= \cosh 2n\theta = C_n(1 + 2a^2) = (-1)^n C_{2n}(ai),\\
x &= \sinh 2n\theta/\cosh\theta = \frac{2a}{n} C_n'(1 + 2a^2) = [(-1)^n i/2n] C_{2n}'(ai)
\end{aligned} \tag{6}
$$

are the integral solutions of (5). The corresponding iterative formulae are

$$
x_{2n+2} = (2a^2 + 1)x_{2n} + 2ay_{2n}, \qquad y_{2n+2} = (2a^2 + 1)y_{2n} + (2a^3 + 2a)x_{2n}, \tag{7}
$$

where $x_0 = 0$ and $y_0 = 1$.

The integers satisfying $y^2 = (a^2+1)x^2 - 1$ are given by $y = \sinh\phi$, $x = \cosh\phi/\cosh\theta$, where $\sinh\theta = a$ and ϕ is an odd multiple of θ. The corresponding reduction formulae are like (7) with odd subscripts and with $x_1 = 1$ and $y_1 = a$.

There exist no solutions of $y^2 = (a^2 - 1)x^2 - 1$, since every square is congruent, mod 4, to 0 or 1.

A rather interesting corollary of the solution of equation (5) is the solution for the problem:

"Find all triangular numbers which are also perfect squares."

We have at once $n(n+1)/2 = s^2$. This reduces upon substitution of $y = 2n+1$, $x = 2s$, $a = 1$ to the form $y^2 = (a^2+1)x^2 + 1$. Applying the solution of this last form we have at once the solution of the problem:

$$
n_{j+1} = 4s_j + 3n_j + 1, \qquad s_{j+1} = 3s_j + 2n_j + 1,
$$

where $s_0 = n_0 = 1$.

Editorial Note. This problem was proposed since this department had received a request for information regarding one of its numerical cases, and since problems concerning some of its special types had been offered. For this reason we shall give an algebraic treatment which leads to the facts forming the basis of the above solution.

Let x, y be a positive integral solution of

$$y^2 = (a^2 - 1)x^2 + 1, \tag{1}$$

where a is an integer greater than unity. Then $y^2 > (a^2-1)x^2$, and $a^2y^2 > (a^2-1)^2 x^2$, or $ay > (a^2-1)x$. We shall suppose further that x is greater than unity. Then $y^2 = a^2x^2 - (x^2-1)$, and $y < ax$. Let us write $y = ax - x'$, where x' is a positive integer, and set this value of y in (1). After solving the resulting quadratic, we find that

$$x = ax' + \sqrt{(a^2 - 1)x'^2 + 1},$$

where the negative sign before the radical must be rejected. For, from the results above, we have $x - ax' = ay - (a^2-1)x > 0$. Hence the expression under the radical must be the square of an integer $y' \geqq 1$; and we have

$$\begin{aligned} x &= ax' + y', \qquad y'^2 = (a^2 - 1)x'^2 + 1, \qquad y' > x' \geqq 1. \\ y &= ax - x', \end{aligned} \tag{2}$$

These equations show that $x > ax' > x' \geqq 1$.

We have proved that, if x, y is a positive integral solution of (1) such that $x > 1$, we can find by (2) another solution x', y' such that x' is smaller than x, in other words we have a smaller solution in positive integers. Suppose we continue, if necessary, this process of finding smaller positive integral solutions until we find the smallest solution x_2, y_2 such that $x_2 > 1$. Then the next step of reduction must give x_1, y_1 for which $x_1 \leqq 1$. But from (2) $x_1 = ax_2 - y_2$, and as shown above we must have $y_2 < ax_2$, since $x_2 > 1$. This shows that $x_1 > 0$, and therefore $x_1 = 1$, $y_1 = a$. Every solution in positive integers for $x > 1$ can therefore be reduced by (2) to the solution $x_1 = 1$, $y_1 = a$. It is easily verified that if x', y' is a solution, then (2) gives x, y as a larger solution. It is not necessary to perform any calculation, since solving (2) for x' and y' gives the same formulae as (2) with the signs of the y's reversed. Thus every positive integral solution is given by

$$\begin{aligned} x_{n+1} &= ax_n + y_n, \qquad x_0 = 0, \qquad y_0 = 1; \\ y_{n+1} &= ax_{n+1} - x_n, \qquad x_1 = 1, \qquad y_1 = a. \end{aligned} \tag{3}$$

A simple combination gives the relations

$$x_{n+2} = 2ax_{n+1} - x_n, \qquad y_{n+2} = 2ay_{n+1} - y_n,$$

and we have the result that the x's and y's satisfy the single difference equation

$$(4) \qquad P(n+2) - 2aP(n+1) + P(n) = 0,$$

with the two sets of initial conditions as given in (3).

We can find solutions of (4) by setting $P(n) = t^n$, and taking for t one of the roots α, β of

$$(5) \qquad f(t) = t^2 - 2at + 1 = 0.$$

Then any solution of (4) has the form

$$P(n) = A\alpha^n + B\beta^n,$$

where A and B are constants depending upon the initial conditions. It is easily found that for the initial conditions in (3) we have

$$(6) \qquad x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad y_n = \frac{\alpha^n + \beta^n}{2}.$$

It is easily shown that x_n, y_n satisfy (1) for any value of a.

In order to find an explicit expression for y_n in terms of a we may proceed as follows: We have

$$\frac{f'(t)}{f(t)} = \frac{2(t-a)}{t^2 - 2at + 1} = \frac{1}{t-\alpha} + \frac{1}{t-\beta},$$

$$\frac{2\left(1 - \dfrac{a}{t}\right)}{1 - \dfrac{1}{t}\left(2a - \dfrac{1}{t}\right)} = \sum_{n=0}^{\infty} \frac{\alpha^n + \beta^n}{t^n} = 2\sum_{n=0}^{\infty} \frac{y_n}{t^n},$$

where t is chosen large enough for absolute convergence. The development of the left side gives

$$(7) \qquad \begin{aligned} y_n &= \frac{\alpha^n + \beta^n}{2} \\ &= \frac{1}{2} \sum_{i=0}^{[n/2]} (-1)^i \frac{n(n-i-1)(n-i-2)\cdots(n-2i+1)}{i!} (2a)^{n-2i}, \end{aligned}$$

where the upper limit is the greatest integer in $n/2$. It is not necessary to develop x_n, since it will be seen later how to derive it from (7).

We may now drop the condition that a is an integer; then x_n and y_n are not necessarily integers, but x_n, y_n satisfy (1), (3), (4), (7). If we set $a = \cosh\theta$, then $\alpha = e^\theta$, $\beta = e^{-\theta}$, and $y_n = \cosh n\theta$. Thus (7) becomes the polynomial expansion of $\cosh n\theta$ in terms of $\cosh\theta$. The differentiation with respect to θ of both sides of (7) will then give $\sinh n\theta/\sinh\theta$ and the expression for x_n. It is obvious from (3) or (4) that x_n and y_n are polynomials in a with integers for coefficients, if the initial values are integers. Thus the coefficients of the polynomial expansion of $\cosh n\theta$ and $\sinh n\theta/\sinh\theta$ in powers of $\cosh\theta$ are all integers.

If the roots of (5) are unequal in absolute value then $P(n)/P(n+1)$ approaches the root of smaller absolute value as n becomes infinite, irrespective of the initial conditions. Hence, if a is real and greater than unity,

$$\text{(8)} \qquad \operatorname*{Limit}_{n\to\infty} \frac{2P(n)P(n+1)}{P^2(n+1) - P^2(n)} = \frac{1}{\sqrt{a^2-1}},$$

for any selection of the initial conditions. Thus, if $a=2$, and if we choose integers for $P(0)$ and $P(1)$, we shall obtain a sequence of integers such that $P^2(n+1)+P^2(n)$, $2P(n)\,P(n+1)$, $P^2(n+1)-P^2(n)$ are the sides of a right triangle one of whose angles approaches 30°. If some of the $P(n)$'s should happen to be negative, as a result of the initial conditions, it is obvious that for n greater than some integer they will have the same sign for all such values of n. See the solution of 3594 [1934, 53].

Let x, y be a positive integral solution of

$$\text{(9)} \qquad y^2 = (a^2+1)x^2 + 1,$$

where a is a positive integer. We have in turn

$$\text{(10)} \qquad \begin{aligned} a^2y^2 &= a^2(a^2+1)x^2 + a^2 < a^2(a^2+1)x^2 + (a^2+1)x^2 < (a^2+1)^2x^2, \\ ay &< (a^2+1)x. \end{aligned}$$

Also $y>ax$, and we shall set $y=ax+h$, where h is an integer, $h\geqq 1$. After inserting this value of y in (9) we find that

$$x = ah + \sqrt{(a^2+1)h^2-1},$$

where the negative sign before the radical has been dropped since $x-ah=x-a(y-ax)=(a^2+1)x-ay>0$. The expression under the radical must be the square of an integer k which is taken as positive. Thus we have

$$\text{(11)} \qquad \begin{aligned} x &= ah + k, \qquad k^2 = (a^2+1)h^2 - 1, \qquad h \geqq 1, \qquad k \geqq 1. \\ y &= ax + h, \end{aligned}$$

We find also from (11) that $1\leqq h<x$, and $1\leqq k<y$. The smallest positive values for h and k are $h=1$, $k=a$. Hence the smallest solution of (9) in positive integers is $x=2a$, $y=2a^2+1$.

Suppose now that x is greater than $2a$, then $h>1$. We have $k^2=a^2h^2+h^2-1>a^2h^2$; and, if we set $k=ah+x'$, then $x'\geqq 1$. After inserting this value of k in (11) we obtain

$$h = ax' + \sqrt{(a^2+1)x'^2+1}.$$

The negative sign must be omitted, since $h-ax'=(a^2+1)h-ak>0$. The inequality is true since $a^2k^2=a^2(a^2+1)h^2-a^2<(a^2+1)^2h^2$. Hence we have

$$\text{(12)} \qquad \begin{aligned} h &= ax' + y', \qquad y'^2 = (a^2+1)x'^2 + 1, \qquad y' > x' \geqq 1. \\ k &= ah + x', \end{aligned}$$

Hence, if x, y is a positive solution of (9) and $x>2a$, there exists a smaller solution x', y' such that

$$2a \leqq x' < h < x, \qquad 2a^2+1 \leqq y' < k < y.$$

If $x'>2a$, we can continue to find smaller solutions of (9); and we shall prove that after a finite number of steps we reach the solution for which $x=2a$. The smallest x we get by such reductions is greater than zero and hence greater than or equal to $2a$. But it would not be the smallest if greater than $2a$, for we could then make a further reduction. We have therefore proved that every solution in positive integers can be reduced by (11) and (12) to a solution for which $x=0$ or $x=2a$. If x', y' is a solution, it is easily verified that (12) gives h, k as a solution of the equation in (11), and finally (11) gives x, y as a solution of (9). Hence the formulae

$$(13)\qquad \begin{aligned} x_{n+1} &= ax_n + y_n, \qquad & y^2 &= (a^2+1)x^2 \pm 1, \qquad & x_0 &= 0, \qquad & y_0 &= 1 \\ y_{n+1} &= ax_{n+1} + x_n, & & & x_1 &= 1, & y_1 &= a \end{aligned}$$

give all the solutions of the two equations on the right. The even subscripts give the solutions of the equation with the last term $+1$, while the odd subscripts give those for -1. The proof that we get all the solutions for the equation with -1 is similar to the above.

We find as before that the sequence of x's, and also the sequence of the y's, satisfy each the difference equation

$$(14)\qquad P(n+2) - 2a\,P(n+1) - P(n) = 0.$$

The initial values for the sequence of x's are $P(0)=0$, $P(1)=1$; for the y's, they are $P(0)=1$, $P(1)=a$.

We now find explicit expressions for $P(n)$ in the two cases by setting $P(n) = t^n$. Then t must be one of the roots α, β, of

$$(15)\qquad f(t) \equiv t^2 - 2at - 1 = 0, \qquad \alpha+\beta = 2a, \qquad \alpha\beta = -1.$$

We easily find from the initial conditions that

$$y_n = \frac{\alpha^n + \beta^n}{2}, \qquad x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The development of y_n as a polynomial in a is given by (7) after omitting $(-1)^i$ from the summation. In this case, if we set $a=\sinh\theta$, y_n is $\cosh n\theta$ for n even and $\sinh n\theta$ for n odd. The coefficients of the powers of a, or of $\sinh\theta$, are all integers for the same reason as before.

3678 [1934, 270]. *Proposed by B. W. Jones, Cornell University.*

Prove the following theorem:

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, where n is a positive integer, $a_n \neq 0$, and all coefficients are real, has only real roots, then

1. Descartes' Rule of Signs gives exactly the number of positive and the number of negative roots of the equation.

2. $a_r=0(n>r>0)$ implies $a_{r+1}\cdot a_{r-1}<0$.

The characteristic equation of a real symmetrical matrix is such an equation.

Solution by E. E. Strock, Yale University

1. Since a change of sign of two consecutive non-vanishing coefficients of $f(x)$ is a permanence of sign in $f(-x)$ and a permanence of sign in $f(x)$ is a change of sign in $f(-x)$, the number of changes of sign in $f(x)$ and $f(-x)$ does not exceed n.

Let n_1 be the number of changes of sign in $f(x)$ and n_2 be the number of changes of sign in $f(-x)$. Then

$$n_1 + n_2 \leqq n.$$

Hence, if the equation has n real roots, the number of positive roots cannot be less than n_1, nor can the number of negative roots be less than n_2, if $a_0 \neq 0$.

2. If $a_r=0$ and if a_{r+1} and a_{r-1} have like signs, they will have like signs in $f(-x)$. Hence the number of changes of signs in $f(x)$ and $f(-x)$ does not exceed $n-2$ and n real roots are not possible. Therefore a_{r+1} and a_{r-1} have unlike signs and $a_{r+1}\cdot a_{r-1}<0$.

If two or more consecutive coefficients are 0, it is impossible to have n real roots.

Solved also by the proposer.

Editorial Note. The proposer considered the case where $a_t x^t$ is the last non-vanishing term of $f(x)$ and showed that the total number, $V_+ + V_-$, of changes of sign in the coefficients of $f(x)$ and $f(-x)$ does not exceed $n-t$, disregarding as usual the intermediate missing terms. Descartes' rule then gives the theorem. There is no loss of generality in considering $a_n\cdot a_0 \neq 0$. The proof may be arranged to cover the general case. If no term is missing $V_+ + V_- = n$. In the case of r terms missing between the non-vanishing terms $a_{m+r+1}\, x^{m+r+1}$ and $a_m\, x^m$ there may be a loss of changes of sign. If there were no missing terms the $r+2$ terms would contribute $r+1$ changes of sign, and we must consider four cases corresponding to r even or odd and to the sign of $a_{m+r+1}\cdot a_m$. If r is even, the missing terms cause a loss of r changes of sign in $f(x)$ and $f(-x)$. If r is odd, there is a loss of $r-1$, if the above product is negative; and a loss of $r+1$, if the product is positive. Let M denote the total number of missing terms; and O, the number of cases in which an odd number of terms are missing, of which U of these cases are for $a_{m+r+1}\cdot a_m<0$ and $O-U$ of them are for the positive product. Then

$$n = V_+ + V_- + M + O - 2U = P + N + I,$$

where P, N, I denote the number of positive, negative, and imaginary roots. This may be written

$$(V_+ - P) + (V_- - N) = I - (M + O - 2U),$$

and by Descartes' rule no one of the two parentheses on the left can be negative. Hence there are at least $M+O-2U$ imaginary roots. This rule for imaginary roots is attributed to De Gua in Burnside and Panton's *Theory of Equations*, vol. 1, 1899, page 197. If $I=0$, then $M=O=U$, $V_+=P$, $V_-=N$; and the first part of the result says that there can be only cases of precisely one missing term and for such cases the remaining two adjacent terms must have opposite signs.

These results are given in works on the theory of equations, but some do not mention the theorem of the problem. It is stated in Pascal's *Repertorium der Höheren Mathematik*, I. *Analysis*, 1910, page 346, as a consequence of Gauss' corollary to Descartes' rule or of certain remarks following Fourier's theorem. Gauss gave in 1828 a proof of Descartes' rule, *Werke*, vol. 3, page 65, which he says is commonly called Harriot's rule; he wrote as a preface that the proof was published because the prevailing proofs lacked clearness, brevity and complete generality. This proof is reproduced in Weber and Wellstein's *Encyklopädie der Elementar-Mathematik*, 1909, vol. 1, page 325. Here it is stated that the theorem is often erroneously credited to Thomas Harriot (1560–1621).

If we replace x by $a+y$, we obtain a new equation in y whose coefficients, disregarding certain positive numerical factors, are in ascending order

$$f(a), f'(a), f''(a), \cdots, f^{(n)}(a).$$

Thus, if all the roots of $f(x)$ are real and $f(a)\neq 0$, the number of its roots which exceed a is precisely equal to the number of changes of sign in the above sequence. If also $b>a$ and $f(b)\neq 0$, the same is true of the sequence replacing a by b. Hence the number of roots of $f(x)$ between a and b is equal to the loss of changes of sign in such a sequence in passing from a to b. This is the form of statement in Fourier's theorem, from which all these results, including Descartes' rule, may be deduced. There are other similar rules seldom mentioned such as those of Laguerre, Newton, and Newton-Sylvester. The last two are difficult to prove and too laborious for practical use; for exact information when simple methods fail one turns to Sturm's theorem. This method may also prove to be a severe test of patience even in fairly simple cases.

3680 [1934, 333]. *Proposed by William Hoover, Columbus, Ohio.*

A given ellipse moves in a plane so that it is always tangent to a fixed straight line at a given point. Derive the equation of the locus of its center.

I. Solution by Abe Gelbart, Student, Central High School, Paterson, N.J.

The equation of the desired locus, regardless of whether we consider the given straight line as fixed and the ellipse as moving, or the ellipse as fixed and the straight line as moving, is merely the relation between the distance to the center of the ellipse from the given line and the distance from the given point to the foot of the perpendicular drawn from the center to the given line. Let these distances be Y and X respectively.

Let the axes be so chosen that the equation of the ellipse becomes

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{1}$$

The tangent to this ellipse at (x_1, y_1) (the given point) is

$$b^2x_1x + a^2y_1y - a^2b^2 = 0$$

$$Y = \frac{-a^2b^2}{(b^4x_1^2 + a^4y_1^2)^{1/2}}. \tag{2}$$

If d is the distance from the center or origin to (x_1, y_1),

$$d^2 = x_1^2 + y_1^2 = X^2 + Y^2. \tag{3}$$

Now, eliminating x_1 and y_1 from (1) (for $x=x_1$, $y=y_1$), (2), and (3), we obtain

$$X^2Y^2 = (Y^2 - b^2)(a^2 - Y^2)$$

which is the equation of the locus of the center when referred to the given line as x-axis and the given point as origin.

II. Solution by A. Pelletier, Montreal, Canada

Let MN be the given straight line containing the fixed point P of tangency; and let O, F, F' be the center and foci of the ellipse with the axes $2a$, $2b$. Set $PF=r$, $PF'=r'$, $F'O=OF=c$. Let K be the projection of O on MN; then, if we take rectangular axes with the x-axis along MN and the y-axis along the perpendicular at P directed towards the ellipse, the coordinates of O are $x=PK$, $y=PO$. We easily find that

$$r = a + \frac{ax}{\sqrt{a^2 - y^2}}, \qquad r' = a - \frac{ax}{\sqrt{a^2 - y^2}}. \tag{1}$$

Also $2(OP^2+c^2)=r^2+r'^2$, $OP^2=x^2+y^2$, $c^2=a^2-b^2$; and we have from (1)

$$2(x^2 + y^2 + a^2 - b^2) = \left(a + \frac{ax}{\sqrt{a^2 - y^2}}\right)^2 + \left(a - \frac{ax}{\sqrt{a^2 - y^2}}\right)^2.$$

This reduces to

$$(a^2 - y^2)(y^2 - b^2) - x^2y^2 = 0.$$

The complete locus consists of two closed curves symmetrical with respect to MN.

Solved also by E. F. Allen, B. Le F. Brown, J. W. Clawson, L. Green, Harry Langman, Otto J. Ramler, A. V. Richardson, E. P. Starke, C. W. Trigg, G. A. Williams, and Margaret Young-Woodbridge.

Editorial Note. A proof of (1) in solution II may be stated as follows: Let M and N be the projections of F' and F on MN, the x-axis; and set $MN=2d$. Produce $F'M$ to F'' so that $F''M=MF'$: then $MO=F''F/2=a$, since r and r' make equal angles with PN and PM. Hence, from the similar right triangles MKO and PNF, we have

$$\frac{r}{a} = \frac{d + x}{d}, \qquad d^2 = a^2 - y^2,$$

$$\frac{r}{a} = 1 + \frac{x}{\sqrt{a^2 - y^2}}, \qquad \frac{r'}{a} = 2 - \frac{r}{a}.$$

The derivation of the equation of the locus by Ramler also used in a different manner only simple properties of the ellipse as in II. He also stated that this problem forms part of an earlier problem 3160 [1927, 99]. The printed solution (l.c.) of that problem by Roscoe Woods referred to similar problems in W. H. Besant's *Roulettes and Glissettes*.

Clawson, Williams and Young-Woodbridge gave an interesting derivation by finding the new equation of the ellipse, given in the ordinary form, by translating and rotating the rectangular axes so that the new coordinates of the center are α, β and the major axis makes an angle θ with the new x-axis. It is then easy to write the conditions that the ellipse passes through the new origin and is tangent there to $y=0$. The equations are rather long and there is a rather involved reduction at the end.

Richardson gave an interesting variation of this idea by writing the equation of an ellipse tangent to the x-axis at the origin in the form

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2fy = 0.$$

The center is given by the solution of the pair of equations $\partial\phi/\partial x = 0$, $\partial\phi/\partial y = 0$. Now, using the invariants of the general conic, $a+b$, $ab-h^2$, and the discriminant, we obtain equations between the fixed lengths of the semiaxes α and β and the coefficients of ϕ. These equations lead to the equation of the locus.

Some of the solvers made use of the equation of the ellipse in polar coordinates with the major axis as polar axis and the pole at the center. The formula for the tangent of the angle between the curve and the radius vector at the point was then used to derive the equation of the locus in polar coordinates. Clawson and Langman added the information that each oval of the locus is tangent to the sides of a rectangle of lengths $a-b$ and $2(a-b)$.

It is interesting to consider the theorem stated in the solution of 3586 [1934, 589] as applied to the area of the locus in this problem. The polar equation of the pedal curve for the ellipse with respect to the center is easily found to be $\rho^2 = a^2 - c^2 \sin^2 \theta$. Observing that $\rho = c \sin \theta$ is the equation of a circle with the diameter c, we see without integration that the area of the pedal curve is the difference between the area of the major auxiliary circle and twice the area of a circle with the diameter c. Since $c^2 = a^2 - b^2$, the area of the pedal curve is the average of the areas of the major and minor auxiliary circles. The area between the pedal curve and the ellipse is therefore $\pi(a-b)^2/2$; and the general theorem referred to tells us that the area of the two ovals is twice this, or that the area of each is the expression above for the area between the pedal and the ellipse. This shows that the area of each oval is the same as that of an ellipse inscribed

in the rectangle circumscribing the oval and having axes parallel to the sides of the rectangle. If the equation of the two ovals is written in polar coordinates and the corresponding area integral is written, this latter has the same form as the integral for $A/4a$ in 3586 [1933, 565] at the top of the page. Hence it may be evaluated in the same manner without performing any integration.

3679 [1934, 333]. *Proposed by J. M. Feld, Brooklyn College.*

Prove that for any positive integer p

$$p!h^p\sum_{k=0}^{n-1}(a+kh)^{p-1}$$

$$= \begin{vmatrix} (a+nh)^p - a^p & \binom{p}{2}h^2 & \binom{p}{3}h^3 & \cdots & \binom{p}{p-1}h^{p-1} & h^p \\ (a+nh)^{p-1} - a^{p-1} & \binom{p-1}{1}h & \binom{p-1}{2}h^2 & \cdots & \binom{p-1}{p-2}h^{p-2} & h^{p-1} \\ (a+nh)^{p-2} - a^{p-2} & 0 & \binom{p-2}{1}h & \cdots & \binom{p-2}{p-3}h^{p-3} & h^{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (a+nh)^2 - a^2 & 0 & 0 & \cdots & \binom{2}{1}h & h^2 \\ (a+nh) - a & 0 & 0 & \cdots & 0 & h \end{vmatrix}.$$

Solution by Harry Langman, Brooklyn, N.Y.

Let

$$v_p = \sum_{k=0}^{n-1}(a+kh)^{p-1}. \tag{1}$$

Then

$$\begin{aligned} \sum_{r=0}^{j}\binom{j}{r}v_{j+1-r}h^r &= \sum_{r=0}^{j}\binom{j}{r}\sum_{k=0}^{n-1}(a+kh)^{j-r}h^r \\ &= \sum_{k=0}^{n-1}\sum_{r=0}^{j}\binom{j}{r}(a+kh)^{j-r}h^r = \sum_{k=0}^{n-1}[a+(k+1)h]^j \\ &= v_{j+1} + (a+nh)^j - a^j, \end{aligned} \tag{2}$$

whence

$$\sum_{r=1}^{j}\binom{j}{r}v_{j+1-r}h^r = (a+nh)^j - a^j. \tag{3}$$

If we write (3) for all values of j from 1 to p, we have p equations in the p un-

known v's. The determinant of their coefficients clearly has the value $p!h^p$. If D be the given determinant, we have then

$$v_p = \frac{D}{p!h^p},$$

from which the identity follows.

Solved also by Frank Ayres, Jr.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The Section (A) on Mathematics of the American Association for the Advancement of Science met on June 25 at the University of Minnesota in conjunction with the Section (B) on Physics. Professors W. L. Hart, W. E. Brooke, and J. T. Tate presided successively. About sixty members and guests were present. The program consisted of three invited addresses, on the general theme of properties of differential equations important in quantum mechanics. The titles of the papers and their authors were as follows: *Oscillation theorems associated with boundary value problems*, by Professor R. W. Brink of the University of Minnesota; *The Stokes phenomenon*, by Professor R. E. Langer of the University of Wisconsin; *Applications of approximation methods to physical problems, with particular reference to molecular spectra*, by Professor D. M. Dennison of the University of Michigan. A joint luncheon of Sections A, B, and D in the Minnesota Union Tuesday noon was attended by forty persons. The company was addressed briefly by President Compton of the Association, who was introduced by the Chairman of Section B, Professor J. T. Tate.

The honorary degree of Doctor of Science was conferred on Professor Marston Morse, of Harvard University, by Colby College.

Professor Oystein Ore, of Yale University, delivered lectures on structure theorems and problems in abstract algebra at Zürich, Göttingen, and Hamburg, as well as in the Academy of Sciences at Oslo during his trip abroad. He has also been recently appointed a member of the editorial board of the Duke Mathematical Journal.

Professor P. R. Rider, of Washington University, has been awarded a General Education Board Fellowship. He will spend the year 1935–36 carrying on research in biomathematics and mathematical statistics, principally in the Galton Laboratory at University College, London.

Professor D. J. Struik, of the Massachusetts Institute of Technology, delivered lectures on the theory of probability and the imbedding of a Riemann manifold in a euclidean space of higher dimensions at the Mathematical Insti-

tute of the University of Moscow, U.S.S.R. Other guests of the Mathematical Institute were Professors J. A. Schouten of Delft and T. Levi-Civita of Rome.

Dr. R. S. Zug, assistant professor of mathematics and astronomy at Drake University, has been awarded a grant from the National Research Council in order to work on galactic star clusters.

Dr. Richard Brauer of the Institute for Advanced Study has been appointed assistant professor of mathematics at the University of Toronto.

Dr. J. W. Cell has been appointed assistant professor at North Carolina State College, Raleigh.

Dr. H. T. Davis of Indiana University has been promoted to a professorship.

Dr. H. L. Dorwart of Williams College has been appointed assistant professor at Washington and Jefferson College.

Associate Professor H. K. Fulmer of Georgia Institute of Technology, on leave of absence for 1935–36, is spending the year at Cornell University.

Dr. M. L. Hartung of the University of Wisconsin has been promoted to an assistant professorship of the teaching of mathematics. He is on leave of absence for the first semester of 1935–36 at Ohio State University.

Professor B. A. Hazeltine of Middlebury College is exchanging posts for the year 1935–36 with Professor Harry H. Barnum of Robert College, Istanbul, Turkey.

Dr. Alfred Hume of the University of Mississippi has been made Chancellor Emeritus; he will continue his teaching there as professor of mathematics.

Professor C. C. MacDuffee of Ohio State University, has been appointed to a professorship of mathematics at the University of Wisconsin.

Dr. W. M. Miller of Tufts College has been appointed assistant professor at Massachusetts State College.

Dr. C. B. Morrey, of the University of California, Berkeley, has been promoted to an assistant professorship in mathematics.

Dr. M. E. Mullings has been appointed professor of mathematics and physics at Abilene Christian College, Abilene, Texas.

Rev. G. A. O'Donnell has been appointed professor of mathematics at Boston College, Chestnut Hill, Mass.

A. J. O'Leary has been appointed head of the department of mathematics at St. Anselm's College, Manchester, N.H.

Professor W. F. Osgood, formerly of Harvard University, will remain at the National University of Peking for another year.

Dr. A. L. O'Toole has been appointed head of the department of mathematics at the College at St. Catherine, St. Paul.

Dr. Gordon Pall, of McGill University, has been appointed to an assistant professorship.

Dr. A. E. Pitcher, of Harvard University, has been appointed an assistant at the Institute for Advanced Study.

Dr. P. K. Rees of Westmoorland College has been appointed assistant professor at New Mexico State College.

Associate Professor J. B. Rosenbach, of the Carnegie Institute of Technology, has been promoted to a professorship.

Associate Professor G. W. Smith of the University of Kansas has been promoted to a professorship.

Professor J. H. VanVleck, of the University of Wisconsin, has been appointed to a professorship of mathematical physics at Harvard University.

The following promotions to associate professorships are announced:

Dr. H. A. Davis, West Virginia University
Dr. B. P. Gill, College of the City of New York
C. A. Keeler, Albany College, Albany, Oregon
Dr. H. H. Pride, New York University
Dr. A. W. Richeson, University of Maryland
Dr. Arthur Tilley, New York University
Dr. Morgan Ward, California Institute of Technology.

The following promotions to assistant professorships have been announced:

Dr. A. B. Brown, Columbia University
Dr. W. M. Davis, Armour Institute of Technology
C. H. Frick, Valparaiso University
Dr. E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N.J.
Dr. Marguerite Lehr, Bryn Mawr College
S. L. Mason, University of North Dakota
Dr. R. C. Stephens, Knox College
Dr. F. H. Steen, Georgia Institute of Technology
Dr. Alexander Tartler, Drexel Institute
C. R. Wilson, Rutgers University
Dr. H. P. Wirth, College of the City of New York.

The following appointments to instructorships in mathematics are announced:

University of Alabama: Dr. P. M. Hummel
Antioch College: Max Astrachan
Armour Institute of Technology: Dr. Rufus Oldenburger

Bethany College: Dr. W. H. Erskine, Jr.
Hood College: Dr. Ruth G. Mason
Kent State College, Kent, Ohio: Dr. Frances Harshbarger
University of Kentucky: Dr. E. D. Jenkins
Massachusetts Institute of Technology: Dr. R. H. Cameron; and Dr. C. W. MacGregor in mechanical engineering.
Mississippi State College: W. E. Cox, Jr., Dr. Arthur Ollivier
North Carolina State College: Dr. Jack Levine, L. S. Winton
Ohio State University: Dr. C. E. Rhodes, Dr. C. R. Wylie, Jr.
U.S. Naval Academy: Dr. N. H. Ball, Dr. W. R. Church, Dr. S. B. Littauer, Dr. T. W. Moore
University of Washington: Dr. Mary E. Haller
Xavier University, Cincinnati: J. F. Butler
Yale University: Marshall Hall, J. W. Wrench, Jr.

W. W. Garnett, of the Central Life Assurance Society, has been appointed a teaching fellow at the University of Washington.

H. W. Emmons has been appointed research assistant in mathematics at Harvard University.

The following is a list of National Research Fellows for 1935–36, together with the institutions at which they will work:

Clarkson, James Andrew Levinson, Norman Martin, William Ted Murray, Francis Joseph Myers, Sumner Byron	Princeton and the Institute for Advanced Study.
Rosser, John Barkley	Harvard
Webber, G. Cuthbert	Brown

Dr. E. D. Grant, Dean and professor of mathematics and astronomy at Earlham College, died September 2, 1935, at the age of sixty-two. He had taught for twenty years at the Michigan College of Mines before coming to Earlham College fifteen years ago. He was a charter member of the Mathematical Association.

Professor E. H. Jones, for eighteen years in the department of mathematics at Southern Methodist University and in later years the head of the department, died at Dallas, Texas, August 7, 1935. He was a charter member of the Mathematical Association.

Dr. W. Paul Webber, professor of mathematics at Louisiana State University, died June 26, 1935. He was a charter member of the Mathematical Association.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4; Beaver Falls, Pa., Oct. 26.
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11; College Park, Md., Dec. 7.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The Chauvenet Prize

The ASSOCIATION in 1926, with the aid of its president, Professor J. L. Coolidge, established a prize of one hundred dollars for a noteworthy expository paper published in English during successive periods of five years by a member of the ASSOCIATION. In 1928 Professor W. B. Ford, then president of the ASSOCIATION, gave $500, the income from which was to supplement the original fund so that the prize could be awarded every three years.

The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE should tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics*. The second award, covering the five years preceding 1930, was made to THEOPHIL HENRY HILDEBRANDT for his paper on "The Borel Theorem and its Generalizations" published in the *Bulletin* of the American Mathematical Society in 1926. The third award, covering the three years preceding 1933, was made to GEORGE H. HARDY for his paper on "An Introduction to the Theory of Numbers" published in the *Bulletin* of the American Mathematical Society in 1929.

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

FOR THE ACADEMIC YEAR

1935–1936

MENASHA, WIS., AND ITHACA, N. Y.
PUBLISHED BY THE ASSOCIATION
1935

For a list of all officers and trustees for preceding years, see page 63

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* Not a member of the Board.

Representatives on Councils of Scientific Organizations

National Research Council—W. R. Longley, Yale University, for the term ending July 1, 1938.
American Association for the Advancement of Science—C. S. Atchison, Washington and Jefferson College, and W. D. Cairns, Oberlin College, for the year 1935.

Associate Editors of the Official Journal for 1935

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Representing the Association 1935

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SASKATOON. Ferns, Ling, Pyke.
TORONTO. Beatty, Brauer, Pounder, Roseburgh, Synge.
VANCOUVER. Buchanan, Gage, Nowlan.
WINNIPEG. Warren, Wilson.
WOLFVILLE. Jeffery.

COLORADO. (28)

BOULDER. Britton, DeLong, Hutchinson, Johnson, Kempner, Kendall, Light, McMaster, Nelson.
COLORADO SPRINGS. Lovitt, Roos, Sisam.
DENVER. Carmichael, Dickinson, Gorrell, Lewis, McGinley, Rainville, Recht, Sabin.
FORT COLLINS. Aroian, Clark, Gunder, Macdonald.
GOLDEN. Everett, Fitterer, Hebel.
LORETTO. Cook.

CONNECTICUT. (35)

HARTFORD. Bronstein, Dadourian, Donchian, Elston, Flynn, Keffer, Mitchell, Rosenbaum, Welling.
MIDDLETOWN. Arnold, Camp, Howland.
MILFORD. Burgess.
NEW BRITAIN. Fuller.
NEW HAVEN. Barney, Engstrom, Fisher, Hille, Kovarik, Longley, Miles, Ore, Raudenbush, P. F. Smith, Thompson, Tracey, Uhler, Whittemore, Wilson, Wrench.
NEW LONDON. Bower, Dimick, Leib.
STORRS. Cheney.
WEST HAVEN. Anderton.

CUBA. (1)

HAVANA. Corral y Alemán.

DELAWARE. (9)

CLAYMONT. Sromovsky.
DAGSBORO. Short.
LAUREL. Stauffer.
NEWARK. Harding, Harter, R. W. Jones, McDougle, Rees, Wilson.

DISTRICT OF COLUMBIA. (42)

BROOKLAND. Landry.
WASHINGTON. H. S. Adams, O. S. Adams, Atchison, Avers, Blake, Claire, Cromwell, Darling, Duerksen, Edmonston, Ewin, Federico, Fleming, Goldberg, Greville, Hamilton, Johnston, Lambert, Larrivee, Lennahan, Maloney, Mears, Nagle, Ramler, Reynolds, Rice, Richmond, Sasuly, Sohon, Stevens, Taylor, Tunell, Tyler, Van Orstrand, Wallis Watts, Weida, Winston, Woolard, Woolhiser, Zoch.

FLORIDA. (14)

DE LAND. Faulkner.
GAINESVILLE. Davis, Dostal, Kokomoor, Kusner, Messick, McInnis, Pirenian, Simpson, Wilson.
LAKELAND. Reinsch.
TALLAHASSEE. Larson, E. R. Smith.
WINTER PARK. Weinberg.

GEORGIA. (27)

ATHENS. Barrow, Beckwith, Callaway, Cox, Cumming, Hill, Stephens.
ATLANTA. Ballou, Field, Fulmer, Hefner, Howe, Parker, D. M. Smith, Starrett, Steen.
DAHLONEGA. Barnes, Rogers.
DECATUR. Gaylord, Robinson.
DEMOREST. W. B. Smith.
DOUGLAS. Reed.
EMORY UNIVERSITY. Messick.
MACON. Wade.
MONROE. Allen.
NORMAN PARK. Ledford.
ROME. Hightower.

HAWAII. (1)

HONOLULU. Yap.

IDAHO. (1)

CALDWELL. Rankin.

ILLINOIS. (115)

BLOOMINGTON. Hunt.
BLUFFS. Carter.
CARDONDALE. Gadske, Wright.
CARTHAGE. Van Velzer.
CHAMPAIGN. Hartley, Moore.
CHARLESTON. Hendrix, Taylor.
CHICAGO. Albert, Aschenbrenner, Barnard, Bartky, Bay, Bibb, Bliss, Breslich, Burdick, Campbell, Christman, J. E. Davis, W. M. Davis, Dickson, Ettinger, Everett, Georges, Gerst, Goldstine, Gore, Graves, Haggard, Kinney, Krathwohl, Kurzin, Lane, Lange, Lasser, Logsdon, Lunn, MacMillan, Moulton, Oldenburger, Olson, Petty, Rasmusen, Reid, Rood, Sanger, Schweitzer, Slaught, Stone, Young.
CLINTON. Querfeld.
DECATUR. Kiefer.
DE KALB. Parson, Stelford, Storm.
ELMHURST. Ollmann.
EUREKA. Newson.
EVANSTON. Curtiss, Holgate, MacMartin, Moulton, Newell, Simmons, Wescott, Wood.
FREEPORT. Martin, Mensenkamp.
GALESBURG. Heren, Sellew, Stephens.
GREENVILLE. Scott.
JACKSONVILLE. Miller.
LAKE FOREST. Curtis.
LA SALLE. Carus.
LEBANON. Stowell.
LINCOLN. Balof.
MACOMB. Ginnings, Schreiber.
MANHATTAN. Holloway.
MAYWOOD. Hildebrandt.
MONMOUTH. Beveridge, Winbigler.
NORMAL. Atkin, Flagg, Mills.
OAK PARK. Escott.
PEORIA. Comstock, Gault.
POPLAR GROVE. Newsom.
QUINCY. Braun.
RIVER FOREST. Dobbin.
ROCKFORD. McGavock.
ROCK ISLAND. Cederberg.
SANDWICH. Rumney.
TAYLORVILLE. Dappert.
URBANA. Armstrong, Bailey, Bower, D. M. Brown, Carmichael, Coble, Crathorne, Dennis, Emch, Gentry, Hazlett, Levy, Lytle, Miles, Miller, Peters, Rumsey, Shreve.
WHEATON. Boyce.

INDIANA. (49)

BLOOMINGTON. Davis, Hacker, Hanna, Hennel, Rothrock, White, Williams, Wolfe.
CRAWFORDSVILLE. Carscallen.
DANVILLE. Cole.
EARLHAM. Long.
FORT WAYNE. Paxton, Reising.
GARY. Copp.
GOSHEN. Lehman.
GREENCASTLE. Arnold, Edington, Greenleaf.
HANOVER. Meyer.
INDIANAPOLIS. Butchart, Lutz, Suter.
LA FAYETTE. Marshall, Mason, Miller.
MARION. Porter.
MISHAWAKA. McColgin.
MUNCIE. Edwards, Shively.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparó, Maurus, Reynolds, Erwin.
TERRE HAUTE. Kennedy, Shriner, Sousley.
UPLAND. Draper.
VALPARAISO. Frick.
WEST LAFAYETTE. Black, Doan, Graves, Hadley, Hardman, Hazard, Hodge, Hughes, Klinger, Robbins, Stone.

IOWA. (49)

AMES. Brandner, Colpitts, Daniells, Fleming, Gouwens, Herr, J. V. McKelvey, M. M. McKelvey, Roberts, Robertson, E. R. Smith, Snedecor, Stagner, Turner.
CEDAR FALLS. Kearney, Wester.
CEDAR RAPIDS. Coffin, Yothers.
CORNING. Stephenson.

DAVENPORT. Wilson.
DES MOINES. Corey, Neff, Weinberger, Westemeier.
DUBUQUE. Theobald, Zimmerman.
FAYETTE. Deming.
FOREST CITY. Mundhjeld.
GRINNELL. McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. Baker, Chittenden, Conkwright, Craig, Reilly, Rietz, Ward, Woods, Wylie.
IOWA FALLS. Kreider.
KEOKUK. West.
MOUNT VERNON. McGaw, Moots.
ODEBOLT. Wilmer.
OSKALOOSA. Pyle.
SIOUX CITY. Graber, Gwinn.

KANSAS. (44)

ATCHISON. Pretz, Sullivan.
BALDWIN. Garrett.
BETHEL COLLEGE. Richert.
EL DORADO. Wrestler.
EMPORIA. Peterson, Philips.
HAYS. Colyer, Grabbe.
HESSTON. Driver.
INDEPENDENCE. Bell.
KANSAS CITY. Dougherty, Thornton.
LAWRENCE. Ashton, Babcock, Black, Jordan, Mitchell, G. W. Smith, Stouffer, Wheeler.
LEAVENWORTH. Ann Elizabeth.
LINDSBORG. Marm.
MANHATTAN. Babcock, Daugherty, Hyde, Janes, Lewis, Lyons, Mossman, Remick, Stratton, White.
PITTSBURG. Shirk, R. G. Smith.
SALINA. Arnoldy, Ploenges.
STERLING. Bell.
TOPEKA. Harshbarger, Householder.
WICHITA. Hoare, Longenecker, Read, Reagan.
WINFIELD. Bird.

KENTUCKY. (31)

BARBOURVILLE. Haswell.
BEREA. Hutcherson, Pugsley.
BOWLING GREEN. Yarbrough.
COVINGTON. Thuener.
DANVILLE. Fehn.
GEORGETOWN. Hatfield.
HOPKINSVILLE. Nowlan.
LEXINGTON. Allison, Boyd, Brown, Cohen, Davis, Downing, Latimer, LeStourgeon, Mathis, Pence, Rees, South, Wright.
LOUISVILLE. Bullitt, Moore, Morrison, Simester, Stevenson.
MOREHEAD. Black, Fair.
MURRAY. Carman.
RICHMOND. Caldwell, Park.

LOUISIANA. (26)

BATON ROUGE. Daspit, Nichols, Sanders, H. L. Smith.
LAFAYETTE. Buchanan.
NATCHITOCHES. Blair, Killen, Maddox.
NEW ORLEANS. Anderson, Blumenthal, Buchanan, Dinwiddie, Duren, Frankenbush, Many, Menuet, Monasterio, Spencer, Thomson, Titsworth, Weiss.
PINEVILLE. Temple.
RUSTON. Hutcheson, P. K. Smith.
SCOTLANDVILLE. James.
SHREVEPORT. Maizlish.

MAINE. (12)

BRUNSWICK. Hammond, Holmes, Korgen, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
LISBON FALLS. Schultz.
ORONO. Bryan, Hart, Jordan.
WATERVILLE. Ashcraft.

MARYLAND. (49)

ABERDEEN. Dederick.
ANNAPOLIS. Ball, Bingley, Bramble, Capron, Church, Clements, Dillingham, Kells, Lamb, Leiper, Littauer, Lyle, Moore, Rawlins, Root, Scarborough, Tyler.
BALTIMORE. Bacon, Cohen, Curtiss, Dorroh, C. H. Harry, S. C. Harry, Haviland, Keppler, Lewis, Love, Majella, Mary Cordia, Morrill, Murnaghan, Reed, Reynolds, Richeson, Thomsen, Torrey, Williamson, Zariski.
CHELTENHAM. Hartnell.
COLLEGE PARK. Gwinner, Richeson, Spann, Taliaferro.
EMMITTSBURG. Burke.
FREDERICK. Brown, Mason, Weeber.
PORT DEPOSIT. Haviland.

MASSACHUSETTS. (86)

AMESBURY. Dame.
AMHERST. Esty, Miller, Moore.
ANDOVER. Newton.
ATTLEBORO. Holt.
BELMONT. Douglass, Rutledge.
BOSTON. Andrew, Benander, Bruce, Gould, Laurentine, Leavens, Mode, Spear, Weaver, Wilson.
BROOKLINE. Miller.
CAMBRIDGE. Beatley, Birkhoff, Bradley, Brown, Cameron, Coolidge, Cooper, Crum, Emmons, Franklin, Gaylord, Graustein, Harvey, Hestenes, Huntington, Kennelly, Langer, MacGregor, Passano, Stone, Street, Walsh, Widder, Wilson, Woods, Zeldin.
CHESTNUT HILL. Marcou, O'Donnell.
DANVERS. Aurelius.
DORCHESTER. Davis.
GROTON. Nash.
NATICK. Willis.
NEWTON CENTRE. Garabedian.
NORTHAMPTON. Benedict, McCoy, Munroe, Rambo, Wood.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
READING. Skofield.
SCITUATE. Gillespie.
SOUTHBRIDGE. Boeder.

SOUTH HADLEY. Baker, Doak, Martin.
SWAMPSCOTT. Evans.
TYNGSBORO. Richmond.
TUFTS COLLEGE. Mergendahl, Ransom.
WELLESLEY. Copeland, Merrill, C. E. Smith, Stark, Young.
WESTON. Burke.
WILLIAMSTOWN. Agard, Hardy, Wells.
WINCHESTER. Lockwood.
WOLLASTON. Dennison.
WORCESTER. Brown, Gay, Melville, Morley, Rice, Wheeler.

MICHIGAN (61)

ALBION. Ingalls, Sleight.
ANN ARBOR. Ayres, Baten, Bradshaw, Churchill, Coe, Copeland, Craig, Field, Ford, Glover, Hildebrandt, Hopkins, Kaltenborn, Karpinski, Love, MacKay, Nyswander, Raiford, Rainich, Rouse, Running, Schorling, Welmers, Wilder.
BAY CITY. Shellenbarger.
DETROIT. Borgman, Darnell, Denton, Fischer, Fisk, Folley, Johnson, Johnston, McCarthy, Mary Paula, Muehlman, Nelson, Pixley, Shires, Thome.
EAST LANSING. Crowe, Grove, Olson, Plant, Powell, Speeker.
FLINT. Swanson.
IRONWOOD. Field.
JACKSON. Richards.
KALAMAZOO. Ackley, Blair, Everett, Walton.
MARQUETTE. Spooner.
MOUNT PLEASANT. Richtmeyer.
YPSILANTI. Barnhill, Erikson, Lindquist, Matteson.

MINNESOTA. (45)

COLERAINE. Fattu, Tangjerd.
COLLEGEVILLE. Danzl, Hansen, Winkelmann.
DULUTH. Cothran, Strane.
EVELETH. Pollard.
GILBERT. Schey.
MINNEAPOLIS. Brink, Brooke, Bussey, Carlson, Dalaker, Gibbens, Hart, Hartig, Jackson, Jensen, Kirchner, Ness, Priester, Quaid, Scammon, Scherberg, Shuman, Shumway, Thorp, Underhill, Wilder.
MOORHEAD. Olson.
NORTHFIELD. Carlson, Gingrich, Solum, White.
ROCHESTER. Cruise.
ST. JOSEPH. Claudette.
ST. PAUL. Blackall, Bush, Mary Aloysius, O'Toole, Rysgaard, Taylor.
ST. PETER. Rundstrom.
VIRGINIA. Hancock.

MISSISSIPPI. (18)

BLUE MOUNTAIN. Hutchins.
CLEVELAND. Hickey.
CLINTON. Hitt.
DUCK HILL. Caruthers.
ELLISVILLE. McFarland.
GRENADA. Harris.
HATTIESBURG. Baker, Dearman.
JACKSON. Babbitt, McCoy.
RAYMOND. McDonald.
STATE COLLEGE. Ollivier, C. D. Smith.
UNIVERSITY. Bickerstaff, Hume, Quarles, Scott.
WESSON. Felder.

MISSOURI. (40)

CANTON. Ingold.
CAPE GIRARDEAU. Knepper.
CARTHAGE. Murto.
CLAYTON. Haertter, Rosskopf, Rule.
COLUMBIA. Callaway, Robinson, Wahlin, Westfall.
FULTON. Christian, Sweazey.
JEFFERSON CITY. Jason.
KANSAS CITY. Bennett, Cutting, Pierson, Sigley.
KIRKSVILLE. Cosby, Jamison.
KIRKWOOD. Harris.
ROLLA. Hinsch.
ST. CHARLES. Karr.
ST. LOUIS. Buell, Callaghan, Case, Dunkel, Gove, King, Osborn, Pennell, Rider, Roever, Siroky, E. Stephens, J. Y. Stephens, Szegö.
SPRINGFIELD. Finkel, H'Doubler.
SULLIVAN. Beasley.
WEBSTER GROVES. Clarke.

MONTANA. (5)

BOZEMAN. Hurst.
HELENA. Canning.
MISSOULA. Carey, Lennes, Merrill.

NEBRASKA. (24)

BEAVER CROSSING. Thompson.
COLUMBUS. Nicolet.
GILEAD. Erwin.
HASTINGS. McDill.
LINCOLN. Basoco, Brenke, Camp, Candy, Gaba, Hansen, Howie, Pierce, Runge, Specht, Stafford.
OMAHA. Bettinger, Daum, Earl, Fitzpatrick, Gunn.
PERU. Hill.
WAYNE. Boyce, Hove.
YORK. Feemster.

NEVADA. (2)

RENO. Roman, Wood.

NEW HAMPSHIRE. (18)

CONCORD. Conwell.
DURHAM. Slobin, Wilbur.
EXETER. Butterfield, Funkhouser. Sweet.
HANOVER. Beetle, Brown, Forsyth, Mathewson, Morgan, Perkins, Robinson, Silverman, Wilder.
MANCHESTER. O'Leary.
PLYMOUTH. G. M. Smith.
STRATHAM. Wiggin.

NEW JERSEY. (47)

BELLEPLAIN. Durell.

BLOOMFIELD. Hussey.
EAST ORANGE. Nordgaard, Stanwick.
ENGLEWOOD. Echols.
HACKENSACK. Dresher.
HIGHTSTOWN. Litterick.
HOBOKEN. Hazeltine, Murray.
JERSEY CITY. J. P. Smith.
LAWRENCEVILLE. Kimball, Mikesh.
MONTCLAIR. Davis, Fehr, Mallory.
NEWARK. Conkling.
NEW BRUNSWICK. Bunyan, Meder, Morris, Nelson, Starke, Walter, Wilson.
ORANGE. Strock.
PRINCETON. Adams, Alexander, Chittenden, Clifford, Cutler, Eisenhart, Flood, Franklin, Gill, Gillespie, Kimball, Knebelman, Lefschetz, Morse, Pitcher, Thomas, Veblen, von Neumann, Wedderburn.
SOUTH ORANGE. Loveridge.
TRENTON. Shuster.
UPPER MONTCLAIR. Hildebrandt.
WORTENDYKE. F. E. Smith.

NEW MEXICO. (13)

ALBUQUERQUE. Bauer, Graham, Larsen, Munn, Munro, Newsom.
LAS VEGAS. Roberts, Rodgers.
PORTALES. MacKay.
ROSWELL. Harp.
SILVER CITY. Mickelson.
SOCORRO. Reece.
STATE COLLEGE. Rees.

NEW YORK. (238)

ALBANY. Alice Irene, Beaver, Birchenough, Do Bell, Lester, Lowenstein, Stokes, Weeber.
ALFRED. Polan, Seidlin, Titsworth, Whitford.
ANNANDALE-ON-HUDSON. Garabedian, Phalen.
AURORA. Hollcroft, Rusk.
BALDWIN. Grove.
BROOKLYN. Berry, Bowden, Charosh, Cowles, Fleisher, Francis Xavier, Griffin, Hertzler, R. A. Johnson, Kaplan, Karnow, Kennison, Koch, A. W. Landers, M. K. Landers, Langman, Lieber, Locke, MacNeish, Mary Thecla, Milkman, Moore, Penn, Ruderman, Schuyler, F. E. Smith, Thompson, Walter, Welkowitz, Whitford, Young-Woodbridge.
BUFFALO. Gehman, Harrington, Montague, Ott, Pound.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CORONA. Hanson.
ELMIRA. Suffa, Wright.
FISHERS ISLAND. Phillips.
FLUSHING. Lehmann.
FOREST HILLS. Walker.
GARDEN CITY. Rice.
GENEVA. Durfee, Hubbs.
HAMILTON. Aude, A. W. Smith.
HOUGHTON. Davison.
ITHACA. Agnew, Barbour, Blackall, Boothroyd, Campbell, Carver, Flexner, Fulmer, Hurwitz, B. W. Jones, Karapetoff, Randolph, Snyder, Trevor, Walker, Wray.
JORDAN. Howe.
KENMORE. Brockett.
NEW YORK. Adams, Alfieri, Allen, Allison, Anderson, Archibald, Berger, Bergstresser, Berkeley, Bernstein, Berry, Bradley, Brewster, A. B. Brown, Burgess, J. C. Bushey, J. H. Bushey, G. A. Campbell, G. C. Campbell, Clark, Cooley, Doermann, Echols, Eisele, Farnum, Feld, Fiske, Fite, Flanders, Foster, Frankel, Fry, Gentzler, Gilder-Claude, Gill, Ginsburg, Graham, Hall, Hamilton, Hawkes, Hayes, Henderson, Hill, Hopper, Hurwitz, Jablonower, Joffe, P. C. Jones, Kasner, Koopman, Kutman, Larkin, Linehan, MacColl, Miller, Mirick, Molina, Mullins, Nehrbas, Oehler, Payne, Pedersen, Penney, Peters, Plimpton, Pride, Putnam, Quilty, Reddick, Rees, Reeve, Richmond, Ritt, Rosinger, Roth, Schelkunoff, Schlauch, Shaw, Shewhart, Siceloff, Simons, Skelding, D. E. Smith, R. F. Smith, Stabler, Streator, Tanzola, Tilley, Turner, Upton, Velton, Wahlert, Walker, Weaver, Wechsler, Weisner, Whelan, Whitford, Wirth, Wood, Wright.
NIAGARA FALLS. O'Connor.
ODGENSBURG. Vaughan.
ONEONTA. Sanford, Schoonmaker.
PARISH. Church.
POTSDAM. Waltz.
POUGHKEEPSIE. Cummings. Wells.
ROCHESTER. Betz, Byrnes, Eastham, Gale, Gergen, Harding, Long, Mestler, Price, Seidel, Smyth, Watkeys.
ROME. Wardwell.
ST. BONAVENTURE. Nickol, Scheier, Wheeler.
SCARSDALE. Lawton, Mac Neish.
SCHENECTADY. Fox, Morse, Poritsky, Snyder, Ulrich, Vedder.
STATEN ISLAND. Andersen.
SYRACUSE. Campbell, Carroll, Church, Decker, Harwood, Roe, Ryan, Taylor.
TROY. Allen, Crockett, McGiffert.
WEST POINT. Echols, Jones.
WILLSEYVILLE. Seely.
YONKERS. Hubert, John, Yanosik.

NORTH CAROLINA. (26)

CHAPEL HILL. Browne, Henderson, Lasley, Linker, Mackie.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM. Dressel, Elliott, Hickson, Rankin, Thomas.
GREENSBORO. Barton, Pegram, Strong.
GREENVILLE. Graham, ReBarker.
MARS HILL. Robinson.
RALEIGH. Cell, Levine, Winton.
STATESVILLE. Stokes.
WILMINGTON. Downing.
WINGATE. Hendricks.

NORTH DAKOTA. (6)

FARGO. Householder, I. W. Smith.
GRAND FORKS. Mason, Staley.

UNIVERSITY. Hitchcock.
VALLEY CITY. Meyer.

OHIO. (115)

ADA. Whitted.
AKRON. Bender, Zarich.
ALLIANCE. Hildner.
ATHENS. Marquis, Reed.
BEREA. Baur, Dustheimer.
BLUFFTON. Hirschler.
BOWLING GREEN. Mathias, Overman.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Baker, Barnett, Brand, Butler, Hancock, Justice, Kennedy, Kersten, Kindle, Lubin, Merriman, Moore, Newlin, Reed, Salkover, E. S. Smith, Yowell.
CLEVELAND. Boyce, O. E. Brown, Burington, Focke, Hadley, Johnson, Jonah, Justin, Margraf, Morris, Musselman, Nassau, Sauté, Simon, Thomas, Torrance, Van Engen.
COLUMBUS. Bailey, Bamforth, Beatty, Blumberg, Horn, M. E. Jones, Kuhn, LaPaz, Manson, Morris, Radó, Rasor, Rhodes, Rickard, Singer, Toops, Wildermuth.
DAYTON. McGee.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
FINDLAY. Roots.
GAMBIER. Allen, Bumer.
GRANVILLE. Ladner, Wiley.
HILLIARD. Weaver.
HIRAM. Clarke.
KENT. Harshbarger, Manchester, Rogers, Stelson.
LOUDONVILLE. Feinler.
MARIETTA. Cope, Sandt.
MOUNT ST. JOSEPH. Corona.
NEW LEXINGTON. Hoops.
NORTH CANTON. Schug.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Anderson, Pollard, Spenceley, Tappan, Wolfe.
PAINESVILLE. Lewis.
ROSS. Haldeman.
SOUTH EUCLID. Garvin.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Mercedes, Winslow, Yeager.
TROY. Keller.
WESTERVILLE. Glover.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan, Dwyer.
YOUNGSTOWN. Foard.

OKLAHOMA. (26)

ADA. Heimann.
ALVA. Hall.
CHICKASHA. Miller.
DURANT. Bridges.
EDMOND. Johnson.
NORMAN. Brixey, Court, Dolezal, Duval, Hassler, LaFon, McFarland, Reaves, Springer.
SHAWNEE. Dwight, Short.
STILLWATER. Allen, Barnett, Flanders, Garretson, Gundersen, H. W. Smith, Zant.
TULSA. Veatch, West.
WEATHERFORD. McCormick.

OREGON. (9)

ALBANY. Keeler.
CORVALLIS. Beaty, Kirkham, Milne, Williams.
EUGENE. De Cou.
McMINNVILLE. Ramsey.
PORTLAND. Griffin, Merriss.

PANAMA. (1)

PANAMA CITY. Linares.

PENNSYLVANIA. (136)

ALLENTOWN. Deck, Hallett.
ANNVILLE. Wagner.
BARNESBORO. Fisanick.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Cairns, Fort, Latshaw, Lehmer, Rau, Raynor, Reynolds, Shook, Smail, Van Arnam.
BRYN MAWR. Lehr, Peterson, Shover, Wheeler.
BUTLER. Robb.
CAMP HILL. Foberg.
CARLISLE. Ayres, Landis.
COLLEGEVILLE. Clawson, Manning.
DENVER. Marburger.
EASTON. Benner, Beverley, Cawley, Free, Hall, Hatch, W. M. Smith.
ERIE. Benedicta, R. E. Smith, Oergel, Wells.
GREENSBURG. McNeil.
GROVE CITY. Grimes, Renwick.
HARRISBURG. Whited.
HAVERFORD. Oakley, Wilson.
HUNTINGDON. Hess, Shively, Stayer.
KUTZTOWN. Knedler, Kunkel.
LANCASTER. Charles, Long, Worthington.
LATROBE. Seubert.
LEETSDALE. Buker.
LEWISBURG. Gold, Lindemann, MacCreadie, Richardson.
LOCKHAVEN. S. J. Smith.
MEADVILLE. Beisel.
NEW KENSINGTON. Sturm.
NEW WILMINGTON. Black.
PHILADELPHIA. Caris, Constable, Davis, Eggert, Evans, Kline, Latshaw, Linton, Mitchell, Rosengarten, Roulton, Safford, Shohat, Spencer, Tartler.
PITTSBURGH. Aberle, Baird, Calkins, Calvert, Cowley, Dines, Hicks, Hoover, Johnson, Karpov, Leifer, Moskowitz, Neelley, Olds, Petrie, Riggs, Rosenbach, Saibel, Swartzel, Taber, Taylor, Wagner, Whitman.
SCRANTON. Bertrand, Mary Daniel, Sheridan.
SEWICKLEY. Miller.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Hamilton,

Moody, F. W. Owens, H. B. Owens, Rupp, Sheffer, Wagner, West.
SWARTHMORE. Brinkmann, Dresden, Kovalenko, Marriott, Schoenberg, Williams.
SWISSVALE. Foraker.
UPPER DARBY. McDonough.
WAYNESBURG. Moston.
WASHINGTON. Atchison, Bert, Dorwart, Rasel, Shaub, Thomas.

PHILIPPINE ISLANDS. (1)

MANILA. Jimenez.

RHODE ISLAND. (17)

EAST PROVIDENCE. Tyler.
NEWPORT. Chase.
PROVIDENCE. Adams, Adkins, Archibald, Bennett, Carlen, Currier, Gilman, Hill, Manning, Reves, Richardson, Smiley, Sperry, Tamarkin, Watt.

SOUTH CAROLINA. (11)

CHARLESTON. Dye, Hair.
COLLEGE PLACE. Weber.
COLUMBIA. Coleman, Jackson, Williams.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Grant.
SALUDA. Ramage.
SPARTANBURG. Peck.

SOUTH DAKOTA. (10)

BROOKINGS. Miller, Walder, Wente.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Bowles.
SIOUX FALLS. Fuller.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILION. Ekman.

TENNESSEE. (19)

CLEVELAND. Hutto.
COOKEVILLE. Hutchinson.
GREENVILLE. Allen.
JOHNSON CITY. Carson.
KNOXVILLE. Bond, Ghormley, Sisk.
MARYVILLE. Knapp.
MEMPHIS. Locke.
NASHVILLE. Blair, Hyden, S. I. Jones, McPherson, N. P. Miser, W. L. Miser, Morrel, Peterson, Wren.
TOWNSEND. Keller.

TEXAS. (63)

ABILENE. Burnam, Mullings, Tate.
ALPINE. Gilley.
AUSTIN. Batchelder, Benedict, Cooper, Craig, Decherd, Dodd, Ettlinger, Lubben, Moore, Robinson, Vandiver.
BROWNSVILLE. De la Garza.
CANYON. Murray.
COLLEGE STATION. Binney, Blumberg, Edmonson, Porter.
COMMERCE. Box.
DALLAS. Mouzon.
DENTON. Barksdale, Brown, White.
EDINBURGH. Searcy.
EL PASO. Turrittin.
FORT WORTH. Howard, Sherer.
GALVESTON. Underwood
GEORGETOWN. Wapple.
GREENVILLE. Rogers.
HEREFORD. Rice.
HOUSTON. Blau, Bray, Dean, Dix, Ford, Kennedy, Lovett, W. A. Rees.
HUNTSVILLE. Querry.
LUBBOCK. May, Michie, Sparks, Thompson, Underwood.
NACOGDOCHES. Ferguson.
OGLESBY. Thomas.
PORT ARTHUR. Winn.
PRAIRIE VIEW. Randall, Turner, Wilson.
SAN ANTONIO. Hurry, McNelly, Schnepp, Wunder.
STEPHENVILLE. Cromwell, McSweeny, Redden.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams.

UTAH. (4)

EPHRAIM. Horsfall.
SALT LAKE CITY. Gibson, Hayes, Pehrson.

VERMONT. (11)

BURLINGTON. Bullard, Butterfield, Millington, Swift, Thomas.
MIDDLEBURY. Bowker, Hazeltine, Perkins, Wiley.
NORTHFIELD. Dix.
PUTNEY. Alliot.

VIRGINIA. (47)

ASHLAND. Simpson.
AYLOR. Aylor.
BLACKSBURG. Hatcher, O'Shaughnessy, Rasche, Williams.
BLUEFIELD. Berry, Wright.
CHARLOTTESVILLE. McShane.
EAST RADFORD. McCain.
EMORY. Miller.
FARMVILLE. Taliaferro.
HAMPTON. Perkins.
LANGLEY FIELD. Pinkerton.
LEXINGTON. Byrne, Paxton, Purdie, L. W. Smith.
LYNCHBURG. Berry, Garland, Larew, Pattillo, Wiggin.
MONTEREY. Colaw.
RICHMOND. Drew, Gaines, Harris, Parker, Whaley, Wheeler.
SALEM. Carpenter, Puckett.
SOUTH BOSTON. Patten.
STAUNTON. Taylor.
SWEET BRIAR. Morenus.
UNIVERSITY. Blincoe, Buchanan, Linfield, Luck, Oglesby, Sparrow, Watson, Whyburn.
WILLIAMSBURG. Calkins, Gregory, Russell, Stetson.

WASHINGTON. (16)

PULLMAN. Butler, Isaacs.
SEATTLE. Ballantine, Beegle, Carlson, Cramlet, Garnett, Haller, McFarlan, Moritz, Mullemeister, Neikirk, Winger.
TACOMA. Martin.
WALLA WALLA. Bratton.
YAKIMA. Whitney.

WEST VIRGINIA. (14)

BELLE. Bennett.
ELKINS. Vest.
HUNTINGTON. Hackney.
INWOOD. Mish.
MONTGOMERY. W. F. Smith.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Turner, Vehse.
SMITHERS. Bell.
WEST LIBERTY. Kiplinger.
WHEELING. Bagby.

WISCONSIN. (51)

BELOIT. Bigelow, Conwell, Huffer.
GREEN BAY. Kalcik.
MADISON. Allen, Arnold, Bennett, Cook, Evans, Hart, Hartung, Ingraham, Langer, Lowney, MacDuffee, March, Sokolnikoff, Trump, Van Vleck, Ward, Wegner.
MILWAUKEE. Axen, Battig, Beckwith, Ericson, Knight, Ledesma, Luteyn, Marden, Mary Felice, Norris, Parkinson, Pettit, Quarles. Rasor, Rose, Roth, Vass, Wilczewski.
OSHKOSH. Bartlett, Beenken, Price.
RIPON. Woodmansee.
RIVER FALLS. Eide.
SHEBOYGAN. McNair.
SUPERIOR. Flogstad, C. W. Smith.
WAUKESHA. Hopkins.
WEST ALLIS. Wolf.
WEST DE PERE. De Cleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (5)

LARAMIE. Barr, Bellamy, Neubauer, Rechard, Stewart.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINA. (1)

BUENOS AIRES. Baidaff.

BELGIUM. (1)

LOUVAIN. Vanhee.

BRITISH HONDURAS. (1)

COROZAL. Zimmerman.

BURMA. (1)

RANGOON. Campbell.

CHILE. (1)

SANTIAGO. Salas-Edwards.

CHINA. (6)

AN-CHING. Chang.
CANTON. MacDonald, Woo.
PEKING. Konantz, Shen-Fu.
SHASI. Reilly.

FRANCE. (3)

LE MANS. Thébault.
PARIS. Fréchet, Hadamard.

GERMANY. (1)

GÖTTINGEN. Bond.

GREAT BRITAIN. (6)

BELFAST. Todd.
CAMBRIDGE. Hardy.
DUBLIN. Rowe.
EDINBURGH. Horsburgh.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville.

INDIA. (3)

ALLAHABAD CITY. Mitra.
BOMBAY. Dalal.
POONA. Banerji.

ITALY. (5)

BOLOGNA. Bortolotti, Pincherle.
PALERMO. Crudeli.
ROME. Enriques, Laboccetta.

JAPAN. (2)

SENDAI. Hayashi.
TOKYO. Kobayashi.

KOREA. (1)

PYENGYANG. Parker.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

PALESTINE. (1)

RAM ALLAH. Tarazi.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SIAM. (1)

BANGKOK. Hadlock.

SOUTH AFRICA. (2)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.

SOUTH AUSTRALIA. (1)

ADELAIDE. Wilton.

STRAITS SETTLEMENTS. (1)

SINGAPORE. Oppenheim.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHÂTEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (2)

ISTANBUL. Hazeltine, Mourad.

UKRAINE. (1)

KIEFF. Kryloff.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement of such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the Presdent, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-President such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings, provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (⅔) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

Presidents

E. R. Hedrick..........1916
Florian Cajori..........1917
E. V. Huntington..........1918
H. E. Slaught..........1919
D. E. Smith..........1920
G. A. Miller..........1921
R. C. Archibald..........1922
R. D. Carmichael..........1923
H. L. Rietz..........1924
J. L. Coolidge..........1925
Dunham Jackson..........1926
W. B. Ford..........1927–1928
J. W. Young..........1929–1930
E. T. Bell..........1931–1932
Arnold Dresden..........1933–1934
D. R. Curtiss..........1935–

Vice-Presidents

E. V. Huntington..........1916
G. A. Miller..........1916
D. N. Lehmer..........1917, 1918
Oswald Veblen..........1917
J. W. Young..........1918, 1926
R. G. D. Richardson..........1919
H. L. Rietz..........1919
Helen A. Merrill..........1920
E. J. Wilczynski..........1920
R. C. Archibald..........1921
R. D. Carmichael..........1921, 1922
B. F. Finkel..........1922
A. B. Chace..........1923
L. P. Eisenhart..........1923
J. L. Coolidge..........1924
Dunham Jackson..........1924, 1925
A. A. Bennett..........1925, 1933, 1934
W. B. Ford..........1926
A. J. Kempner..........1927, 1928, 1935
Clara E. Smith..........1927
F. D. Murnaghan..........1928
E. T. Bell..........1929, 1930
W. C. Graustein..........1929, 1930
Arnold Dresden..........1931
C. N. Moore..........1931
W. H. Bussey..........1932
G. C. Evans..........1932
E. B. Stouffer..........1933
E. P. Lane..........1934
L. L. Dines..........1935

Secretary-Treasurer

(Appointed by the Trustees after 1918)

(W. D. Cairns..........1916-

Committee on Official Journal

(Appointed by the Trustees)

H. E. Slaught..........1916–
R. D. Carmichael..........1916–1918
W. H. Bussey..........1916–1918
R. C. Archibald..........1919–1921
W. A. Hurwitz..........1919–1921
A. A. Bennett..........1922
H. P. Manning..........1921–1922
W. B. Ford..........1923–1925
J. L. Coolidge..........1923
A. J. Kempner..........1924–
W. H. Bussey..........1926–1931
W. B. Carver..........1932–

Elected Members of the Board

D. N. Lehmer..........1916–1918, 1922–1924, 1930–1932
R. E. Moritz..........1916–1918
K. D. Swartzel..........1916
Oswald Veblen..........1916, 1920–1922, 1926–1931
R. C. Archibald..........1916–1917, 1923–1930
Florian Cajori..........1916, 1918–1923, 1926–1930
M. B. Porter..........1916–1917
J. W. Young..........1916–1917, 1920–1922
B. F. Finkel..........1916–1921, 1930–
E. H. Moore..........1916–1921, 1923–1928
Alexander Ziwet..........1916–1918
E. R. Hedrick..........1917–1922, 1924–1929, 1932–
J. N. Van der Vries..........1916–1918
Helen A. Merrill..........1917–1919
D. E. Smith..........1917–1919, 1921–1926
Elizabeth B. Cowley..........1918–1920
G. A. Miller..........1918–1920, 1922–1924
E. J. Wilczynski..........1918–1919, 1922–1926
L. P. Eisenhart..........1919–1922, 1925–1930
E. V. Huntington..........1917, 1919–1927, 1933–
E. L. Dodd..........1920
R. D. Carmichael..........1920, 1924–1929
A. A. Bennett..........1921, 1930–1932
H. L. Rietz..........1921–1923, 1925–1930, 1934–
C. F. Gummer..........1921–1925
Dunham Jackson..........1923–1929
Clara E. Smith..........1923–1925
A. B. Chace..........1924–1925
J. L. Coolidge..........1926–1931
E. T. Bell..........1927–1928
E. P. Lane..........1928–1933
W. B. Ford..........1929–1934
E. R. Smith..........1929
W. L. Hart..........1930–
Lao G. Simons..........1930–1931
L. L. Dines..........1931–1933
T. C. Fry..........1931–1933
J. W. Glover..........1931–1933
H. E. Buchanan..........1932–
W. R. Longley..........1932–1934
E. J. Moulton..........1933–
R. W. Brink..........1934–
D. R. Curtiss..........1934
J. L. Walsh..........1934–
Arnold Dresden..........1935–
J. O. Hassler..........1935–
F. D. Murnaghan..........1935–

The Carus Mathematical Monographs

The CARUS MONOGRAPHS are fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS, and her son, DR. EDWARD H. CARUS.

Somewhat more than one-half the members of the ASSOCIATION have taken advantage of the distribution at cost of these Monographs. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at this special price.

It would be a great tribute to the donor and an honor to the ASSOCIATION if a large majority of the members would subscribe for the complete series.

It is believed that the ASSOCIATION is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

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No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSOR DAVID EUGENE SMITH and DOCTOR JEKUTHIEL GINSBURG. (First Impression, 1934.)

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XLII, 1935

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THE MAY MEETING OF THE INDIANA SECTION

The twelfth meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday May 3–4, 1935, at Clifty Falls State Park, Madison, Indiana, and Hanover College, Hanover, Indiana.

A total of seventy-six from seventeen schools registered, including the following twenty-five members of the Association: P. P. Boyd, J. H. Butchart, H. T. Davis, J. E. Dotterer, Olive M. Draper, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, S. G. Hacker, Laurence Hadley, W. R. Hardman, C. T. Hazard, Cora B. Hennel, H. K. Hughes, Florence Long, Juna M. Lutz, T. E. Mason, H. A. Meyer, G. T. Miller, Mary S. Paxton, L. S. Shively, W. O. Shriner, Anna K. Suter, and K. P. Williams.

At the business session the following officers were elected for next year: Professor H. A. Meyer, Hanover College, Chairman; Professor P. D. Edwards, Ball State Teachers College, Secretary.

It was decided to hold a meeting of the Section in connection with the fall meeting of the Indiana Academy of Science to be held at Wabash College, Crawfordsville, Indiana, in November. The spring meeting of 1936 will be held at Manchester College, North Manchester, Indiana.

A committee appointed in 1934 to formulate plans to encourage and recognize well prepared teachers of secondary mathematics made its report. The committee suggested that the Indiana Section of the Association create a "Certificate of Merit in Mathematical Preparation" to be awarded on the basis of examinations taken by prospective teachers before graduation. The report was adopted and the committee instructed to work out details for holding the first examinations during the school year 1935–1936. Professor K. P. Williams of Indiana University is chairman of the committee.

A dinner was held Friday evening at the Inn in Clifty Falls State Park. The visitors were welcomed by Professor Meyer of Hanover College. After the dinner a program was given by members of the mathematics department of Purdue University, the first number of which was the address of the retiring chairman, Professor T. E. Mason: "Is there a popular appeal in mathematics?"

Professor Mason gave a number of instances showing a wide interest in numbers and things mathematical. He believes that teachers of mathematics should take advantage of this interest to develop interest in their subject matter. This is possible in the elementary school, in the high school, and in the college. The natural sciences are making increasing use of mathematics. Economic and educational studies are becoming more and more statistical. Hence, the individual who does not elect to study mathematics is limited more and more in the fields of work that he may enter. This brings to the teacher of mathematics the responsibility for using all possible means of interesting students in the subject matter so that those with ability shall not shun mathematics because of lack of interest.

After Professor Mason's address the following series of papers were given as

suggestions to teachers for awakening popular interest in mathematics:

1. "Driving across the solar system" by Professor Laurence Hadley.
2. "As we number our days" by Professor C. T. Hazard.
3. "Getting out of our own world" by Professor G. H. Graves.
4. "An example of symbolism" by Stanley Bolks, introduced by Professor Mason.
5. "The gambler's chance" by G. T. Miller.
6. "The highly honored elephant" by Professor H. K. Hughes.
7. "The tiring irons" by Neil Little, introduced by Professor Mason.
8. "String figures" by W. R. Hardman.

Abstracts of the papers follow.

1. Professor Hadley built the solar system in miniature with the sun at the monument in the Circle at Indianapolis and with Pluto not far from Clifty Falls.

2. Professor Hazard gave a brief historical sketch of the development of the calendar and made some observations on current proposals to reform it.

3. Professor Graves pointed out that the study of geometry, particularly of four dimensions, gives one experience in drawing conclusions from unfamiliar assumptions. By proper attention to transfer of training, a contribution may be made toward meeting the conditions of life in a rapidly changing world.

4. Mr. Bolks showed some examples of the symbolism used by mathematicians of the seventeenth century and illustrated their method of extracting roots.

5. Mr. Miller gave some examples to show that from a mathematical point of view the gambler with limited capital always loses.

6. Professor Hughes discussed a problem, given in an Algebra of 1692, in the form of a story about a pet elephant belonging to a king. The method and notation used in solving the problem seem very clumsy to us of the present.

7. Mr. Little gave a brief description of the ancient puzzle of the Tiring Irons, demonstrated its operation, and applied mathematics to the solution of a problem concerning it.

8. Mr. Hardman discussed briefly the nature and history of string figures, and demonstrated the method of construction of some of the simpler types.

The session on Saturday morning was held at Hanover College. The following papers were presented:

1. "Reminiscences of forty-four years as a teacher of mathematics" by Professor S. C. Davisson, Indiana University, by invitation.
2. "What about mathematics in the junior high school? One answer" by Vivian R. Ely, George Washington High School, introduced by Professor Mason.
3. "Mathematics as a personal experience" by Professor P. P. Boyd, University of Kentucky, by invitation.
4. "Problems in the training of teachers of mathematics" by Professor L. H. Whitcraft, Ball State Teachers College, introduced by Professor Mason.

5. "Early Indiana mathematics and mathematicians" by Professor W. E. Edington, DePauw University.

Abstracts of the papers follow.

1. Professor Davisson discussed the changes that have taken place in the teaching of mathematics at Indiana University during the past 44 years. He was one of four mathematics majors in the first class to graduate after the adoption of the plan to require students to major in some chosen field. He discussed important contributions of various mathematicians to the development of mathematics in Indiana.

2. Miss Ely gave a brief sketch of the junior high school movement in the Indianapolis Public Schools with a detailed description of the new course of study recently written for the course in general mathematics. She concluded with some comments on the success of the venture and suggestions for future procedure.

3. Dean Boyd mentioned some of the current misunderstandings concerning the nature and usefulness of the mathematician's work. He pointed out the advantages that the mathematical thinker possesses in dealing with public questions because of his loyalty to ideals of accuracy and logical procedure, but warned against the dangers of intolerance and egotism and of failure to "dress up" his social and political argument so as to appeal to the emotion and the will. An attempt was then made to bring out the contributions of mathematical study to one's personal enrichment through understanding of the world and human life and through the "elevation and composed delight" that reward the devotee.

4. Professor Whitcraft discussed three problems which confront teacher training institutions, namely, (1) who should be admitted to teacher training and the method of selecting those to be admitted; (2) the selection of the curriculum which will be of greatest value to the teacher; and (3) the placement in a teaching position of the individual who has completed his training.

5. Professor Edington traced the development of mathematics in Indiana during the nineteenth century. Indiana University, Hanover College, Wabash College, Franklin College, and DePauw University were all founded between 1820 and 1840, the latter four being strictly sectarian in organization and intent at the time of their founding. The presidents and many of the professors of all five institutions were for many years preachers, the work offered was classical, and the mathematics offered was most elementary since there were no high schools and few academies to prepare students for college. However, fluxions or calculus was offered before 1850, but the number of students taking such work was small, and, as in the east, mathematics was taught as a preparation for astronomy. In 1856 a scientific course of three years in which mathematics was stressed was organized at Indiana University, but it was 1868 before this became a standard four year course and the formal choice of a major subject was not declared until 1887. The first Master's degree with mathematics as the major subject was given at Indiana University in 1888 and the first Ph.D. with

mathematics as the major was given in 1912. The development in the other colleges was parallel to that of Indiana University.

P. D. EDWARDS, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The fourth regular meeting of the Allegheny Mountain Section was held at Bethany College, Bethany, West Virginia, on Saturday, May 4, 1935. Sessions were held at 10:30 and at 1:30, with a luncheon at 12:45. Professor C. S. Atchison, chairman of the Section, presided at both sessions. Following the afternoon meeting those in attendance were entertained at a very delightful tea as guests of Bethany College.

Sixty-seven representatives of twenty-one educational institutions and research laboratories attended the meetings, including the following twenty-four members of the Association: C. S. Atchison, L. C. Bagby, O. F. H. Bert, Helen Calkins, W. E. Cleland, Elizabeth B. Cowley, L. L. Dines, N. C. Grimes, E. E. Hess, H. C. Hicks, B. P. Hoover, W. W. McCormick, W. I. Miller, T. W. Moore, L. T. Moston, J. H. Neelley, E. G. Olds, J. B. Rosenbach, E. A. Saibel, C. S. Shively, J. C. Stayer, J. S. Taylor, R. W. Thomas, E. A. Whitman; and two institutional member representatives, H. L. Black and W. H. Cramblet.

The fall meeting was set for Saturday, October 26, 1935, at Geneva College, Beaver Falls, Pennsylvania.

The following seven papers were read:

1 "Secondary mathematics on the college level" by President W. H. Cramblet, Bethany College.

2. "The problem of Chasles for $n=4$" by Professor W. A. Hallam, West Virginia Wesleyan College, introduced by Professor Atchison.

3. "An old Euclid of 1537" by Professor O. F. H. Bert, Washington and Jefferson College.

4. "Some implicit functional theorems" by Professor Helen Calkins, Pennsylvania College for Women.

5. "From the simple to the involved and back again" by A. M. Dudley, Westinghouse Electric and Manufacturing Company, introduced by the Secretary.

6. "Some examples from operational calculus" by Professor M. M. Culver, University of Pittsburgh, introduced by the Secretary.

7. "Seventeenth century calculus" by Professor E. A. Whitman, Carnegie Institute of Technology.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Following a cordial welcome to Bethany College, President Cramblet presented many reasons supporting the opinion that college work in mathematics should be made available to a selected group of graduates from approved high

schools who have taken commercial or vocational subjects and lack part or all of the traditional mathematics units for college entrance. He announced an experimental course of this kind for Bethany College for next year, a course which has been approved by and will be conducted under the supervision of the Commission on Higher Education of the North Central Association. Over a two year period it is hoped that the work will encompass the field regularly covered in the first two years of college mathematics.

2. After a summary of the problem of Chasles for $n=2$ and $n=3$, Professor Hallam gave a preliminary report outlining a method of determining necessary and sufficient conditions that it be possible to pass a plane curve of the fourth order through fifteen given points. The plan of attack involves a reduction of the problem for $n=4$ to a residual problem for $n=3$ and offers suggestions for the solution of the general case.

3. In exhibiting a copy of Euclid printed in 1537 Professor Bert indicated certain features of the volume associated with the period in question and called attention to many interesting sections of the text. As far as has been ascertained certain portions of Euclid's works contained in this edition were here printed for the first time.

4. Professor Calkins stated and proved the conditions under which a functional equation of the type

$$G(x, y(x), y'(x)) - \int_{x_1}^{x} P(x, s, y(s), y'(s))ds = 0 \tag{1}$$

has one and only one continuous solution $y(x)=f(x)$ which reduces to an initial value y_1 when $x=x_1$. The first step is accomplished by establishing an introductory existence theorem for an equation of the form (1) where $y'(x)$ is replaced by $z(x)$. This reduces (1) to an implicit functional similar to that previously discussed by V. Volterra and G. C. Evans.* While the general idea of the proof used by Evans appears in the discussion, the details vary considerably; and in order to exhibit a neighborhood in which a solution exists it was found necessary to adjust Evans's theorem. The second step is carried out by proving a lemma stating the conditions under which the solution of the introductory theorem satisfies a Cauchy-Lipschitz condition.

5. Mr. Dudley called attention to a need of further analysis and effort to make mathematics more generally useful to engineers. He suggested that proper familiarity with and use of mathematics beyond the calculus will greatly facilitate the use of analytic geometry, calculus, and differential equations. Definite suggestions were made as to class room procedure, the use of practical applications, and learning by doing. In conclusion he referred to recent papers by Professor R. G. Minarik, of the University of California, "Are we teaching engineers mathematics?" in the March 1935 issue of the *Journal of Engineering Education*;

* G. C. Evans, *Functionals and their applications*, Cambridge Colloquium (1916), vol. 5, part 1, pp. 70–72.

and by Professor P. W. Ott, of Ohio State University, "The usefulness of mathematics to engineers," in the March 1935 issue of the *General Electric Review*.

6. In discussing some examples of operational calculus, Professor Culver exhibited the nature of the operators involved and illustrated the economy with which the operational method achieves the solution of certain types of problems. In conclusion he called attention to certain existence theorems on which the validity of the operational method is based.

7. In his paper on seventeenth century calculus Professor Whitman mentioned the methods used early in this century to find the tangents to curves and the area under curves, methods considered as steps preliminary to the calculus of Newton and Leibniz. The speaker also exhibited a chart furnishing a convenient means of comparing the life spans of the more prominent mathematicians of the period and summarized their principal contributions to the field in question.

J. S. TAYLOR, *Secretary*

LINEAR GROUPS AND FINITE GEOMETRIES*

By H. H. MITCHELL, University of Pennsylvania

The term "Linear Group" refers to a set of linear transformations, finite or infinite in number, that satisfy the usual postulates for an abstract group. As an example of an infinite group, we have the set of all real rotations about a point in three-dimensional space. An example of a finite group is the set of all rotations of a regular solid into itself, e.g., the 60 rotations that return a regular icosahedron to its original position.

Groups of finite or infinite order may each be subdivided into classes according to the character of the coefficients of the transformations. Thus the group of all rotations about a point in three-dimensional space may be termed "continuous," as the set of transformations which it contains may be expressed in terms of three real parameters, of which two may be considered as fixing the axis of rotation and the third the angle through which the turning is made.

An infinite group of quite a different character, however, is that consisting of all transformations of the form

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where the four coefficients represent rational integers with determinant $+1$. This "discontinuous" group plays an important rôle in connection with the theory of the "equivalence" of binary quadratic forms and of the closely allied theory of elliptic modular functions. Many illustrious names, such as those of Lagrange, Gauss, Klein and Weber, associate themselves with one or the other

* This paper is based on an address with the same title given by the author as retiring Vice-President and Chairman of Section A of the A.A.A.S. at Cambridge, Mass., December, 1933.

of the two topics just mentioned. The name of Sophus Lie, not to mention others, arises on the other hand whenever the subject of continuous groups is mentioned. His interest in them arose mainly from their connection with differential equations.

Either of these two types of groups of infinite order might have been chosen as the subject of a paper of this sort, as the contributions to both theories have been rich and varied. We shall however restrict our attention to the subject of finite groups, except where an occasional reference to one or the other of the two types of infinite groups might seem appropriate.

Finite groups of linear transformations fall naturally into two categories according as the coefficients belong to the field of ordinary complex numbers or are "marks" of a finite field. In the latter case, any set of transformations in a given number of variables must necessarily generate a group of finite order. While we shall return to the consideration of groups of this type later, it seems natural to consider, first, finite groups in which the coefficients are ordinary complex numbers.

Let us deal with linear, homogeneous transformations in n variables. Any such transformation that is of finite order must possess elementary divisors that are exclusively of the first degree and its "multipliers," i.e., the roots of its characteristic equation, must be roots of unity. It follows that by a proper change of variables, it may be written in the "canonial form," $x_i' = \alpha_i x_i$ $(i=1, 2, \cdots, n)$, where each α_i is a root of unity. If we consider not a single transformation, but an "abelian" (i.e., commutative) group of linear transformations, it has been shown that by a proper selection of the variables, all the transformations may simultaneously be written in canonical form.

If we pass to the case of a group that is not abelian, it is evident that the statement just made is not true. For a group whose order is a power of a prime, however, it has been shown that all the transformations may be written simultaneously in "monomial form," i.e., one in which each variable is replaced by a constant times the same, or some other, variable. In general, no such representation is possible and the problem of constructing the possible groups that exist in a given number of variables is one of considerable difficulty, that has been solved completely only for the cases of two, three, and four variables.

For the case of two variables, several methods lead to the desired goal. By use of the familiar stereographic projection of the sphere on the complex plane, it may be shown that there are groups of linear fractional transformations in one variable that are (1, 1) isomorphic with the groups of rotations of the regular solids. Excluding the relatively trivial cyclic and dihedral groups, there are thus at least three types of so-called "primitive" groups corresponding to the regular tetrahedron, octahedron and icosahedron, the solids dual to the last two leading naturally to the same groups. The corresponding binary homogeneous groups are found to be multiply isomorphic with these linear fractional groups.

When the question is raised as to the possible existence of additional binary groups that do not correspond to the regular solids, the answer is in the nega-

tive. It is obviously not possible to represent any linear fractional transformation, the coefficients of which involve essentially three independent complex coefficients, by means of a real rotation of the sphere, which involves three real parameters. On the other hand, if we make use of the theorem (due, among others, to E. H. Moore) that every finite group of linear transformations has as an absolute invariant at least one positive definite Hermitian form, it follows that any finite group of the type considered corresponds to a group of real rotations.

Another method of determining the possible types of binary groups depends on the solution of a diophantine equation due to Jordan, and is well worth mention in view of the possibility of its application to finite groups of other types. It consists essentially of counting the transformations in the various conjugate sets of cyclic groups contained in the given group in terms of the orders of these cyclic groups, the orders of the maximal groups under which they are invariant, and the order of the entire group. By means of this principle it may be proved that no linear fractional group can contain more than three conjugate sets of maximal cyclic subgroups. If there are exactly three, it follows also that those in one set must be of order 2, and those of a second set of order 2 or 3, from which the possible orders of the cyclic groups in the third set and of the entire group may be readily determined.

Let us consider now the case of three homogeneous variables, and let us regard as identical two transformations whose coefficients differ only by a factor of proportionality. From the geometrical point of view therefore we may suppose that we are dealing with the collineation group in the projective plane, provided imaginary elements are not excluded. Any such collineation group may be regarded as the quotient group of a corresponding linear group with respect to its invariant subgroup consisting of transformations whose matrices are scalar.

The first serious attack on the problem of determining the possible ternary groups seems to have been made by Jordan, his chief weapon being the type of diophantine equation referred to above in connection with the discussion of the binary groups. The problem, however, is considerably more complex than in that case. For one thing, a cyclic group of order d may now be invariant under a group of order fd, where $f=1, 2, 3, 6$, depending on the permutations that are made on the invariant triangle. In addition, if the invariant triangles of two cyclic groups have one side and the opposite vertex in common, these groups may have transformations in common other than the identity, i.e., homologies having for axis and center respectively the common side and vertex of the fixed triangles.

In spite of the brilliant method that he devised for his attack on the problem, Jordan's attempt to determine the ternary groups must be considered a failure, as he overlooked two of the most interesting groups. One, the simple group of order 168, that has an invariant quartic curve, was discovered a few years later by Klein. The other, a group of order 360, simply isomorphic with the alternating group on six letters, was later discovered by Valentiner in his attempt to

make a complete determination of the ternary groups. The latter, however, evidently unaware of Jordan's similar attempt, omitted from his enumeration one of the groups found by Jordan, namely, the so-called "Hessian group" of order 216, which permutes the inflexional points of a pencil of cubic curves.

A notable advance in the theory of linear groups, and in the theory of groups in general, occurred near the end of the nineteenth century with the development of the theory of group characters. This theory, due in the main to Frobenius, considers the problem of the construction of linear groups from a somewhat different angle, the question being primarily with the linear groups that give "representations" of a given abstract group, rather than with the linear groups that can be constructed in a given number of variables.

A variety of relations have been proved to connect the numbers which make up the several characters of a group. By use of these relations, together with abstract properties of the group in question, it has been found possible in some cases to determine the numbers of the variables by means of which it may be represented as an irreducible linear group without constructing the groups. This has even been done for certain infinite systems of groups.

It can hardly be denied that any worker in the theory of abstract groups would find a knowledge of group characters virtually indispensable. Thus, by use of it, Burnside proved that any group whose order is a product of powers of two primes is composite, a problem that had appeared too difficult of attack by other methods. A somewhat parallel theory has been developed in recent years for continuous groups.

A new, and highly ingenious, method of attack on the problem of the determination of the linear groups (in a given number of variables) has been developed by Blichfeldt. By forming the products of one transformation by powers of a second one, he obtains relations connecting the multipliers of the various transformations, which are then shown to lead to certain congruences. The latter often give valuable information concerning the orders of the transformations involved. Perhaps the most striking result he has obtained in this way is that the order of a primitive group in n variables cannot be divisible by a prime greater than $(n-1)(2n+1)$.

The question naturally arises whether this limit which Blichfeldt has obtained for a prime that may divide the order of a primitive group can be approximately attained. For $n=2$ there is actually a group, namely, the icosahedral group, where such is the case. There is, however, no primitive group in either three or four variables whose order is divisible by a prime greater than 7, which compares with the limits 14 and 27, respectively, given by the formula. As far as the author is aware, also, there is no known primitive group whose order is divisible by a prime greater than $2n+1$. Blichfeldt has himself shown by a more detailed examination of the congruences referred to that for particular values of n, e.g., $n=3$, 4, this limit may be substantially lowered.

Another ingenious method of analysis, used by Valentiner, Bieberbach and Frobenius, and somewhat refined by Blichfeldt, leads to an upper bound for the

order of any abelian subgroup that may be contained in a primitive group. By use of it Blichfeldt has shown that no primitive group can contain a transformation whose multipliers, when located on the unit-circle, occupy an arc extending not more than 60° on either side of some one of them, a result that goes somewhat further than those obtained by Bieberbach and Frobenius. The application of this result to the determination of an upper bound to the order of any abelian subgroup of a primitive linear group in a given number of variables is not difficult.

The theorems referred to among others enabled Blichfeldt to make the first accurate determination of the groups in three variables, and to make the first determination of the groups in four variables. The latter are too numerous to attempt to enumerate here. One of the most interesting among them is the collineation group of order 25,920, isomorphic with the group of the equation for the twenty-seven lines on a cubic surface.

Some of the groups in two, three, and four variables belong to one of three closely associated infinite systems of groups that have been investigated by Jordan, Klein, Burkhardt and others. Jordan constructed a group in p^m variables, where p is a prime, which may be regarded as a generalization of the Hessian group of order 216 in three variables. Associated with this group there are two other groups in $(p^m-1)/2$ and $(p^m+1)/2$ variables, which represent a quotient group of the first group with respect to an invariant subgroup. For $m=1$, $p=7$, the groups in the last two systems reduce to the known ternary and quaternary groups of order 168, while, for $m=2$, $p=3$, they become quaternary and quinary groups of order 25,920.

Outside of these three systems of groups, comparatively few groups of interest appear to be known in more than four variables. Burnside determined all the collineation groups in $n(>4)$ variables with rational coefficients that contain as a subgroup the symmetric group on those variables. Outside of the rather obvious group of order $(n+1)!$, which exists for any n, the only primitive groups he found were three groups for $n=6$, 7, 8 respectively. The first of these is isomorphic with the group of the equation for the 27 lines on a cubic surface and had already been found by Burkhardt. The one in seven variables he proved to give a representation of the group of the 28 bitangents to a quartic curve.

Shortly after this work by Burnside, the author undertook the solution of a more general problem, namely, the determination of all primitive groups in more than four variables that contain homologies, i.e., transformations in which all the multipliers are equal with the exception of one. A transformation in the symmetric group on the variables that interchanges two of them, and leaves the others unaltered, has -1 for one multiplier and $+1$ for each of the others, being therefore an homology of period 2. In three variables all the primitive groups contain homologies, and in four variables there are several that do, the groups in this category having been determined by Bagnera independently of Blichfeldt's enumeration of all the groups in four variables.

In spite of the seemingly more general character of this problem as com-

pared with that solved by Burnside, no restrictions being placed on the character of the coefficients, the results were chiefly negative. In addition to the groups in Burnside's list only two other primitive groups were found, one a group in five variables belonging to one of the systems referred to above, and the other a group in six variables, the existence of which had apparently not been noted previously. This was found to be isomorphic with Jordan's so-called "first orthogonal group" on six variables with the modulus 3.

Before leaving the subject of linear groups whose coefficients are ordinary complex numbers, it would be inappropriate not to mention a recent result obtained by Hasse. It had long been believed that by a proper choice of coordinates any finite linear group could be represented in a form where the coefficients all belonged to a cyclotomic field. Making use of a result due jointly to R. Brauer, H. Hasse and E. Noether that every normal division algebra over an algebraic field is a cyclic (Dickson) algebra, Hasse was able to show that such a representation of a group always exists. The door to further investigations along this line does not seem to have been completely closed, however, As far as the author is aware, the groups that are known to exist can be represented with coefficients in the field defined by the multipliers of their transformations. Hasse's result does not go this far.

So far in the paper no mention has been made of Finite Geometries. It is when we pass to the consideration of linear groups with coefficients that belong to a finite field of numbers that this concept appears as a natural and useful accompaniment to the algebraic principles involved.

Turning our attention first to the latter, we note as an example of a finite field of numbers the set of residues $0, 1, 2, \cdots, p-1$ with respect to a rational prime p. These are readily seen to obey the laws of combination associated with the usual concept of "field," division by the zero element being of course excluded. This is the type of field used by Jordan in his construction of various systems of modular groups. The set of all linear non-singular substitutions in n variables and with coefficients in this field is easily shown to form a group of order $(p^n-1)(p^n-p) \cdots (p^n-p^{n-1})$. This group has an obvious invariant subgroup of order $p-1$ consisting of the transformations having scalar matrices. The quotient group with respect to this invariant subgroup is analogous to the collineation group in ordinary $(n-1)$-space.

Jordan showed that, if, instead of considering the set of all the linear substitutions defined above, only those with determinant unity (or an nth power in the field) are employed, the quotient group of this modified group with respect to its scalar subgroup is simple with the two exceptions $n=2$, $p=2, 3$. For $n=2$, $p=5$, it is, for example, the familiar icosahedral group of order 60, and for $n=3$, $p=2$, it is the simple group of order 168.

If further restrictions be placed on the coefficients, e.g., if it be assumed that the transformations have certain invariants analogous to those that are associated with the well known continuous groups in the ordinary case, additional groups of interest are obtained. Thus, in the case of an even number of

variables, if bilinear functions of the variables be introduced analogous to the Plücker line coordinates of ordinary 3-space, and the condition be imposed that a linear function of these be left invariant by the transformations, an interesting group results, of which the quotient group with respect to its invariant scalar subgroup in the case of four variables is analogous to the 10-parameter continuous group in ordinary 3-space that has an invariant linear complex.

In spite of these analogies to certain concepts of ordinary geometry, it does not seem as if Jordan attempted to carry them through with their logical implications. No such effort appears indeed to have been made until 1905 when Veblen noted that many of the axioms of ordinary geometry are satisfied by certain finite sets of elements. He first constructed an example of a finite geometry in connection with an independence proof needed for his dissertation. The idea was much elaborated in a joint paper by Veblen and Bussey published a year or so later.

A simple example of a finite geometry in two dimensions is the triple system

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2. \end{matrix}$$

Here the "points" may be considered to be the seven numbers that appear in the table, and the "lines" the seven sets of three numbers each that appear in the columns.

Veblen and Bussey found that the elementary part of the synthetic theory may be developed from certain axioms quite independently of the hypothesis that the number of points is finite. A particularly interesting point occurs in connection with the configuration of Pappus. It had been shown by Hilbert that the assumption of the commutative law of multiplication for the number system associated with the geometry is equivalent to the assumption that the three pairs of opposite sides of the hexagon in this configuration, intersect on a line. In view of Wedderburn's theorem that for an algebra of a finite number of elements, in which every element except 0 possesses an inverse, the commutative law holds, it follows therefore that the three points referred to are necessarily collinear in any finite geometry.

Wedderburn's result supplements in an interesting way the theorem due to E. H. Moore that every finite field is a Galois field. By use of these results, Veblen and Bussey were able to exhibit the collineation groups of the finite geometries by use of modular linear groups of the sort described above, except that the field of the coefficients has now a more general form. By a (finite) Galois field is meant any field that is abstractly identical with the set of residues obtained by reducing any polynomial in an indeterminate x with rational integral coefficients by use of two moduli, one a prime number p, and the other a polynomial in x that is irreducible, modulo p. Moore was able to prove not only that every finite field is a Galois field, but that any two Galois fields containing the same number of elements are abstractly identical. There is thus one,

and only one, finite field containing p^k elements for each prime p and positive integer k.

Using this more general type of field, Dickson was able to generalize the results of Jordan, a comprehensive treatment of the subject appearing in his book, "Linear Groups." An idea of the importance of the subject from the point of view of finite groups in general may be gained from the fact that, with the exception of the alternating groups on more than four letters and a few others, all the known simple groups belong to one or more of the several triply infinite systems that have been thus represented.

Antedating this work of Dickson's by a short time were determinations of the subgroups of the linear fractional group in one variable with coefficients in a general Galois field by Moore and Wiman. For the case of a field of order p this problem had been solved by Gierster. The simple group of order $p(p^2-1)/2$ that appears here for any prime >3 was found to have as subgroups certain metacyclic groups of order fp, where $f=(p-1)/2$, or is some divisor thereof, certain cyclic and dihedral groups, the tetrahedral and octahedral groups of orders 12, 24 respectively, and, for $p\equiv \pm 1$, mod 5, the icosahedral group of order 60. For the case of the general field the problem proved considerably more formidable. All of those who investigated these groups found an indispensable aid in the type of diophantine equation devised by Jordan, which was discussed earlier in this paper. The author found it possible to make an even more extensive use of this equation in a later determination of the same groups.

For the case of the binary groups, the geometrical notions involved appear too simple to make them particularly useful in their determination. If a group be represented in homogeneous form, the symbol (x_1, x_2), where x_1, x_2 represent any pair of elements, or "marks," of the Galois field of order p^k except (0, 0), may be considered to represent a point on the finite line, two such pairs representing the same point if one be obtainable from the other by multiplying both numbers of the pair by the same non-zero mark of the field. In this way p^k+1 points are obtained. Such notions as the cross-ratio of four points, while useful in constructing the possible groups, do not seem adequate to enable one to make an exhaustive enumeration of them.

When one proceeds to the groups in a larger number of variables, however, the situation changes very materially, and the value of the geometric approach to the problem seems much more marked. The truth of this statement will become more evident when one considers how many of the powerful instruments available to the worker in the theory of ordinary linear groups become useless, or nearly so, when he turns his attention to the modular groups. Thus, in the case of group characters, while one may introduce such notions as the traces of the transformations, and obtain certain congruences connecting them, there seems no possibility of developing a theory at all comparable with that in the ordinary case.

As another example, let us consider the theorems that flow from the invariance of a definite Hermitian form. In the modular case there is a natural

analogue of this form, namely, the so-called "hyperorthogonal" invariant introduced by Jordan and later extended to the case of the general Galois field. On the other hand, it is not true that every modular group has such an invariant, although it seems quite possible that this may be the case for the groups whose orders are not divisible by the prime modulus of the field. A useful theorem that follows as a consequence of the invariance of a Hermitian form is to the effect that any group that is reducible is also "intransitive," i.e., completely reducible. Thus any finite collineation group in the ordinary plane that leaves fixed a line must also leave fixed a point not on the line, and similarly for the dual statement. No such assertion can be made, however, for groups in the modular plane.

As one more illustration of the relative paucity of analytical methods available for attempting the determination of the modular groups, let us recall the theorems of Blichfeldt leading to limits to the order of the ordinary groups in a given number of variables. These rest on Kronecker's theorem concerning the irreducibility of the cyclotomic equation, the analogue of which in the case of a Galois field is comparatively trivial and seemingly of no value for the purpose at hand. Somewhat similar remarks might be made concerning the theorems relating to the distribution of the multipliers of transformations that may belong to a primitive group considered as points on the unit circle.

When one turns from the somewhat discouraging considerations of the sort briefly outlined above, to trains of thought suggested by geometrical concepts, prospects appear much brighter. In a sense, finite geometries corresponding to different moduli each present their own geometrical problems and theorems. Thus, in the case of a field of order 2^k, as pointed out in the paper by Veblen and Bussey, to which reference has already been made, there is the somewhat startling result that the diagonal points of a complete quadrangle are collinear! Another theorem for this geometry, obtained by U. G. Mitchell in his Princeton dissertation, is that the tangents to a point conic form a degenerate line conic, i.e., pass through a point. The modulus 2, in fact, seems to lead to such marked differences in the geometries associated with it that it seems necessary to give a separate treatment of the groups.

In the case of odd prime moduli, however, while there are doubtless special theorems for each, there seem also to be a sufficient number that are common to all the associated geometries to make it possible to treat the collineation groups together. When the author was a student at Princeton, Professor Veblen suggested that he undertake a problem along this line, namely, the determination of the ternary collineation groups of the geometry for the field of order 5. It was found possible to carry the problem through, although the methods employed were such as to lead to similar results for any field where the number of elements is a power of an odd prime. The treatment of the groups for the case of a field of order 2^2 was carried out about the same time by U. G. Mitchell in the paper already cited. For a field of order 2^k for any k the problem was later solved by R. M. Hartley.

The first attempt to determine the ternary modular groups appears to have

been made by Burnside, who considered the groups for the case of a field of order p, but his attack on the problem cannot be considered to have been particularly successful. For the same case, a complete determination of those groups which contain operators of period p was made by Dickson. These writers relied on analytical and group-theoretic methods rather than on those suggested by geometrical considerations.

In the attack on the problem by geometrical methods it seems convenient to separate the discussion into two cases according as the groups under consideration do or do not contain transformations whose order is the prime modulus of the field. For the groups of the latter category, it seems reasonable to suppose that the results will be the same as for the ordinary plane. Following a method employed by Valentiner for that case in a paper already cited, one may seek to determine the groups that contain homologies. Any group generated by two homologies must leave fixed the line joining their centers and the point of intersection of their axes, and must therefore be isomorphic with a binary group on each of the two one-dimensional forms so determined. If we are considering therefore the groups whose orders are prime to the modulus of the field, it follows that it is impossible for any such group to contain two homologies having a common center but different axes, a common axis but different centers, or such that the line joining their centers passes through the point of intersection of their axes. In each of these cases one or both of the binary groups referred to will have one and only one invariant element, and hence, by the results for the binary groups obtained by Gierster and generalized by Moore and Wiman, its order cannot be prime to the modulus of the field.

By use of geometrical considerations of the sort described, it is easy to show that any group whose order is prime to the modulus of the field, and that contains homologies of period greater than 3, must either contain an invariant homology, or else exactly three such homologies whose centers and axes form the vertices and sides of a triangle. In neither case therefore is the group "primitive."

In the case of two non-commutative homologies of period 3 the group on the line joining their centers must be the tetrahedral group, as otherwise the fixed points of the cyclic groups of period 3 would be interchanged, leading to the presence of an homology of period 6. The resulting group of order 2.12 contains a reflection, i.e., an homology of period 2, having the line joining the centers of the generating homologies as axis, and the point of intersection of their axes as center. It is then found that the centers and axes of any additional homologies of period 3 must separate harmonically the center and axis of this reflection, and that there can be but eight such homologies, each commutative with one of the four already present. These twelve homologies may then be shown to generate a group of order 216, called the "Hessian Group" by Jordan, who first found it in the ordinary plane, as it is associated with a pencil of cubic curves.

For groups that contain reflections, but no homologies of higher period, it is convenient to separate the discussion into cases according as there are, or are

not, present any four-groups generated by two commutative reflections. In the former case it is found that any reflection not commutative with any one of the three in a four-group leaves invariant a conic in common with the four-group, and must therefore generate with the latter either an octahedral or an icosahedral group, in view of the isomorphism of groups with invariant conics with the binary groups. If no icosahedral group is present, one is led to the simple group of order 168. If an icosahedral group is present, the only other group is found to be the one of order 360 isomorphic with the alternating group on six letters, e.g., the six conics conjugate with the invariant conic of the original icosahedral group.

For the case where no two of the reflections are commutative, geometrical methods of the sort described above no longer seem adequate to meet the situation. Fortunately, however, Jordan's diophantine equation is now much more convenient to employ than where there are, so to speak, overlapping subgroups. It is found by a simple analysis that the only primitive groups are two subgroups of the Hessian group of order 36 and 72. A similar argument shows that there is no primitive group that contains no homologies, and is thus of odd order.

The type of argument that may be employed in determining the groups in the category considered, i.e., those whose orders are prime to the modulus of the field, has been sketched in some detail to show the value of the concepts that Veblen and Bussey have introduced. The argument needs to be only slightly modified to take care of the case where transformations are present in which there is but one invariant point and line. In addition to groups with invariant conics, the only primitive groups found here were of orders 720 and 2520, each of which exists only for a field of order 5^k, where k is even.

In completing the study of the ternary groups, it was only necessary to consider groups that contain elations. It was found possible to treat these in much the same way as those described above. The ones of chief interest found here were the hyperorthogonal groups, analogous to the group of infinite order in the ordinary plane having an invariant Hermitian form, the conjugate imaginary variables being here replaced by suitable powers of the variables themselves.

As a by-product of this investigation, it was possible to determine the minimum number of letters on which two infinite systems of simple groups may be represented as permutation groups, one of them being either the entire collineation group of the plane, or an invariant subgroup of it of index 3; and the other the hyperorthogonal group referred to above. These results were of a character similar to those obtained earlier for the binary groups, for example by E. H. Moore in a paper already cited.

The quaternary groups with coefficients in a finite field do not seem to have been completely determined. The author found it possible, however, to determine those collineation groups in three dimensions whose order is not divisible by the modulus and that do not contain homologies. The groups in ordinary space that contain homologies had been previously determined by Bagnera by

methods applicable to the modular case. Certain classes of quaternary groups had previously been discussed by Dickson for the case of a field of order p. These papers supplement each other and together check Blichfeldt's work on the ordinary quaternary groups.

The author found it possible also to determine the quaternary groups having an invariant linear complex for the case of a field whose order is a power of an odd prime. If the order is the first power of a prime, the group becomes the Galois group of the equation for the p-section of the periods of hyperelliptic functions, and in this connection had been investigated by Jordan, Witting and Burkhardt. It is also isomorphic with certain systems of collineation groups with ordinary coefficients, to which reference was made earlier.

A CORRECTION

Regarding my paper, *A New Theorem Concerning the Rank of a Matrix*, in the December 1934 issue of this MONTHLY, Professor Th. Motzkin of the Hebrew University at Jerusalem points out that "the theorem stated is refuted by the counter example

$$m = 3,\ A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{vmatrix} = 1."$$

The theorem as stated was incomplete. There was inadventently omitted an additional hypothesis as follows:

(4) *If the minor of* α *of order* $(n-m+h)_h$ $(h=1, 2, \cdots, m)$ *every element of which contains the minor* $\left|{}^{g_1}_{g_1}\right|$, *is axisymmetric or skew-symmetric.* The corresponding corrections should be made also in the second statement of the theorem.

This added hypothesis is needed in the proof at line 3 on page 608.

W. H. METZLER

QUESTIONS, DISCUSSIONS AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE ORTHOCENTRIC TETRAHEDRON

By R. GOORMAGHTIGH, Bruges, Belgium

Professor N. A. Court has given, in his *Notes on the orthocentric tetrahedron* (this MONTHLY, vol. 41, 1934, p. 499), this interesting theorem:

The ends of two skew orthogonal diameters of two orthogonal spheres are the vertices of an orthocentric tetrahedron.

Let BC and AD be two diameters of two spheres; denote BC, CA, AB, DA, DB, DC by a, b, c, a', b', c' and $a^2+a'^2$, $b^2+b'^2$, $c^2+c'^2$ by L, M, N respectively. Then if θ is the angle between BC and AD, and α the bimedian joining the midpoints of BC and AD, well known formulas give

$$2aa' \cos \theta = M - N, \quad 4\alpha^2 = -L + M + N.$$

Hence, when $\theta=\pi/2$, $M=N$; and, if further the given spheres are orthogonal, $4\alpha^2=L$, and $L=M=N$; therefore the tetrahedron $ABCD$ is orthocentric.

On the Mean Value Theorem

By H. L. Krall, Pennsylvania State College

A function $y(x)$ which has derivatives of the first n orders can be expanded in the form

$$\begin{aligned} y(x_2) &= y(x_0) + (x_2 - x_0)y'(x_0) + \cdots + \frac{(x_2 - x_0)^{n-1}}{(n-1)!} y^{(n-1)}(x_0) \\ &\quad + \frac{(x_2 - x_0)^n}{n!} y^{(n)}(x_1), \end{aligned} \tag{1}$$

where $x_1=x_0+\phi(x_2-x_0)$ and $0<\phi<1$. The value of ϕ depends on x_0, x_2 and n. It has been shown that if ϕ is independent of x_0 and x_2, its value must be $1/(n+1)$. The ordinate $x=x_1$ divides the arc x_0x_2 of the curves $y=y(x)$ in the ratio arc x_0x_1/arc $x_0x_2=\theta$. We propose to show that if $y^{(n+1)}(x)$ is continuous and θ is independent of the choice of x_0 and x_2, the value of θ must be $1/(n+1)$.

Letting s be the parameter of the arc length, we use the notation

$$\begin{aligned} x_0 &= x(s), & x_1' &= \frac{\partial x}{\partial s}(s + \theta k), \\ x_1 &= x(s + \theta k), & & \\ x_2 &= x(s + k), & x_2' &= \frac{\partial x}{\partial s}(s + k). \end{aligned}$$

The partial derivative of (1) with respect to k gives

$$\begin{aligned} y'(x_2)x_2' &= x_2' \left[y'(x_0) + (x_2 - x_0)y''(x_0) + \cdots + \frac{(x_2 - x_0)^{n-2}}{(n-2)!} y^{(n-1)}(x_0) \right. \\ &\quad \left. + \frac{(x_2 - x_0)^{n-1}}{(n-1)!} y^{(n)}(x_1) \right] + \theta x_1' \frac{(x_2 - x_0)^n}{n!} y^{(n+1)}(x_1). \end{aligned}$$

Expanding $y'(x_2)$ and $y^{(n)}(x_1)$ about the point x_0, this gives

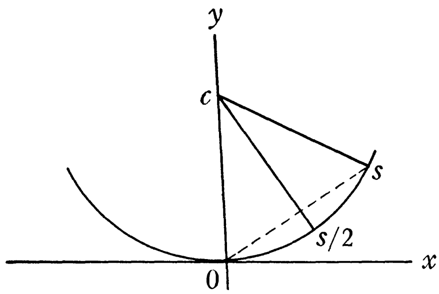
Since

$$\lim_{s\to 0} x'(s) = \lim_{s\to 0} x'(-s) = 1,$$

the plus sign holds in the neighborhood of the origin, and an integration gives us

$$x(s) = -x(-s).$$

Accordingly every normal is also a line of symmetry.



Now let the normal to the curve at s intersect the y axis at the point C. Then since the normal at the point $s/2$ is a line of symmetry, this normal is the perpendicular bisector of the line Os and must also go through the point C. Likewise the normals at $s/4$, $3s/4$, $s/8$, $3s/8$, $5s/8$, $7s/8$, $\cdots$, pass through C and from the continuity of the derivative, it follows that all the normals pass through C. Hence the curve is a circular arc.*

Properties of Parabolas Inscribed in a Triangle

By J. A. Bullard, University of Vermont

The three propositions which follow are merely corollaries of well known theorems concerning the parabola, as a figure will suggest. In the new dress they might be presented to the student in the class-room for proof. They will be used to establish other results.

If, in any triangle, a parabola is inscribed tangent to two sides with the third side forming the chord of contact, then

(1) the parabola bisects the median upon the third side;

(2) the line joining the middle points of the two tangent sides is tangent, at its middle point, to the parabola;

(3) the parabola divides the area of the triangle into two parts in the ratio 1:2. (The triangle is equivalent to the circumscribing parallelogram.)

A Construction. The second proposition gives a simple construction for a parabola when the tangents at two points are given. The writer believes it to be better than those he has found in engineering handbooks for this case. This construction gives the tangent with the point of tangency; those that he has

* The straight line is of course an exceptional case, but in this case θ can have any value.

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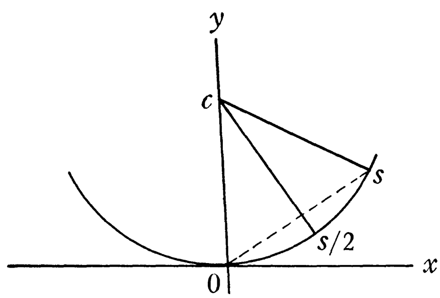

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found mentioned, determine either a tangent without the point of tangency or else merely a point on the curve. For points on the parabola between the two given ones, this construction requires three bisections and the drawing of one line segment; for points not between the given points three line segments must be doubled. Thus, in Fig. 1, given the tangents at B and C intersecting at A, by bisecting AC and AB and then bisecting the line joining the points just found,

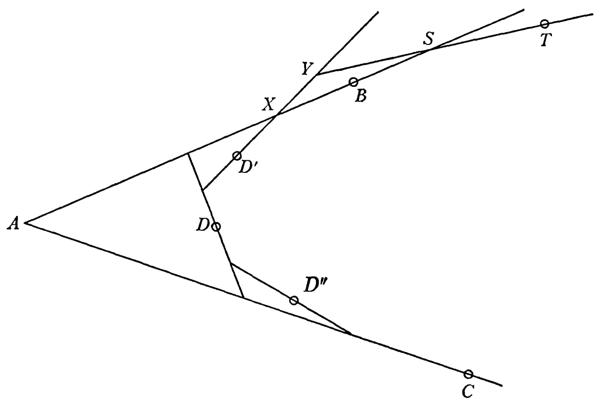


FIG. 1

we determine D, a point on the parabola, and the tangent at that point. By repeating the construction D', D'', etc., can be found. By drawing $D'XY=2D'X$, $XBS=2XB$ and $YST=2YS$, we determine a point, T, and the tangent at that point.

The following theorem will now be established by elementary methods:

THEOREM. *If three parabolas are inscribed in any triangle, each parabola tangent to two sides with the third side forming the chord of contact; then*

(1) *the parabolas intersect, two by two, on the medians at points dividing each median in the ratio* 1:8; *and*

(2) *the parabolas and the medians divide the area of the triangle into eighteen parts, twelve of which are each bounded by two line segments and a parabolic arc and each contains* 5/162 *of the area of the triangle, and six of which are each bounded by one line segment and two parabolic arcs and each contains* 17/162 *of the area of the triangle.*

To prove the first part of the theorem, let the vertices of the triangle be $O(0, 0)$, $S(x_1, y_1)$ and $T(x_2, y_2)$, the equation of the parabola through O and T be

$$(ax + by)^2 + 2dx + 2ey = 0,$$

the equation of the median upon ST be $y=y_0x/x_0$, where $2x_0=x_1+x_2$ and $2y_0=y_1+y_2$, and the point where the parabola crosses the median be denoted by (m, n).

Since T lies on the parabola, we have

$$(1) \qquad (ax_2 + by_2)^2 + 2dx_2 + 2ey_2 = 0;$$

since the tangent at O passes through S,

$$(2) \qquad dx_1 + ey_1 = 0,$$

and since the tangent at T passes through S,

$$(ax_1 + by_1)(ax_2 + by_2) + d(x_1 + x_2) + e(y_1 + y_2) = 0,$$

or,

$$(3) \qquad (ax_1 + by_1)(ax_2 + by_2) + 2dx_0 + 2ey_0 = 0.$$

Adding equation (1) and twice equation (2), we obtain

$$(4) \qquad (ax_2 + by_2)^2 + 4(dx_0 + ey_0) = 0.$$

Eliminating dx_0+ey_0 from equations (3) and (4), we have

$$(ax_2 + by_2)[(ax_2 + by_2) - 2(ax_1 + by_1)] = 0,$$

and hence

$$(5) \qquad ax_2 + by_2 = 2(ax_1 + by_1) = 4(ax_0 + by_0)/3.$$

Solving the equation of the parabola and the median simultaneously, we find

$$(ax + by_0x/x_0)^2 + 2dx + 2ey_0x/x_0 = 0,$$

or

$$(ax_0 + by_0)^2x + 2x_0(dx_0 + ey_0) = 0,$$

where x is the abscissa of the point of intersection desired. Thus

$$m = -2x_0(dx_0 + ey_0)/(ax_0 + by_0)^2,$$

and substituting from equations (4) and (5),

$$m = x_0[4(ax_0 + by_0)/3]^2/2(ax_0 + by_0)^2 = 8x_0/9,$$
$$n = 8y_0/9.$$

It can be shown similarly that the parabola through O and S also passes through this point, and since O is any vertex the conclusion follows.

To prove the second part of the theorem we construct the three parabolas and the three medians as shown in Fig. 2. Let b denote the length of GI, h the altitude upon GI, and K the area of the triangle. We first note that, as a consequence of the first part of the theorem, HE and HF are trisected. Then from simple geometric considerations we find that $EF=b/2$ and $AC=b/3$, that the altitude of triangle HEF is $h/6$ and the altitude of triangle HAC is $h/9$. Now the desired areas can be calculated from combinations of triangles, semi-segments of parabolas, and trapezoids. The equations showing the relations between these areas are follows:

$$
\begin{aligned}
AEG &= MGEB - MGAB - AEB \\
&= 3bh/16 - bh/6 - 7bh/1296 = 5bh/324 \\
&= 5K/162 = CFI, \\
AVG &= MGAB - HAB - HAV - GHM \\
&= bh/6 - 10bh/324 - bh/12 = 17bh/324 \\
&= 17K/162 = ISC.
\end{aligned}
$$

Similarly,

$$NCB = VTG = IST = NBA = 17K/162,$$

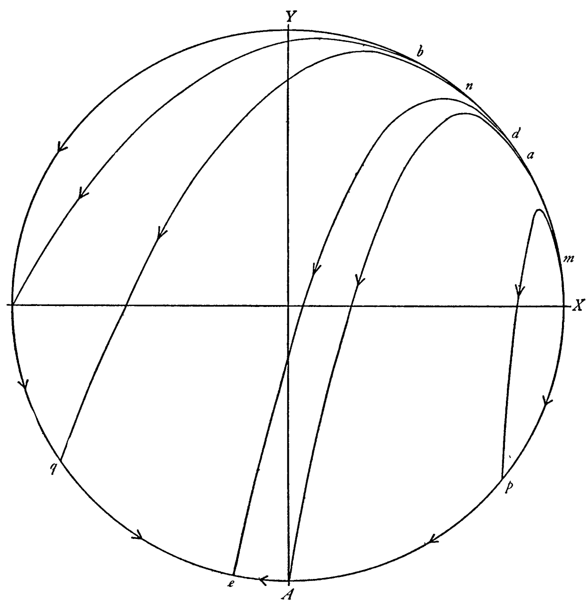
and

$$NFC = GTM = MTI = NAE = 5K/162.$$

Motion of a Twirled Weight

By W. B. Campbell, Ithaca, N.Y.

The familiar problem of the weight twirled by a cord suggests an inquiry as to what occurs if the initial velocity is less than that required to effect complete revolutions, but more than that resulting in repeated oscillations over a minor arc.



The problem is equivalent to that of a particle sliding in a vertical plane on the smooth inner surface of a horizontal cylinder having the equation $x = \cos\theta$, $y = \sin\theta$. If u is the velocity (to the right) at the bottom point $A = (0, -1)$, the

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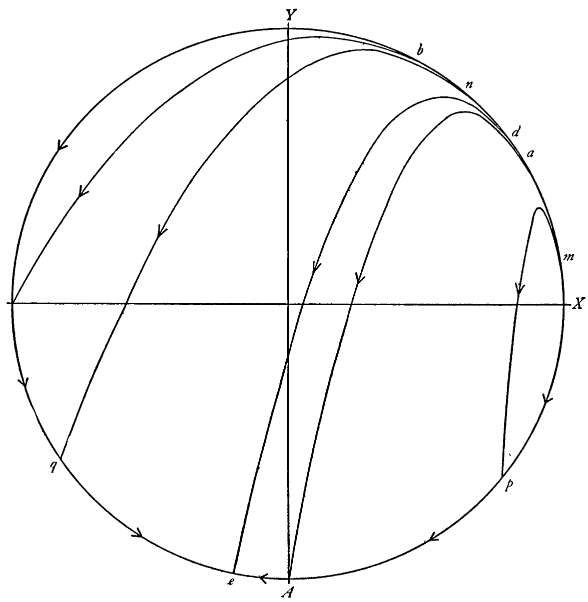

The problem is equivalent to that of a particle sliding in a vertical plane on the smooth inner surface of a horizontal cylinder having the equation $x = \cos\theta$, $y = \sin\theta$. If u is the velocity (to the right) at the bottom point $A = (0, -1)$, the

velocity v at any point (x, y) which may be reached on the circle is given by

$$v^2 = u^2 - 2g(y + 1).$$

If $u^2 < 2g$, v will become 0 while the body is still in the fourth quadrant, and it will then oscillate periodically through the fixed angle $2 \cos^{-1}\{1 - u^2/(2g)\}$.

The total acceleration required to maintain motion in such a circular path is $v^2/r = v^2$, directed toward the center. Whenever the component of gravity in the same direction, $g\sin\theta = gy$, becomes greater than v^2, the body becomes a free projectile. If $u^2 \geqq 5g$, no separation occurs, and the body makes complete revolutions indefinitely, v being a constant function of position. For $2g < u^2 = cg < 5g$, separation occurs at a point P_1 in the first quadrant. At this point P_1, $y_1 = (c-2)/3$, the velocity $v_1 = (y_1 g)^{1/2}$, with horizontal and vertical components $-v_1 y_1$ and $v_1 x_1$. The trajectory becomes

$$x = x_1 - v_1 y_1 t, \qquad y = y_1 + v_1 x_1 t - \tfrac{1}{2} g t^2.$$

The vertex is reached when $t = x_1 (y_1/g)^{1/2}$, and the parabola again meets the circle at the point Q_2, corresponding to $t_2 = 4x_1(y_1/g)^{1/2}$. If the body and wall (or the cord in the case of a twirled weight) are inelastic, the radial component of velocity is destroyed, and the new peripheral velocity v_2 at Q_2 is the algebraic sum of the tangential components there. The coordinates of Q_2 are

$$x_2 = x_1(1 - 4y_1^2), \qquad y_2 = y_1(1 - 4x_1^2) = y_1(4y_1^2 - 3),$$

If $0 < y_1 < \sin 30° = a$; Q_2 is in the fourth quadrant.
If $\sin 30° < y_1 < \sin 60° = b$; Q_2 is in the third quadrant.
If $\sin 60° < y_1 < 1$; Q_2 is in the second quadrant.
Moreover

$$v_2 = (v_1 y_1) y_2 + (v_1 x_1 - g t_2) x_2 = -(y_1 g)^{1/2}(8y_1^4 - 12y_1^2 + 3).$$

Hence $v_2 = 0$ (that is, the end of the trajectory is normal to the circle) if

$$y_1 = \tfrac{1}{2}(3 - 3^{1/2})^{1/2} = 0.563 = d, \qquad y_2 = -3^{1/2} y_1 = -0.974 = e.$$

If $y_1 < d$; $v_2 < 0$, and the initial motion at Q_2 is clockwise; if $y_1 > d$; $v_2 > 0$, and it is counterclockwise.

After circular motion is re-established, the resulting velocity w at A is such that $v_2^2 = w^2 - 2g(y_2 + 1)$. Placing $w^2 = kg$, we have

$$k = 2 + y_1(3 - 64x_1^6 y_1^2) = c - [(4/27)(c-2)(5-c)(1+c)]^3.$$

For $2 < c < 5$, we have $0 < y_1 < 1$, and $0 < k < c < 5$. It can be shown trigonometrically that there are just two values of y_1 *between* 0 and 1 at which $k = 2$; approximately they are $0.236 = m$, and $0.751 = n$, while the corresponding values of y_2 are $-0.655 = p$ and $-0.559 = q$.

For $0 < y_1 < m$; $2 < k < c$, $w < 0$, and the body after regaining circular motion at Q_2 in the fourth quadrant passes to the left through A with a velocity less than the initial velocity u, and rises to a point P_3 in the second quadrant,

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Essentials of Plane Trigonometry and *Essentials of Plane Trigonometry and Analytic Geometry.* By A. H. Sprague. New York, Prentice-Hall, Inc., 1934. viii+124 pages and x+228 pages respectively. \$.80 and \$1.80 respectively.

The first of these books is the first part of the second bound separately. There is little to distinguish them from similar textbooks except the stripping of their contents to the barest essentials for brief courses in trigonometry and plane analytic geometry. The author emphasizes the fact that the law of sines and the law of cosines are adequate for the solution of all triangles, but includes the law of tangents and the half-angle formulas in the final chapter on trigonometry. The first chapter, "Logarithms," contains a brief treatment of the laws of exponents which should be useful. Each book contains four-place tables of logarithms, squares, and sines, cosines, tangents and cotangents and their logarithms. Also there is a summary of trigonometric formulas.

The material on analytic geometry contains no mention of curves other than straight lines, circles and other conics, nor are polar coordinates treated. The conics are defined in terms of focus, directrix and eccentricity. The alleged derivation of the equations of the asymptotes of the hyperbola is fallacious. The chapter on transformations of coordinates contains the usual test for distinguishing the conics.

In spite of the above mentioned fallacy, the definitions, arguments and discussions are generally clear and concise. The supply of problems is abundant and the unusually high quality of paper and printing will recommend the books to those who find their contents adequate in extent for their needs.

J. L. DORROH

Actualités Scientifiques et Industrielles. Paris, Hermann et Cie. No. 139, *Étude des Fonctions Sousharmoniques au Voisinage d'un Point.* By Marcel Brelot. 1934. 55 pages. 14 francs.

During the last decade subharmonic functions have proven to be of considerable usefulness in a number of branches of mathematics as, for example, in the theory of functions of a complex variable, in potential theory, in the study of various partial differential and integro-differential equations and in differential geometry. The former treatments of subharmonic functions have been more or less incidental to the subjects mentioned. The present pamphlet is an attempt to give a connected account of one important aspect of the theory.

The first of the three chapters deals with the theory of convex functions of one variable, the fundamental general properties of subharmonic functions and briefly with the nature of harmonic functions in the neighborhood of a point.

The second and principal chapter of the book is a study of functions subharmonic in the neighborhood of a point, which may or may not be subharmonic at the point. The treatment is quite general, a definition of subharmonicity being adopted which does not require the function to be continuous or even to be everywhere finite. However, a few pages are devoted to the case of a continuous Laplacian, but the function is now studied not merely in the neighborhood of an isolated point but in the neighborhood of a closed point set of zero capacity.

The third and last chapter presents a few applications. One of the most interesting of these is to the investigation of the nature of harmonic functions on the boundary of their region of definition. Other applications to partial differential equations more general than Laplace's are indicated.

The book is carefully and clearly written. Although the reader will find it brief in places numerous references to the literature are given in footnotes. The language of two dimensions is used throughout although, as is indicated, the extension to three or more dimensions is easily made. A few misprints were noticed but they are of such an obvious nature that it does not seem worthwhile to list them.

G. E. RAYNOR

Fundamentals of College Mathematics. By C. H. Helliwell, Arthur Tilley, and H. E. Wahlert. New York, The Macmillan Company, 1935. xiii+406 pages. $3.50.

The reviewer was somewhat disappointed in not finding in this book what he had expected from the title. It seems to him that "College Mathematics," particularly in the freshman year, should be "fundamentally" concerned with a critical study of whatever material is taken up, rather than the acquisition of a fairly large body of technique of problem solving. However, most of the textbooks and most of the teachers leave the real "fundamentals" for advanced courses as being out of place for freshmen. As is seen further on, this text is no exception.

The topics covered are parts of intermediate and advanced algebra, trigonometry, analytics, differentiation and elementary integration. This covering of a variety of topics, largely independently of each other, gives flexibility in use. The general treatment and order of topics are largely standard. The inclusion of a certain amount of "intermediate" algebra seems a little strange for a book written in New York City, but has some advantages.

The following points appealed to the reviewer: an Introduction consisting of "Review Topics for Reference" (elementary algebra, plane geometry); Chapter I on "The Number System of Algebra," including the best elementary definition of an irrational number yet seen by the reviewer, namely—"a number . . . which cannot be expressed as a rational number, but which can be approximated as closely as desired by rational numbers"; the introduction of the trigonometric functions by the general definition; and the early assumption of the formula for differentiating a power, logarithmic proof of which is given later.

The following points did not appeal to the reviewer: the woefully incomplete treatment of determinants; the handling of the functions of quadrantal angles by a limiting process, instead of using the general definition already given; the lack of mention, to say nothing of proof, that $dy/dx=(dy/dv)(dv/dx)$; and the consequent lack of any logical treatment of integration by substitution.

The chapter on "Limits" requires a paragraph to itself. In the first place it is marked with an asterisk, implying that the authors think it may be omitted in a shorter course. To the reviewer it seems unthinkable that the student be subjected to the "Δ-process" without a clear conception of limit, and without even a mention of continuity. Even if the asterisk is disregarded, the distinction between $f(c)$ and $\lim_{x\to c}f(x)$ is never clear—to quote: "*By inspection it can be seen** that as x takes on values nearer and nearer to 2, such as ± 1, ± 1.5, ± 1.9, $\pm 1.99, \cdots$, $3x^2=3$, 6.75, 10. 83, 11.8803, $\cdots$ in succession, and finally

$$\lim_{x\to 2} 3x^2 = 12."$$

In passing, the reviewer is surprised to read that the above sequence for x approaches 2; possibly the ambiguous signs are an oversight. In later chapters, (page 324 last line, and page 337 last line) it is implied that the cosine and logarithm are continuous, without even a statement to that effect. Such slipshodness will invariably lead the student to the belief that evaluation of a limit is the same as mechanical substitution and soon there will crop out the equation, familiar to all teachers of beginning calculus:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{0}{0}\cdot$$

A further quotation from the chapter on limits: "Thus a continuous curve can be drawn without removing the pencil from the paper." One wonders if Weierstrass and Peano have lived in vain!

Four-place tables, answers to alternate problems, and a good index, complete the book.

L. S. KENNISON

Financial Mathematics. By A. W. Richeson. New York, Prentice-Hall, Inc., 1935. xiv+361 pages. $2.50.

In the preface, the author states the text is intended primarily for students majoring in business administration, but that it is designed so that it may be used with profit by those who are not specializing in that subject. The book is divided into four parts: interest and annuities, probability and life insurance, review of subjects from algebra, and 15 tables.

The book is well written in simple language. It appears to be a happy medium between the theoretical and the practical side of the subject. Brief intro-

* Italics mine—L. S. K.

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ductory discussions of the various topics are given before the theory is taken up. The explanations are careful and complete; derivations of the formulas, logical and clar-cut; and definitions, concise. The text is not "cluttered up" with useless formulas and illustrative exercises.

The selection of the problems seems very good. Students will not form the idea that all rates are 4%! The range of interest rates is somewhat greater than appears in similar texts. Review exercises are given at the end of each chapter, and at the conclusion of Part II, 159 practical miscellaneous problems are compiled.

For a first edition there are perhaps a minimum of errors. The book seems to be very usable.

H. A. ROBINSON

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pennsylvania

All reports of club activities should be sent to F. W. Owens, 462 East Foster Ave., State College, Pennsylvania.

CLUB ACTIVITIES

1934–1935

Pi Mu Epsilon of the University of Kentucky

With twenty-three members, a successful year passed under the following officers: Flora E. LeStourgeon, Director; Harry Spragens, Vice-Director; M. C. Brown, Secretary; Anna Bruce Gordon, Treasurer; H. H. Downing, Librarian.

The meetings and programs were as follows:

September 27, 1934: "Tetracyclical co-ordinates" by James Stewart; "More on quadratic transformations" by Dr. P. P. Boyd.

October 25, 1934: "Fermat's method of infinite descent with illustrations" by C. G. Latimer. Election of six new members.

November 16, 1934: Semi-annual banquet and initiation at Tea Cup Inn. "History of Pi Mu Epsilon" by Dr. P. P. Boyd. William Pell spoke for the initiates.

November 22, 1934: "Metric spaces of infinitely many dimensions" by L. W. Cohen.

December 13, 1934: "Astronomical models" by H. H. Downing.

January 17, 1935: "Maxwell's equations" by B. P. Ramsay, Department of Physics.

February 28, 1935: "The law of mortality" by D. E. South. Election of four new members.

March 28, 1935: "Italy and mathematics" by F. E. LeStourgeon.

April 5, 1935: Semi-annual banquet and initiation, Wellington Arms Apartment. The new members were welcomed by Dr. P. P. Boyd; Response "A beginning, not an end" by James Brittain; "Culture versus science" by C. G. Latimer.

April 25, 1935: "What is a matrix?" by Dr. M. H. Ingraham, guest speaker from the University of Wisconsin.

May 23, 1935: Reports and election of officers.

Kentucky Alpha chapter is pleased that it can annually present to the library of the mathematics department of the University of Kentucky a set or collection of worthwhile books of mathematical literature.

M. C. BROWN, *Secretary*

duced.) Now if the areas of the triangles PQR, PDE, QFA and RBC, with proper signs, are denoted by k, l, m and n, show that $1/k=1/l+1/m+1/n$.

E 184. *Proposed by M. O. Reade, Brooklyn, New York.*

The cube root of an eight-digit number was extracted in the usual manner, and then each digit was replaced by a code letter, with the following result. Solve the code.

$$
\begin{array}{ll l l|l}
W & P, & W\,D\,B, & M\,W\,C & \underline{A\;K\;C} \\
B & P & & & \\
\hline
A & S & W\,D\,B & & \\
B & P & K\,P\,B & & A\;M\;K\;M \\
\hline
 & B & C\,M\,S & M\,W\,C & \\
 & B & C\,M\,S & M\,W\,C & M\;M\;S\quad S\;P\;C \\
\hline
\end{array}
$$

E 185. *Proposed by C. A. Richmond, Tyngsboro, Mass.*

A thin, straight wire is marked off into m equal lengths by $m-1$ points. It is then bent at a right angle at each of one or more of these points, making each segment parallel to one of two rectangular axes. The bent wire may be self-intersecting, but not self-coincident over a finite length. How many different shapes may it have? Extend the problem to three dimensions by permitting the segments of the wire to be parallel to any of three rectangular axes.

E 186. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

In a geometric progression whose terms are all positive integers, the first five terms contain nine digits each, the next five terms contain ten digits each, the next four terms contain eleven digits each, and the remaining two terms contain twelve digits each. Reconstruct the progression and show that it is unique.

SOLUTIONS

E 14 [1932, 606]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

At country fairs there occasionally appears a man with five discs, each four inches in diameter, and a table on which is painted a larger red circle. He challenges anyone to place the five discs so they completely cover the red circle on the first attempt. It appears that a correct solution lacks central symmetry. How large is the maximum red circle which may be covered by the five discs, and how must they be placed?

Editorial Note. No solution of this problem, nor of its equivalent, 3574 in the advanced problems, has been received. However, J. E. Trevor of Cornell points out that the problem appears in Ball's *Recreations*, 1922, p. 253; and also in an article by Neville in the Proceedings of the London Math. Soc., vol. 14 (1915), p. 308.

Editorial Note on E 130. Through an error, a solution of this problem [1935, 324] was credited to R. A. Johnson which should have been credited to R. A. Johnson, Jr.

E 154 [1935, 320]. *Proposed by V. Thébault, Le Mans, France.*

Find the smallest positive integer, not beginning with zero, such that if it is written down twice in succession so as to form a number of twice as many digits, that number will be a perfect square.

Solution by G. E. Raynor, Lehigh University

Let N be the required positive integer and n the number of its digits. Also, let S be the number obtained by writing N twice in succession. Then evidently $S=N(10^n+1)$. Since N has n digits and 10^n+1 has $n+1$ digits, $N<10^n+1$, and hence S can not be a perfect square if the prime factors of 10^n+1 are all distinct. We thus seek the smallest value of n for which 10^n+1 has a repeated prime factor.

The smallest value of n which gives 10^n+1 a square factor is found by trial to be $n=11$. Then $10^{11}+1=11^2\cdot 23\cdot 4093\cdot 8779$. Hence N must have eleven digits, and must be a square multiple of $23\cdot 4093\cdot 8779=826{,}446{,}281$. The smallest square by which we may multiply 826,446,281 and get an eleven digit number is 16, so that $N=16\times 826{,}446{,}281=13{,}223{,}140{,}496$. Then $S=1322314049613223140496=36363636364^2$.

Also solved by W. E. Brooke, Mary L. Constable, Daniel Finkel, W. F. Penney, Joseph Milkman, E. P. Starke, C. W. Trigg, Simon Vatriquant, G. W. Wishard and the proposer.

E 155 [1935, 320]. *Proposed by Maud Willey, Gulfport, Mississippi.*

Prove that

$$\sum_{i=0}^{n}\left[{}_nC_i\sum_{j=0}^{i}{}_iC_j\right]=3^n.$$

Solution by W. F. Penney, Union City, N. J.

In the expansion, $(1+x)^i={}_iC_0+{}_iC_1x+{}_iC_2x^2+\cdots+{}_iC_ix^i$ let $x=1$. Then

$$\sum_{j=0}^{i}{}_iC_j=2^i.$$

Similarly, $\sum_{i=0}^{n}({}_nC_i\cdot 2^i)$ is the value of the expansion $(1+y)^n$ when $y=2$. This value is obviously 3^n. Consequently,

$$\sum_{i=0}^{n}\left[{}_nC_i\sum_{j=0}^{i}{}_iC_j\right]=3^n.$$

If this be generalized by making r multiplications and summations, the result will obviously be $(r+1)^n$.

Also solved by B. Le F. Brown, J. F. Locke, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 156 [1935, 320]. *Proposed by J. A. Bullard, University of Vermont.*

If a parabola is inscribed to two sides of an equilateral triangle with the third side forming the chord of contact, then the focus of the parabola lies at the centroid and the latus rectum is equal to the radius of the circumcircle. If three such parabolas are inscribed, what is the ratio of the area inside all three parabolas to the area of the triangle?

Solution by K. W. Crain, Purdue University

Consider the equilateral triangle whose vertices in rectangular coordinates are $A(0, a)$, $B(0, -a)$ and $C(a\sqrt{3}, 0)$ where a is positive. Inscribe the parabola which is symmetric to the x-axis and which is tangent to AC at A and to BC at B. Its equation would be of the form

$$y^2 = -2p(x - h). \tag{1}$$

Then $dy/dx = -p/y$, and since the slope of the tangent to the parabola at A is $-1/\sqrt{3}$, $p = a/\sqrt{3} = a\sqrt{3}/3$. Also, since (1) passes through A, $h = \frac{1}{2}a\sqrt{3}$, and the parabola has the equation

$$y^2 = (-2a/\sqrt{3})(x - a\sqrt{3}/2). \tag{2}$$

From this we see that the focus is the point $F(a/\sqrt{3}, 0)$, or the centroid of the triangle. Furthermore, from the equation (2), the length of the latus rectum is $2a/\sqrt{3}$. In an equilateral triangle, the centroid is also the circumcenter, and $AF = 2a/\sqrt{3}$. Therefore the radius of the circumcircle is equal to the length of the latus rectum.

From the symmetry of the figure, due to the fact that ABC is an equilateral triangle, we see that the area enclosed by the three parabolas is six times the area bounded below by the x-axis, above at the left by the line BF produced, and above at the right by the parabola (2). Hence the area bounded by the three parabolas is given by

$$6\int_0^{a/3}\left[\frac{3a^2 - 3y^2}{2a\sqrt{3}} - \frac{a + y}{\sqrt{3}}\right] dy = 5a^2\sqrt{3}/27.$$

But the area of the triangle ABC is $a^2\sqrt{3}$, so that the required ratio of areas is $5/27$.

Also solved by Leon Recht, E. P. Starke, C. W. Trigg, Simon Vatriquant and the proposer.

E 157 [1935, 321]. *Proposed by Raymond Garver, University of California at Los Angeles.*

Prove that

$$2\sqrt{7}\cos\left[\tfrac{1}{3}\cos^{-1}(1/2\sqrt{7})\right] - 6\cos(2\pi/7) = 1.$$

Solution by E. P. Starke, Rutgers University

If in the identity

$$\cos 7\theta \equiv 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$$

we first set $\theta = 2\pi/7$, so that $\cos 7\theta = 1$, and then set $6 \cos \theta = x$, the resulting equation reduces to

$$(x - 6)(x^3 + 3x^2 - 18x - 27)^2 = 0.$$

Since the equation, $x^3+3x^2-18x-27=0$, has but one positive root, we know that it must be $x=6 \cos 2\pi/7$.

Now let $1/2\sqrt{7} = \cos 3\phi$, which identically equals $4 \cos^3 \phi - 3 \cos \phi$, with $0<\phi<\pi/6$, and put $\cos \phi = (y+1)/2\sqrt{7}$. This will show that $y = 2\sqrt{7} \cos \phi - 1$ is the positive root of $y^3+3y^2-18y-27=0$. Hence $x=y$. If $\cos^{-1}(1/2\sqrt{7}) = 3\phi + 6k\pi$, with k any integer, the proposed equation follows immediately. But if $\cos^{-1}(1/2\sqrt{7}) = 3\phi + (6k \pm 2)\pi$, the proposed equation is not true unless $2\pi/7$ be replaced by $4\pi/7$ or $6\pi/7$.

Also solved by Charles O'Hara, Simon Vatriquant and the proposer.

E 158 [1935, 321]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Several years ago today John Smith borrowed some money from his bank at a normal rate of simple interest and then vanished without paying anything on his debt. Today he suddenly reappeared at the bank and paid off his accumulated indebtedness, which amounted to precisely $204.13. How much did he borrow, at what rate did he borrow it, and how long did he keep it?

Solution by Wm. Douglas, Courtenay, B. C.

Let P, r and t denote the principal, the percentage rate per annum and the time in years respectively. Then $P+Prt/100 = 204.13$, or

$$P = \frac{20413}{100 + rt} = \frac{137 \times 149}{100 + rt}.$$

We note that 137 and 149 are both primes, and as P must have a rational value, t be integral and r a normal rate of interest, we have only the one obvious choice: $r=t=7$. Hence the sum borrowed was $137.00; the rate was 7% and the term was 7 years.

Also solved by L. J. Adams, W. E. Brooke, Mary L. Constable, Daniel Finkel, W. F. Penney, W. R. Ransom, Dorothy Stephenson, C. W. Trigg, Simon Vatriquant, B. C. Zimmerman and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would

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Advanced Problems

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Show that the probability that the selected urn be the jth one is given by the formula

$$P_j = \frac{W_j}{\sum_{i=1}^{n} W_i}$$

where

$$W_i = (w_i + b_i - w - b)!(w+b)!\frac{(w_i)!}{w!(w_i - w)!}\,\frac{(b_i)!}{b!(b_i - b)!}$$

with the usual conventions that

$$\frac{1}{(w_i - w)!} = 0 \text{ if } w > w_i \text{ and } \frac{1}{0!} = 1.$$

It evidently may happen that for some value of j, $P_j = 1/n$. Therefore, our increased knowledge is not always sufficient to increase our ability to predict an event. This particular example (for simple cases) is thus often used to illustrate a possible "faux pas" in reasoning on this subject. See, Poincaré, *Probabilités*, page 26, "Problème des trois coffrets." See also Czuber, *Wahrscheinlichkeitsrechnung*, vol. 1, art. 16. Is the above statement true if each of the urns is known to contain only white balls but not all the same number?

3763. *Proposed by Paul Erdös, The University, Manchester, England.*

Given any simple polygon P which is not convex, draw the smallest convex polygon P' which contains P. This convex polygon P' will contain the area P and certain additional areas. Reflect each of these additional areas with respect to the corresponding added side, thus obtaining a new polygon P_1. If P_1 is not convex, repeat the process, obtaining a polygon P_2. Prove that after a finite number of such steps a polygon P_n will be obtained which will be convex.

SOLUTIONS

3681 [1934, 333]. *Proposed by Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y.*

Construct the triangle ABC, given its orthocentre, and the mid-points of two of its sides.

I. *Solution by Hansraj Gupta, Govt. College, Hoshiarpur (India)*

Analysis. Let $E(e, 0)$ and $F(f, 0)$ be the given mid-points of sides, and $H(0, h)$ the given orthocentre. Let O be the origin of rectangular coordinates. Let the vertex A which lies on OH be $(0, a)$. Then B is $(2e, -a)$ and C is $(2f, -a)$. Since CH is perpendicular to AB, we must have

$$\frac{h+a}{-2f}\cdot\frac{2a}{-2e} = -1, \qquad a^2 + ah + 2ef = 0,$$

$$a = -\frac{h}{2} \pm \sqrt{\frac{h^2}{4} - 2ef}.$$

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$$a = -\frac{h}{2} \pm \sqrt{\frac{h^2}{4} - 2ef}.$$

be a point obtained in the construction; let H' be the midpoint of CH; and L', the midpoint of CL. Then $MPL'H'$ is a parallelogram; and MH' is perpendicular to LC, since by the construction PL' is perpendicular to LC. Thus H' is the orthocentre of LMC, and hence H is the orthocentre of the completed triangle ABC.

The case where the projection of H on LM falls at either end point is rather trivial. If the projection of H falls within that segment, PL is greater than PM, and the circle with center P and radius PL cuts in two points the perpendicular from H to LM. This follows from the fact that the projection of MP on ML is just one-half of that of HL, or less than half of ML. Hence the projection of LP is more than half of ML, and $LP > MP$. In this case there always are two constructions. The condition for real constructions in the remaining cases requires more computation, but of a simple nature, and the result is the same as in I.

In solution III the statement that two solutions are "always possible" is inaccurate. There are always two distinct constructions if the projection of H on MN falls within that segment. For then the angle NMM' is obtuse and M lies within the circle upon NM' as a diameter. In this case the circle cuts in two distinct points the perpendicular from H to MN. As before there may or may not be constructions if the projection falls outside MN.

The remaining solutions reduced the problem to the construction of the roots of a quadratic; and Clawson gave a geometric construction for these roots.

3682 [1934, 333]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

If a prime p has the form $p = 4k+3$, and m is the number of quadratic non-residues less than $p/2$, prove that

(a) $$1\cdot 3\cdot 5\cdots(p-2) \equiv (-1)^{m+k} \bmod p,$$

(b) $$2\cdot 4\cdot 6\cdots(p-1) \equiv (-1)^{m+k+1} \bmod p.$$

Solution by Harry Gershenson, Brooklyn, New York

We shall use the following well-known facts:

(A) Wilson's Theorem: If p is a prime, then

$$(p-1)! \equiv -1 \pmod{p}.$$

(B) The product of any number of quadratic residues is a quadratic residue. The product of an odd number of quadratic non-residues is a quadratic non-residue; the product of an even number of quadratic non-residues is a quadratic residue.

(C) -1 is a quadratic non-residue of a prime $p = 4k+3$.

In the congruence in (A) replace $(p-s)$ by $-s$ for all values s equal to or less than $2k+1$. The result is:

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In the congruence in (A) replace $(p-s)$ by $-s$ for all values s equal to or less than $2k+1$. The result is:

$$(-1)^{2k+1}\{1\cdot 2\cdot 3\cdots(2k+1)\}^2 \equiv -1 \pmod{p},$$

or

$$(2k+1)! \equiv \pm 1 \pmod{p}.$$

To determine the choice of sign, we note that the product $(2k+1)!$ is the product of m quadratic non-residues and $2k+1-m$ quadratic residues. According, therefore, as m is odd or even, the product is a quadratic non-residue or a quadratic residue; this result follows directly from (B). If m is even, that is, if the product is a quadratic residue, the positive sign must be taken, in accordance with (C). If m is odd the negative sign is required. In either case,

$$(2k+1)! \equiv (-1)^m \pmod{p}.$$

In the last result, replace each of the k even integers $E=2, 4, 6, \cdots, 2k$ by $(p-E)$, and arrange the integers in proper order to yield:

$$(-1)^k 1\cdot 3\cdot 5\cdots(p-2) \equiv (-1)^m \pmod{p},$$

which immediately becomes

$$1\cdot 3\cdot 5\cdots(p-2) \equiv (-1)^{m+k} \pmod{p}.$$

This result with (A) gives the second congruence of the problem.

Solved also by Melvin Dresher, Hansraj Gupta, A. S. Peters, E. P. Starke, and the proposer.

Editorial Note. Several of the simpler theorems of integers, such as are used in the solutions of this problem, are fundamental theorems of finite groups. This is an obvious but important fact. The residues of an odd prime p, excluding zero, form an Abelian group G of order $p-1$ with ordinary multiplication and reduction modulus p as the law of combination. The identity element of the group is the integer unity. The period of each element of a finite group is a divisor of the order of the group, and hence

$$(1) \qquad x^{p-1} \equiv 1 \bmod p$$

has precisely $p-1$ distinct solutions. The elements of G in this case are also elements of another Abelian group of order p with ordinary addition as the law of combination and with zero for the unit element, or identity. This group is obviously cyclic. Since the elements of G have a second law of combination, it is possible to say more about its nature. The elements of G may be regarded as $p-1$ classes of integers prime to p, and these classes may be represented in a variety of ways by selecting an individual from each class. We may select as elements the positive residues of p, or the positive and negative residues whose absolute values are less than $p/2$, or the positive odd residues of p less than $2p$.

We easily see that there is only one element of period two and it may be written as $p-1$ or as -1. Every other element, excluding the identity, has a unique inverse element different from the original element. The product of all

$$\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv (-1)^{(p+1)/2}, \bmod p, \tag{6}$$

a result which is useful in this problem.

The test in (3) and (4) for elements of K is due to Euler. There is another test related to this problem which is due to Gauss. If z is any element of G we consider the sequence formed by the products of z and those positive elements of G which are less than $p/2$. These distinct elements may then be expressed as $(p-1)/2$ positive and negative elements of absolute value less than $p/2$. Since neither the sum nor the difference of two of these residues can be divisible by p, there are no two residues of the same absolute value. The product of all the residues is then congruent to $(-1)^\mu[(p-1)/2]!$, where μ is the number of negative residues. But it is also equal to this same factorial times $z^{(p-1)/2}$. Hence

$$z^{(p-1)/2} \equiv (-1)^\mu \bmod p, \tag{7}$$

and z belongs to K or N according as μ is even or odd by (3) and (4). For $z=2$ there are two cases. If $(p-1)/2$ is even it is clear that $\mu=(p-1)/4$; if $(p-1)/2$ is odd, then $\mu=(p+1)/4$. In the first case $(p+1)/2$ is odd and $\mu\equiv(p^2-1)/8$, mod 2. This congruence is also true in the second case, and hence

$$2^{(p-1)/2} \equiv (-1)^{(p^2-1)/8} \bmod p. \tag{8}$$

The solutions of Dresher, Peters, Starke and the proposer made use of this result, which with (6) gives (b) of the problem and then (a) follows easily. Gupta used a theorem of Gauss which says that

$$\begin{aligned} 1\cdot3\cdot5 \cdots (p-2)\cdot(p+2) \cdots (2p-1) &\equiv -1 \bmod 2p, \\ &\equiv -1 \bmod p. \end{aligned}$$

The second of these is merely another way of writing (2) as stated above, and the first results from the second immediately. The integers less than $2p$ and prime to $2p$ form an Abelian group of order $p-1$; and these same integers may also be taken as the elements of G. It is clear that we have thus a one-to-one isomorphism between G and G'. These two groups have therefore the same abstract properties; for example, G' is cyclic.

3683 [1934, 334]. *Proposed by Raphael Robinson, University of California at Berkeley.*

Show that the sum of the medians of a simplex in n dimensions is smaller than $2/n$ and greater than $(n+1)/n^2$ times the sum of the edges of the simplex, and that these are the *best limits* that can be given.

This is a generalization of problem 3618, solved in [1934, 338].

Solution by J. K. Peterson, Nashville, Tenn.

The proposition holds for an n-simplex (in any r-space) which has at least three non-collinear or at least four distinct vertices. If the two limits may be *attained*, it holds for an arbitrary n-simplex, however degenerate.

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Let the vertices $A_0, A_1, \cdots, A_n$ of the n-simplex be fixed by the vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_n$ from the centroid; let m_i be the length of the median from A_i; e_{ij}, the length of the edge A_iA_j; M, the sum of the medians; and E, the sum of the edges. It may be of interest to note first that the relations

$$2/n > M/E > (n+1)/n^2 \tag{1}$$

are equivalent to the

THEOREM. *The mean of the medians of a simplex is less than the mean of the edges, whether the weights are all equal or are proportional to the magnitudes to be averaged.*

For equal weights, $M/(n+1) < 2E/n(n+1)$ is obviously equivalent to the first inequality (1). For weightings proportional to the magnitudes, the theorem states that

$$\frac{\sum_{i=0}^{n} m_i^2}{\sum_{i=0}^{n} m_i} < \frac{\sum_{i<j} e_{ij}^2}{\sum_{i<j} e_{ij}}. \tag{2}$$

But by (3) in the solution of 3656 [1935, 184], (2) is equivalent to

$$\frac{M}{E} > \frac{\sum_{i=0}^{n} m_i^2}{\sum_{i<j} e_{ij}^2} = \frac{n+1}{n^2},$$

or to the second inequality (1).

We first show that the length of a median is not greater than the mean of the lengths of the edges from the same vertex. The mean length, a_0, of the edges from A_0 is

$$\frac{1}{n}\sum_{i=1}^{n} e_{0i} = \frac{1}{n}\sum_{i=1}^{n}\sqrt{\mathbf{b}_i\cdot\mathbf{b}_i},$$

where $\mathbf{b}_i = \mathbf{a}_i - \mathbf{a}_0$. Thus

$$n^2a_0^2 = \sum_{i,j=1}^{n}\sqrt{(\mathbf{b}_i\cdot\mathbf{b}_i)(\mathbf{b}_j\cdot\mathbf{b}_j)}.$$

Since m_0 is the length of the vector

$$\frac{1}{n}(\mathbf{a}_1 + \cdots + \mathbf{a}_n) - \mathbf{a}_0 = \frac{1}{n}\sum_{i=1}^{n}\mathbf{b}_i, \tag{3}$$

we have

vertices, the equality holds if and only if they are collinear, C coincides with the middle one, and none except perhaps C is a multiple vertex; and if there are precisely two distinct vertices, the equality holds if and only if $n=1$.

In case the n-simplex has as many as four distinct vertices, or as many as three non-collinear vertices (in particular, in case the n-simplex is non-degenerate with $n \geqq 2$), both the inequalities (1) hold.

It will now be shown that the range for M/E in (1) cannot be narrowed and still suffice for all non-degenerate n-simplexes. Let $(x_1, x_2, \cdots, x_n)$ be the Cartesian coordinates of a point in n-space, and consider first the non-degenerate simplex with A_0 at the origin and with the other vertices each having but one non-zero coordinate, as follows: A_1 has $x_1=-1$ and A_i has $x_i=\delta$, $\delta>0$, $2 \leqq i \leqq n$. For $\delta=0$ there are only two distinct vertices and $M/E=2/n$. For $\delta>0$ $M/E<2/n$, as shown above. Since M/E as a function of δ is continuous at $\delta=0$, we can, for any given positive ϵ, determine a δ such that for it and smaller positive values, $M/E>2/n-\epsilon$. Hence no smaller upper limit for M/E can be used.

Now consider another non-degenerate simplex in the same space. The vertex A_0 is again at the origin, A_1 has $x_1=2$ and each other coordinate zero, while each coordinate of A_i is zero except $x_1=1$ and $x_i=\delta$, $\delta>0$, $2 \leqq i \leqq n$. Here again M/E is a continuous function of δ at $\delta=0$. For $\delta>0$, $M/E>(n+1)/n^2$, and for $\delta=0$, $M/E=(n+1)/n^2$. This suffices to show that the lower limit for M/E cannot be increased.

Solved also by J. Rosenbaum and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

To celebrate the centenary of the birth of Simon Newcomb, a cairn has been erected by the Historic Sites and Monuments Board of Canada, four miles from Wallace, Nova Scotia, where Newcomb was born. The cairn was unveiled August 30, 1935, by his daughter, Mrs. Joseph Whitney.

C. W. Crockett, professor of mathematics and astronomy at the Rensselaer Polytechnic Institute, has been made professor emeritus.

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CONTENTS

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twentieth Annual Meeting, St. Louis, Mo., Dec. 30-31, 1935.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1935 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Bethany, W.Va., May 4; Beaver Falls, Pa., Oct. 26.
ILLINOIS, Decatur, May 3-4.
INDIANA, Hanover, May 3-4.
IOWA, Grinnell, Apr. 19–20.
KANSAS, Topeka, Mar. 16.
KENTUCKY, Lexington, May 4.
LOUISIANA-MISSISSIPPI, Pineville, La., Mar. 29-30.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D.C., May 11; College Park, Md., Dec. 7.
MICHIGAN, Ann Arbor, Mar. 9.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, May 3.
OHIO, Columbus, Apr. 4.
OKLAHOMA, Tulsa, Feb. 1.
PHILADELPHIA, Easton, Pa., Nov. 30.
ROCKY MOUNTAIN, Golden, Colo., Apr. 19-20.
SOUTHEASTERN, Decatur, Ga., Mar. 22-23.
SOUTHERN CALIFORNIA, Los Angeles, Mar. 2.
TEXAS, Lubbock, Apr. 20.
WISCONSIN, Milwaukee, May.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS